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Weak convergence and weak compactness in the space of almost periodic functions on the real line

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Abstract. We give necessary and sufficient conditions for sequences in the space AP($\mathbb{R}$) of continuous almost periodic functions on the real line to converge in the weak topology. The abstract results are illustrated by a number of examples which show that weak convergence seems to be a rare phenomenon. We also characterize the weakly compact subsets in AP($\mathbb{R}$). In particular, earlier statements made in the monograph by Dunford and Schwartz are refined and completed. We close with some open problems.

Keywords. Almost periodic functions; weak convergence and weak compactness in spaces of continuous functions.

1. Introduction

In the investigation of classical Banach spaces the characterizations of weakly convergent sequences and weakly compact subsets have always been considered an integral part of the desirable knowledge about these spaces [7, p. 374–379]. It is well-known that the investigation of weak compactness has led to some of the deepest results in functional analysis. In particular, spaces such as the space of the Bochner integrable functions, the problem has been under research for more than twenty years and it seems only now that a satisfactory answer has been obtained [4].

In this paper we call the reader's attention to the problems of characterizing weak convergence and weak compactness in the space AP($\mathbb{R}$) of the complex valued continuous almost periodic functions on $\mathbb{R}$. In spite of the extensive literature about AP($\mathbb{R}$) written over several decades these questions seem to have been addressed only in the treatise of Dunford and Schwartz [7, p. 379]. As for weak compactness, the reader is referred to a criterion for relatively weakly compact subsets in the space $B(S)$ of all bounded functions on an arbitrary set S [7, p. 280], which is not appropriate for AP($\mathbb{R}$); weak convergence is characterized by one condition in an exercise [7, p. 345], which is not easy to apply. Earlier, W F Eberlein had termed sequential convergence in $C_b(\mathbb{R})$ "refractory" [8, p. 232]; this remark has retained its truth equally for the subspace AP($\mathbb{R}$). In fact, a characterization of weak convergence or weak compactness in AP($\mathbb{R}$) faces the following difficulties:

1. The reduction of the problem to $C(S)$ (S being a compact Háusdorff space) with the help of the isometric isomorphism AP($\mathbb{R}$) $\equiv C(\mathbb{R}^a)$ (where $\mathbb{R}^a$ is the Bohr compactification of $\mathbb{R}$) is not illuminating because of the involvedness of $\mathbb{R}^a$.
2. It seems to be impossible to make use of the representation of the dual AP($\mathbb{R}$)* of AP($\mathbb{R}$) as a space of finitely additive measures which was given by Hewitt [9]. In fact, obtaining a natural sufficient condition for weak convergence amounts to proving

a theorem for passing to a limit under the integral sign where the integral is taken with respect to a finitely additive measure, and the status of the theory of finitely additive measures [15] gives enough evidence that the question poses insurmountable difficulties.

With the intention to fill the gap in the literature we reinvestigate these problems. Our study shows, that, as for weak convergence, the main problem is to find conditions under which a bounded sequence (f_n) in $\mathrm{AP}(\mathbb{R})$ which converges pointwise on $\mathbb{R}$, converges to a limit f in $\mathrm{AP}(\mathbb{R})$. Considering the corresponding sequence $(\hat{f}_n)$ in $C(\mathbb{R}^a)$, we need to find conditions under which the pointwise convergence of the (f_n) on the dense subset $\mathbb{R}$ leads to pointwise convergence on the compact Hausdorff space $\mathbb{R}^a$ to a limit in $C(\mathbb{R}^a)$. Lemma 2.1 of § 2 answers this question in a more general situation. In Theorem 2.3, we characterize weak convergence in $\mathrm{AP}(\mathbb{R})$ by eight conditions. A tool which is used here is the concept of a Bohr net in $\mathbb{R}$.

We have made particular efforts to give conditions for sequences in $\mathrm{AP}(\mathbb{R})$ as functions on $\mathbb{R}$ rather than as sequences in $C(\mathbb{R}^a)$. According to Arzelà's Theorem [7, p. 268] the limit of a bounded pointwise converging sequence in $C_b(\mathbb{R})$ is continuous iff the convergence is quasiuniform on every compact subinterval. In a similar spirit, we characterize these sequences in $\mathrm{AP}(\mathbb{R})$ for which the convergence is quasiuniform and at the same time the pointwise limit belongs to $\mathrm{AP}(\mathbb{R})$; this is done in Lemma 2.2.

Section 3 contains the main theorem on weak compactness. In § 4 we illustrate the abstract results by a number of examples and counterexamples which may have interest of their own. Weak convergence is studied here also with a look at the ε-periods of the involved functions (the reader may compare Examples 4.6 and 4.7 with the somewhat surprising Proposition 4.8). Using the representation for the characters on $\mathbb{R}$ by *v.* Neumann and a theorem of Lindemann–Weierstrass–Kronecker on the linear independence of a certain class of real numbers over the rationals, we disprove weak convergence for some natural candidates of sequences. The given examples create the impression that weak convergence aside from trivial (e.g. norm convergent or uniformly periodic) examples seems to be a rare phenomenon.

In § 5 we collect some open questions which we hope might appeal to readers interested in classical real analysis.

The standard notation and terminology of [7] and of the literature on almost periodic functions (see e.g. [1], [3], [5], [11]–[13], [16]) has been used. In particular, $\mathrm{AP}(\mathbb{R})$ is the Banach algebra of complex valued continuous almost periodic functions on the real line $\mathbb{R}$ and $\mathbb{R}^a$ is the Bohr compactification of $\mathbb{R}$. For $f \in \mathrm{AP}(\mathbb{R})$, $\hat{f}$ is its continuous extension to $\mathbb{R}^a$ and for $t \in \mathbb{R}$, $f_t(x) := f(x+t)$ will be its translate. $C_b(\mathbb{R})$ is the Banach space of bounded continuous functions on $\mathbb{R}$. For $f \in C_b(\mathbb{R})$ $\|f\|$ will be $\|f\|_\infty$.

2. Weak convergence

It follows from the characterization of weak convergence in $C(S)$ for a compact Hausdorff space S that a sequence $(f_n) \subset \mathrm{AP}(\mathbb{R})$ tends to a limit $f \in \mathrm{AP}(\mathbb{R})$ weakly if and only if it is bounded and $\hat{f}_n \longrightarrow \hat{f}$ pointwise on $\mathbb{R}^a$, which implies $f_n \longrightarrow f$ pointwise on a dense subset of $\mathbb{R}^a$. In the following lemma we treat a more general situation. One may compare the proof with that of [7, Theorem 14, p. 269].

Lemma 2.1. *Let T be a dense subset of a compact Hausdorff space S and let $(h_n) \subset C(S)$ be a bounded sequence converging pointwise on T to a limit function h: $T \longrightarrow \mathbb{C}$. Let e_s be the*

evaluation functional at $s \in S$. Then the following conditions are equivalent:

(a) *h has an extension $\hat{h} \in C(S)$ and $h_n \longrightarrow \hat{h}$ pointwise on S.*
(b) *h has an extension $\hat{h} \in C(S)$ and for all nets $(t_\alpha) \subset T$ with $t_\alpha \to s$ for some $s \in S$ the following double limits exist and are equal:*

$$\lim_n \lim_\alpha h_n(t_\alpha) = \lim_\alpha \lim_n h_n(t_\alpha).$$

(c) *For all nets $(t_\alpha) \subset T$ with $t_\alpha \longrightarrow s$ for some $s \in S$ one has*

$$e_{t_\alpha} \longrightarrow e_s \text{ quasiuniformly on } \{h_n : n \in \mathbb{N}\}.$$

(d) *For all sequences $(t_r) \subset T$ such that $\lim_{r\to\infty} h_n(t_r) =: L_n$ exists for $n \in \mathbb{N}$ one has $e_{t_r} \longrightarrow e_s$ quasiuniformly on $\{h_n : n \in \mathbb{N}\}$ for all s such that $L_n = h_n(s)$, $n \in \mathbb{N}$ (and such s exist).*
(e) *For all sequences $(t_r) \subset T$ and all $s \in S$ such that*

$$\lim_{r\to\infty} h_n(t_r) = h_n(s), \quad n \in \mathbb{N}$$

the limit $\lim_{r\to\infty} h(t_r)$ exists and

$$\lim_{r\to\infty} h(t_r) = \lim_{n\to\infty} h_n(s).$$

(f) *Every sequence $(t_r) \subset T$ contains a subsequence (t'_r) such that*

$$\lim_{r\to\infty} h(t'_r) = \lim_{n\to\infty} \lim_{r\to\infty} h_n(t'_r).$$

Proof. (a)$\Leftrightarrow$(b) is obvious and (a)$\Rightarrow$(c) follows from Arzelà's Theorem [7, p. 268].

(c)$\Rightarrow$(d): Let $\mathscr{E}_0$ be the filter generated by the sets $E_r = \{t_r, t_{r+1}, \ldots\}$, $r \in \mathbb{N}$, in S and $\mathscr{E} = \{K_\alpha : \alpha \in A\}$ be the ultrafilter refining $\mathscr{E}_0$. There exists $s \in S$ with $\mathscr{E} \longrightarrow s$. For all $\alpha \in A$, $E_1 \cap K_\alpha \neq \phi$. Hence there exists some $r(\alpha) \in \mathbb{N}$ with $t_{r(\alpha)} \in K_\alpha$. If we define $\alpha \leqslant \beta$ to mean $K_\alpha \supset K_\beta$ then $(t_{r(\alpha)})_{\alpha \in A}$ is a net in T. We show that $t_{r(\alpha)} \longrightarrow s$. In fact, if U is a neighbourhood of s there exists $\alpha_0 \in A$ such that $U = K_{\alpha_0}$ and for all $\alpha \geqslant \alpha_0$ we have $t_{r(\alpha)} \in K_\alpha \subset K_{\alpha_0} = U$ and the assertion follows. Now for $n \in \mathbb{N}$, $h_n(t_r) \longrightarrow L_n$. Hence $h_n(\mathscr{E}_0) \longrightarrow L_n$ and $h_n(\mathscr{E}) \longrightarrow L_n$. On the other hand, $\mathscr{E} \longrightarrow s$ and $h_n(\mathscr{E}) \longrightarrow h_n(s)$ by continuity. Hence $L_n = h_n(s)$. Let $\varepsilon > 0$ and $r_0 \in \mathbb{N}$. Then there exists α_0 with $E_{r_0} = K_{\alpha_0}$. Hence by (c) there exist $\alpha_1, \ldots, \alpha_k \geqslant \alpha_0$ such that for all $n \in \mathbb{N}$ there is $j \in \{1, \ldots, k\}$ with $|h_n(t_{r(\alpha_j)}) - h_n(s)| < \varepsilon$. For $j \in \{1, \ldots, k\}$ we define $r'_j = r(\alpha_j)$ and hence

$$t_{r'_j} = t_{r(\alpha_j)} \in K_{\alpha_j} \subset K_{\alpha_0} = E_{r_0}.$$

Hence $r'_j \geqslant r_0$, $j = 1, \ldots, k$. Thus we have shown that there exist $r'_1, \ldots, r'_k \geqslant r_0$ such that for all $n \in \mathbb{N}$ there exists $j \in \{1, \ldots, k\}$ with $|h_n(t'_{r_j}) - h_n(s)| < \varepsilon$, which is (d).

(d)$\Rightarrow$(e): Any subsequence of (h_n) contains a further subsequence (again denoted by (h_n)) such that $\lim_{n\to\infty} h_n(s) =: L$ exists. Let $\varepsilon > 0$ and $r_0 > 0$. Then by (d) there exist $r_1, \ldots, r_m \geqslant r_0$ such that for every $n \in \mathbb{N}$ there is $j \in \{1, \ldots, m\}$ with $|h_n(t_{r_j}) - h_n(s)| < \varepsilon/3$. Furthermore there exists $j_0 \in \mathbb{N}$ such that for all $n \geqslant j_0$ and for all $j \in \{1, \ldots, m\}$ we have $|h_n(t_{r_j}) - h(t_{r_j})| < \varepsilon/3$. Also there exists $i_0 \in \mathbb{N}$ such that $|h_n(s) - L| < \varepsilon/3$ for all $n \geqslant i_0$. Hence there exists an $r_j \geqslant r_0$ for some $j \in \{1, \ldots, m\}$ with $|h(t_{r_j}) - L| < \varepsilon$. It follows that

there exists a subsequence (t'_r) of (t_r) such that $h(t'_r) \longrightarrow L$. This implies that

$$\lim_{r\to\infty} h(t_r) = L = \lim_{n\to\infty} h_n(s).$$

But then this is true for the original sequence (h_n) also. This shows (e) to be true.

(e)$\Rightarrow$(f): Let $(t_r) \subset T$. There exists a subsequence (t'_r) such that $\lim_{r\to\infty} h_n(t'_r) = h_n(s)$ exists for $n \in \mathbb{N}$ for some $s \in S$ (see the proof of (c)$\Rightarrow$(d)). By (e) we have

$$\lim_{r\to\infty} h(t'_r) = \lim_{n\to\infty} h_n(s) = \lim_{n\to\infty} \lim_{r\to\infty} h_n(t'_r).$$

This is (f).

(f)$\Rightarrow$(e): Consider $(t_r) \subset T, s \in S$ such that

$$\lim_{r\to\infty} h_n(t_r) = h_n(s) \quad \text{for } n \in \mathbb{N}.$$

Let (t'_r) be any subsequence of (t_r). By (f) there exists a further subsequence (t''_r) such that

$$\lim_{r\to\infty} h(t''_r) = \lim_{n\to\infty} \lim_{r\to\infty} h_n(t''_r) = \lim_{n\to\infty} h_n(s).$$

This shows that $\lim_{r\to\infty} h(t_r)$ exists and is equal to $\lim_{n\to\infty} h_n(s)$.

(e)$\Rightarrow$(a): First let S be metrizable. Let $s \in S$ and choose $(t_r) \subset T$ such that $t_r \longrightarrow s$ (such sequences exist by denseness of T). Then $h_n(t_r) \to h_n(s), n \in \mathbb{N}$ and the conclusion of (e) implies (a). The general case is reduced to the first case by considering the compact metric quotient $\tilde{S} = S/\sim$ with respect to the equivalence relation $s \sim s'$ iff $h_n(s) = h_n(s')$ for $n \in \mathbb{N}$. The canonical surjection $\sigma: S \to \tilde{S}$ is continuous. The set $\tilde{T} = \sigma T$ is dense in $\tilde{S}$, the sequence $(\tilde{h}_n) \subset C(\tilde{S})$ given by $\tilde{h}_n(\sigma s) = h_n(s)$ converges pointwise on $\tilde{T}$ to the limit $\tilde{h}: \tilde{T} \to \mathbb{C}$ given by $\tilde{h}(\sigma s) = h(s)$. $(\tilde{h}_n)$ satisfies (e) on $\tilde{S}$ and (a) follows for $(\tilde{h}_n)$ which implies that (a) holds for (h_n) on S.

Before applying Lemma 2.1 we note the following result which is in the spirit of Arzelà's Theorem.

Lemma 2.2 Let $(f_n) \subset \mathrm{AP}(\mathbb{R})$ be a bounded sequence converging pointwise on $\mathbb{R}$ to a limit function $f: \mathbb{R} \longrightarrow \mathbb{C}$. Then the convergence is quasiuniform and $f \in \mathrm{AP}(\mathbb{R})$ iff the following condition holds:

$$\text{(A)} \begin{cases} \forall \varepsilon > 0 \exists l > 0 \, \forall a \in \mathbb{R} \, \exists \tau \in [a, a, +l] \, \forall n_0 \in \mathbb{N} \, \exists n_1, \ldots, n_k \geqslant n_0 \, \forall x \in \mathbb{R} \\ \exists i, j \in \{1, 2, \ldots, k\} \text{ with } |f_{n_i}(x) - f(x)| < \varepsilon \text{ and } |f_{n_i}(x) - f_{n_j}(x+\tau)| < \varepsilon. \end{cases}$$

Proof: If $f \in \mathrm{AP}(\mathbb{R})$, then $\forall \varepsilon > 0 \, \exists l > 0 \, \forall a \in \mathbb{R} \, \exists \tau \in [a, a+l]$: $\|f_\tau - f\| < \varepsilon/3$. If the convergence is quasiuniform, then $\forall n_0 \in \mathbb{N} \, \exists n_1, \ldots, n_k \geqslant n_0 \, \forall x \in \mathbb{R} \, \exists i, j \in \{1, \ldots, k\}$ with

$$|f_{n_i}(x) - f(x)| < \frac{\varepsilon}{3} \quad \text{and} \quad |f_{n_j}(x+\tau) - f(x+\tau)| < \frac{\varepsilon}{3}.$$

This implies $|f_{n_i}(x) - f_{n_j}(x+\tau)| < \varepsilon$ and proves (A). Conversely, (A) implies that the convergence is quasiuniform and that $f \in C_b(\mathbb{R})$. To prove that $f \in \mathrm{AP}(\mathbb{R})$, let $x \in \mathbb{R}$. By (A),

for all $p \in \mathbb{N}$ there exist $n_{i_p}, n_{j_p} \geqslant p$ such that

$$|f_{n_{i_p}}(x) - f(x)| < \varepsilon, \quad |f_{n_{i_p}}(x) - f_{n_{j_p}}(x + \tau)| < \varepsilon,$$

which gives $|f_{n_{j_p}}(x + \tau) - f(x)| < 2\varepsilon$. We may assume $n_{j_1} < n_{j_2} < \cdots < n_{j_p} \uparrow \infty$ as $p \to \infty$. Since $f_{n_{j_p}}(x + \tau) \to f(x + \tau)$ as $p \to \infty$ it follows that

$$|f(x + \tau) - f(x)| \leqslant 2\varepsilon \quad \text{for } x \in \mathbb{R}.$$

This proves that $f \in \mathrm{AP}(\mathbb{R})$.

The nets $(x_\alpha) \subset \mathbb{R}$ which converge to some $\xi \in \mathbb{R}^a$ in $\mathbb{R}^a$, will be called Bohr nets. Obviously, a net $(x_\alpha) \subset \mathbb{R}$ is a Bohr net iff for all $f \in \mathrm{AP}(\mathbb{R})$, $(f(x_\alpha))$ converges, or for all $\lambda \in \mathbb{R}$, $(e^{i\lambda x_\alpha})$ converges in $\mathbb{C}$, which means

$$\forall \varepsilon > 0 \exists \delta > 0 \forall \lambda \in \mathbb{R} \quad \exists \alpha_0 = \alpha_0(\varepsilon, \lambda) \quad \forall \alpha, \beta \geqslant \alpha_0:$$
$$|\lambda(x_\alpha - x_\beta)| < \delta \,(\mathrm{mod}\ 2\pi).$$

For all $\xi \in \mathbb{R}^a$ there is a Bohr net (x_α) converging to $\xi \in \mathbb{R}^a$ in $\mathbb{R}^a$. Furthermore, one sees that a net $(x_\alpha) \subset \mathbb{R}$ is a Bohr net iff (x_α) is a Cauchy net in the precompact uniform space $(\mathbb{R}, \mathscr{U})$, where the translation invariant uniformity $\mathscr{U}$ is generated by the sets $\{(s, t) \in \mathbb{R} \times \mathbb{R} : \|f_s - f_t\| < \varepsilon\}$ for $f \in \mathrm{AP}(\mathbb{R})$ and $\varepsilon > 0$.

The proof of the following theorem which characterizes weak convergence in $\mathrm{AP}(\mathbb{R})$ is clear from the preceding two lemmas. The condition (g) is similar to the one given in [7, Theorem IV. 6.31, p. 281].

Theorem 2.3. *A bounded sequence $(f_n) \subset \mathrm{AP}(\mathbb{R})$ is weakly convergent iff (f_n) converges pointwise on a dense subset D of $\mathbb{R}$ to a limit $f : D \to \mathbb{C}$ and one of the following conditions is satisfied*:

(a) *f has an extension in $\mathrm{AP}(\mathbb{R})$ and $\hat{f}_n \to \hat{f}$ pointwise on $\mathbb{R}^a$.*

(b) *f has an extension in $\mathrm{AP}(\mathbb{R})$ and for all Bohr nets (x_α) the following double limits exist and are equal*:

$$\lim_n \lim_\alpha f_n(x_\alpha) = \lim_\alpha \lim_n f_n(x_\alpha).$$

(c) *For every Bohr net $x_\alpha \to \xi$ for some $\xi \in \mathbb{R}^a$ one has*

$$e_{x_\alpha} \to e_\xi \text{ quasiuniformly on } \{\hat{f}_n : n \in \mathbb{N}\}.$$

(d) *For all sequences $\{x_r\} \subset D$ such that $\lim_{r\to\infty} f_n(x_r) =: L_n$ exists for $n \in \mathbb{N}$ one has*

$$e_{x_r} \to e_\xi \text{ quasiuniformly on } \{\hat{f}_n : n \in \mathbb{N}\}$$

for all $\xi \in \mathbb{R}^a$ such that $L_n = \hat{f}_n(\xi)$, $n \in \mathbb{N}$ (and such ξ exist).

(e) *For all sequences $(x_r) \subset \mathbb{R}$ and all $\xi \in \mathbb{R}^a$ such that $\lim_{r\to\infty} f_n(x_r) = \hat{f}_n(\xi)$, $n \in \mathbb{N}$, the limit $\lim_{r\to\infty} f(x_r)$ exists and $\lim_{r\to\infty} f(x_r) = \lim_{n\to\infty} \hat{f}_n(\xi)$.*

(f) *Every sequence $\{x_r\} \subset D$ contains a subsequence $\{x_r'\}$ such that*

$$\lim_{r\to\infty} f(x_r') = \lim_{n\to\infty} \lim_{r\to\infty} f_n(x_r').$$

(g) *$f_n \to f$ quasiuniformly on D together with all subsequences.*

(h) *Every subsequence of (f_n) satisfies condition* (A) *of Lemma* 2.2.

3. Weak compactness

We can characterize the relatively weakly compact subsets in AP($\mathbb{R}$) as follows.

Theorem 3.1. *For a bounded subset $F \subset$ AP($\mathbb{R}$) the following conditions are equivalent:*

(i) *F is relatively compact in the weak topology of* AP($\mathbb{R}$).
(ii) *For all Bohr nets (x_α) with $x_\alpha \to \xi$ one has $e_{x_\alpha} \to e_\xi$ quasiuniformly on $\hat{F}$.*
(iii) *For all countable subsets $F_0 \subset F$, all sequences $(x_r) \subset \mathbb{R}$ such that $\lim_{r\to\infty} f(x_r) =: L_f$ exists for $f \in F_0$, one has $e_{x_r} \to e_\xi$ quasiuniformly on $\hat{F}_0$ for all $\xi \in \mathbb{R}^a$ such that $L_f = \hat{f}(\xi)$ for $f \in F_0$ (and such ξ exist).*
(iv) *Every sequence in F which converges pointwise on $\mathbb{R}$ (equivalently, on a dense subset of $\mathbb{R}$) is weakly convergent in* AP($\mathbb{R}$).

Proof. To prove (iy)$\Rightarrow$(i) we remark that each sequence $(f_n) \subset F$ contains a subsequence which converges pointwise on the set $\mathbb{Q}$ of rationals and is weakly convergent by (iv). The other implications follow from arguments already given in the proof of Lemma 2.1.

4. Further results and examples

This section is devoted to miscellaneous results and examples which throw some light on the nature of weak convergence in AP($\mathbb{R}$).

Example 4.1. For a sequence $(a_n) \subset \mathbb{R}$, let $f_n(x) = e^{ia_n x}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. If $a_n \to a$ for some $a \in \mathbb{R}, f(x) = e^{iax}$ is the pointwise limit of f_n. By Theorem 2.3, $f_n \to f$ weakly in AP($\mathbb{R}$) iff for all Bohr nets (x_α)

$$\lim_n \lim_\alpha e^{ia_n x_\alpha} = \lim_\alpha e^{iax_\alpha},$$

equivalently, $\chi(a_n) \to \chi(a)$ for every character χ of $\mathbb{R}$. According to a general device given by *v.* Neumann [13, Example IV, 15.4] a discontinuous character χ on $\mathbb{R}$ can be constructed as follows. Let B be a basis for $\mathbb{R}$ over the set $\mathbb{Q}$ of rationals. B contains exactly one rational which we may take as 1. Define

$$\chi(t) = \exp(i(u_1 + \cdots + u_r)),$$

if $t = u_1 t_{\alpha_1} + \cdots + u_r t_{\alpha_r}$ with $t_{\alpha_i} \in B, u_i \in \mathbb{Q}, i = 1, \ldots, r$ is the unique representation of $t \in \mathbb{R}$. There exists $a \in B \setminus \{2n\pi + \frac{1}{2}, n \in \mathbb{Z}\}$ and a sequence (a_n) in $\mathbb{Q}$ with $a_n \to \dfrac{a}{2}$, and we have

$$\chi(a_n) = e^{ia_n} \to e^{ia} \neq e^{1/2} = \chi(a/2).$$

For this sequence (a_n), $(e^{ia_n(\cdot)})$ does not converge weakly in AP($\mathbb{R}$) to $e^{ia(\cdot)/2}$.

The following example shows that even for a sequence of periodic functions the concept of quasiuniform convergence and quasiuniform convergence together with all subsequences are two distinct concepts. It also illustrates the condition (h) of Theorem 2.3. For $a \in \mathbb{R}$ and $\lambda > 0$, $\phi_{a,\lambda}$ will denote the function in $C_b(\mathbb{R})$ vanishing outside $[a-\lambda, a+\lambda]$ with $\phi_{a,\lambda}(a) = 1$ and linear in between.

Example 4.2. Consider the sets

$$M_1 = \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \dots\}$$
$$M_3 = \{\pm 3, \pm 9, \pm 15, \pm 21, \pm 27, \dots\}$$
$$M_5 = \{\pm 9, \pm 27, \pm \dots\}$$
$$\vdots$$
$$M_2 = \{\pm 2, \pm 6, \pm 10, \pm 14, \pm 18, \dots\}$$
$$M_4 = \{\pm 6, \pm 18, \pm 30, \pm \dots\}$$
$$M_6 = \{\pm 18, \pm 54, \pm \dots\}$$
$$\vdots$$

and define the sequence (f_n) as follows:

$$f_{2n+1} = \sum_{j \in M_{2n+1}} \phi_{j,1/2},$$

$$f_{2n} = \sum_{j \in M_{2n}} \phi_{j,1/2}, \quad n \in \mathbb{N}.$$

This sequence has the following properties:

(1) $f_n \to 0$ pointwise on $\mathbb{R}$.
(2) For all $m, n \in \mathbb{N}$, $\operatorname{supp} f_{2n} \cap \operatorname{supp} f_{2m+1} = \emptyset$, hence $\min(f_{2n}, f_{2m+1}) = 0$ and $f_n \to 0$ quasiuniformly on $\mathbb{R}$.
(3) If $m, n \in \mathbb{N}$ are both even or both odd, then for $n \leqslant m$, $\operatorname{supp} f_n \supset \operatorname{supp} f_m$ and $f_n = f_m$ on $\operatorname{supp} f_m$. Then $f_{2n+1} \not\to 0$ quasiuniformly on $\mathbb{R}$. In fact, for arbitrary odd numbers $n_1 < n_2 \cdots < n_k$ and for $x \in \mathbb{R}$ with $f_{n_k}(x) = 1$, one also has $f_{n_i}(x) = 1, i = 1, \dots, k$. Similarly, $f_{2n} \not\to 0$ quasiuniformly on $\mathbb{R}$.
(4) $f_n \not\to 0$ weakly in $\mathrm{AP}(\mathbb{R})$. This follows from Theorem 2.3: One can use (g) or disprove (h) for the sequence with odd indices, $\varepsilon = 1$ and $a = 1$: For all $l > 0$ there exists an odd n_0 such that for $\tau \in [1, 1 + l]$ and all odd $n \geqslant n_0$ one has $\operatorname{supp} f_n \cap \operatorname{supp} f_{n,\tau} = \emptyset$. Together with (3) it follows that for all odd $m > n \geqslant n_0$ there exists $x \in \mathbb{R}$ with $|f_m(x) - f_n(x + \tau)| = 1$.
(5) The sequence of periods of the f_n tends to infinity.

Let $(f_n) \subset \mathrm{AP}(\mathbb{R})$ be a bounded sequence which is pointwise convergent to $f: \mathbb{R} \to \mathbb{C}$. Such a sequence is strongly convergent iff it is equicontinuous and equi-almost periodic, which means

$$\bigcap_{n \in N} \{\tau \in \mathbb{R} : \|f_{n,\tau} - f_n\| < \varepsilon\}$$

is relatively dense in $\mathbb{R}$ [7, p. 345]. The equicontinuity implies the uniform convergence on every compact subinterval and then the equi-almost periodicity gives uniform convergence on $\mathbb{R}$. If one weakens equicontinuity to quasi-equicontinuity and replaces equi-almost periodicity by the following condition (B) one gets weak convergence of (f_n) in $\mathrm{AP}(\mathbb{R})$ according to the following proposition.

PROPOSITION 4.3

Let $(f_n) \subset \mathrm{AP}(\mathbb{R})$ be bounded and converge pointwise to $f:\mathbb{R}\to\mathbb{C}$. Let (f_n) be quasi-equicontinuous on every compact interval and satisfy the following condition

$$\text{(B)} \qquad \begin{cases} \forall\,\varepsilon>0\,\exists\,L=L(\varepsilon)>0\,\forall\,\pi\subset\mathbb{N} \text{ finite}, \quad \forall\,x\in\mathbb{R}\,\exists\,y_\pi(x)\in[0,L]\,\forall\,n\in\pi \\ |(f_n-f)(y_\pi(x))-(f_n-f)(x)|<\varepsilon. \end{cases}$$

Then $f_n \to f$ weakly in $\mathrm{AP}(\mathbb{R})$.

Proof. Because the assumptions of the proposition are satisfied by any subsequence it is sufficient to show that $f_n \to f$ quasiuniformly on $\mathbb{R}$. Let $S\subset\mathbb{R}$ be any compact interval. Because (f_n) is quasi-equicontinuous on S, f is continuous on S by Lemma 2.1 (with $T=S$) and hence on $\mathbb{R}$. In particular $f_n\to f$ quasiuniformly on $[0,L]$:

$$\forall\,\varepsilon>0\,\forall\,n_0\in\mathbb{N}\,\exists\,n_1,\ldots,n_k\geqslant n_0\,\forall\,y\in[0,L]\,\exists\,i\in\{1,\ldots,k\} \text{ with } |f_{n_i}(y)-f(y)|<\varepsilon.$$

Hence by (B) we have

$$\forall\,\varepsilon>0\,\forall\,n_0\in\mathbb{N}\,\exists\,n_1,\ldots,n_k\geqslant n_0\,\forall\,x\in\mathbb{R}\,\exists\,i\in\{1,\ldots,k\} \text{ with } |(f_{n_i}-f)(x)|<2\varepsilon.$$

This means $f_n\to f$ quasiuniformly on $\mathbb{R}$. By Theorem 2.3, $f_n\to f$ weakly in $\mathrm{AP}(\mathbb{R})$.

One can verify easily that the condition (B) is satisfied if (f_n-f) is equi-almost periodic. This is the case if (f_n) is equi-almost periodic, $f_n\to f$ pointwise on $\mathbb{R}$ and f is continuous on $\mathbb{R}$. Hence we have

COROLLARY 4.4

Let $(f_n)\subset\mathrm{AP}(\mathbb{R})$ be bounded and equi-almost periodic and converge pointwise to $f\in C_b(\mathbb{R})$. Then $f_n\to f$ weakly in $\mathrm{AP}(\mathbb{R})$.

The following example shows that (B) is not necessary for weak convergence.

Example 4.5. Consider the sequence (f_n) where

$$f_n=\sum_{i=-\infty}^{\infty}\phi_{(2i+1)2^{n-1},1/2},\, n\in\mathbb{N}.$$

Then (f_n) has the following properties:

(1) f_n has period 2^n for all $n\in\mathbb{N}$.
(2) For all $L>0$ there exists $n_0\in\mathbb{N}$ such that for all $n\geqslant n_0$ we have $f_n=0$ on $[0,L]$.
(3) For all $n,m\in\mathbb{N}$, $n\neq m$, $\inf(f_n,f_m)=0$, because the sets $\{(2i+1)2^{n-1}, i\in\mathbb{N}\}$, $n\in\mathbb{N}$, are pairwise disjoint.
(4) $f_n\to 0$ quasiuniformly on $\mathbb{R}$ together with all subsequences and hence $f_n\to 0$ weakly in $\mathrm{AP}(\mathbb{R})$.
(5) For all $L>0$ there exists $n_L\in\mathbb{N}$ with $f_{n_L}=0$ on $[0,L]$ and $x_L\in\mathbb{R}$ with $f_{n_L}(x_L)=1$ and hence the condition (B) is not satisfied.

The following two examples show that boundedness, equicontinuity and pointwise convergence to a limit $f\in\mathrm{AP}(\mathbb{R})$ together with the periodicity of the functions having periods converging to a limit in $\mathbb{R}$ or diverging to infinity do not ensure weak convergence in $\mathrm{AP}(\mathbb{R})$.

Example 4.6. Define $\phi := \Sigma_{n\in\mathbb{Z}} \phi_{n+1/2,\,1/2}$. Then there exists a sequence $(\lambda_n) \subset \mathbb{R}$ such that $\lambda_n \downarrow 1$ and the functions f_n, $n\in\mathbb{N}$, defined by $f_n(x) = \phi(x/\lambda_n)$ have the following properties:

(1) (f_n) is equicontinuous.
(2) $f_n \to \phi$ pointwise on $\mathbb{R}$.
(3) $f_n \not\to \phi$ quasiuniformly on $\mathbb{R}$ and hence $f_n \not\to \phi$ weakly in AP($\mathbb{R}$). In fact, let $\lambda_1 = \frac{2}{3}$. There exist two odd positive numbers p_2, q_2 such that if we let $\lambda_2 = p_2/q_2$, then $1 < \lambda_2 < \frac{1+\lambda_1}{2}$. Then by induction we can define two sequences (p_i), (q_i) of odd positive integers such that the sequence (λ_i) defined by $\lambda_i = (p_i/q_i)\lambda_{i-1}$ satisfies $1 < \lambda_i < \frac{1+\lambda_{i-1}}{2}$, $i = 2, 3, \ldots$. It then follows that $\lambda_n \downarrow 1$. Now let $t_k = p_k p_{k-1} \cdots p_2 \lambda_1 = p_k p_{k-1} \cdots p_3 q_2 \lambda_2 = \cdots = q_k q_{k-1} \cdots q_2 \lambda_k$. Clearly $t_k \in \mathbb{N} + \frac{1}{2}$. Hence by periodicity, $\phi(t_k) = \phi(\frac{1}{2}) = 1$. But $t_k/\lambda_n \in \mathbb{N}$ and hence $f_n(t_k) = \phi(t_k/\lambda_n) = 0$ for all $n \in \{1, \ldots, k\}$. Hence the convergence is not quasiuniform.

Example 4.7. Let ϕ be as above, $f_n(x) = \phi(x/2^n)$, $n\in\mathbb{N}$. Then

(1) (f_n) is equicontinuous.
(2) $f_n \to 0$ pointwise on $\mathbb{R}$.
(3) $f_n \not\to 0$ quasiuniformly on $\mathbb{R}$. To prove this, let $t_k = \frac{2}{3}2^k$ for $k \geqslant 0$. Then it is easy to see that $t_k = \pm\frac{2}{3}$ (mod 2) according to k is even or odd. Hence for all $k \geqslant n$ we have $f_n(t_k) = \phi(\frac{2}{3}2^{k-n}) = \phi(t_{k-n}) = \phi(\pm\frac{2}{3}) = \frac{2}{3}$. Now take $\varepsilon = \frac{2}{3}$ and let $n_1, n_2, \ldots, n_k$ be arbitrary in $\mathbb{N}$. Suppose $n = \max(n_1, \ldots, n_k)$ and $k \geqslant n$. Then $f_{n_j}(t_k) = \frac{2}{3}$, $j = 1, \ldots, k$. This shows that the pointwise convergence $f_n \to 0$ is not quasiuniform on $\mathbb{R}$.

Examples 4.6 and 4.7 show that the pointwise convergence of a bounded, equicontinuous sequence of periodic functions to a continuous periodic limit function may not even imply quasiuniform convergence on $\mathbb{R}$. However, the periods of the differences $f_n - f$ in the examples tend to infinity. If these periods are bounded, one at once gets uniform convergence by the following proposition. Here for $f\in$ AP($\mathbb{R}$) and $\varepsilon > 0$, $l(f, \varepsilon)$ denotes the infimum of all the ε-periods of f.

PROPOSITION 4.8

Suppose $(f_n) \subset$ AP($\mathbb{R}$) is equicontinuous on compact intervals and $f_n \to f$ pointwise on $\mathbb{R}$ with $f\in$ AP($\mathbb{R}$). Suppose further that for every $\varepsilon > 0$, $L(\varepsilon) = \sup_{n\geqslant 1} l(f_n - f, \varepsilon) < \infty$. Then $f_n \to f$ uniformly on $\mathbb{R}$.

Proof. Assume $\| f_n - f \| \not\to 0$. For convenience let $\Delta_n = f_n - f$. Then for $\varepsilon > 0$ there exists $(n_i) \subset \mathbb{N}$ and $(t_i) \subset \mathbb{R}$ such that

$$|\Delta_{n_i}(t_i)| \geqslant \varepsilon \quad \text{for all } i\in\mathbb{N}. \tag{1}$$

Now for every $i\in\mathbb{N}$ there exists $\sigma_i \in [-t_i, -t_i + L(\varepsilon/2)]$ with $\| \Delta_{n_i}(\cdot + \sigma_i) - \Delta_{n_i} \| < \varepsilon/2$, which together with (1) gives

$$|\Delta_{n_i}(t_i + \sigma_i)| > \varepsilon - \varepsilon/2 = \varepsilon \quad \text{for all } i\in\mathbb{N}. \tag{2}$$

Since $0 \leqslant t_i + \sigma_i \leqslant L(\varepsilon/2)$ for all $i \geqslant n$, there exists a $y\in[0, L(\varepsilon/2)]$ and a subsequence $(\sigma_{i_k} + t_{i_k})$ with $\sigma_{i_k} + t_{i_k} \to y$ as $k \to \infty$. Since $\Delta_n \to 0$ pointwise, there exists $k_0 \in\mathbb{N}$ with

$$|\Delta_{n_{i_k}}(y)| < \varepsilon/4 \quad \text{for all } k \geqslant k_0 = k_0(y). \tag{3}$$

Now by assumption (f_n) and therefore (Δ_n) is equicontinuous on $[0, L(\varepsilon/2)]$. Hence for all sufficiently large k, $|\Delta_{n_{i_k}}(\sigma_{i_k}+t_{i_k})-\Delta_{n_{i_k}}(y)|<\varepsilon/4$. Then (3) gives $|\Delta_{n_{i_k}}(\sigma_{i_k}+t_{i_k})|<\frac{\varepsilon}{2}$ for sufficiently large k, which contradicts (2). This completes the proof.

Example 4.9. Let $\phi\in C(\mathbb{R})$ have period 2π. Let $\phi(0)=0$ and assume $\phi(x)\neq 0$ in $(0,2\pi)$. Let (λ_n) be a sequence of positive numbers with $\lambda_n\to\infty$. Define f_n on $\mathbb{R}$ by $f_n(x)=\phi(x/\lambda_n)$, $n\geqslant 1$, $x\in\mathbb{R}$. Assume: $\exists\delta\in(0,1)\,\forall i\in\mathbb{N}\;\exists\tau_i\in\mathbb{R}\;\forall n=1,\ldots,i$

$$\text{(C)}\qquad \left|\frac{\tau_i}{\lambda_n}-\pi\right|<\delta\pi(\mathrm{mod}\ 2\pi).$$

Then $f_n\to 0$ pointwise on $\mathbb{R}$, but not quasiuniformly.

Proof. $f_n\to 0$ pointwise on $\mathbb{R}$ is trivially true. For the second assertion observe that for a fixed $i\in\mathbb{N}$ there exist $j_1,\ldots,j_i\in\mathbb{Z}$ with the following property: if we define $\delta_n=\frac{\tau_i}{\lambda_n}-\pi+2\pi j_n$, then $|\delta_n|<\delta\pi, n=1,\ldots,i$. Hence for $n=1,\ldots,i$

$$\begin{aligned}|f_n(\tau_i)|&=|\phi(\tau_i/\lambda_n)|=|\phi(\pi+\delta_n)|\\&\geqslant\inf\{|\phi(\pi+x)|:|x|\leqslant\delta\pi\}=:\varepsilon.\end{aligned}$$

Then $\varepsilon>0$ since $\phi(x)\neq 0$ in $[\pi-\delta\pi,\pi+\delta\pi]$. Thus we have shown that $\exists\varepsilon>0$ $\forall i\in\mathbb{N}\,\exists\tau_i\in\mathbb{R}\;\forall n=1,\ldots,i$ we have $|f_n(\tau_i)|\geqslant\varepsilon$, which proves the assertion.

Remark. In the light of Kronecker's theorem [11, Theorem 3.1] the condition (C) is satisfied if $\{\lambda_n:n\in\mathbb{N}\}$ is linearly independent over the rationals. The existence of sequences satisfying (C) is then guaranteed, for example, by the theorem of Lindemann–Weierstrass–Kronecker [2, Theorem 1.4].

We conclude this section with an example which shows that if $F\subset\mathrm{AP}(\mathbb{R})$ is weakly compact, the set of all its translates may not be so. This is in contrast with the corresponding norm case.

Example 4.10 [10]. Let (f_n), $n\geqslant 2$, be the sequence of continuous periodic functions of period 1 defined on $\mathbb{R}$ by

$$\begin{aligned}f_n(x)&=\phi_{1/n,1/n}(x)\quad && 0\leqslant x\leqslant 2/n\\&=0 && 2/n<x\leqslant 1,\end{aligned}$$

and extended by periodicity. Then $f_n\to 0$ weakly in $\mathrm{AP}(\mathbb{R})$ but the set of its translates $\{f_{n,a}:n\geqslant 2, a\in\mathbb{R}\}$ is not relatively weakly compact.

Proof. It is easy to see that $f_n\to 0$ pointwise and quasiuniformly on $\mathbb{R}$ together with all subsequences. Hence $f_n\to 0$ weakly in $\mathrm{AP}(\mathbb{R})$. For the other assertion observe that for all $n\geqslant 2$, we have $(f_n)_{1/n}(-\frac{1}{n})=0$ while $(f_n)_{1/n}(0)=1$ so that $\lim_{n\to\infty}(f_n)_{1/n}$ is not even continuous.

5. Open questions

5.1 Let $(f_n)\subset\mathrm{AP}(\mathbb{R})$ be bounded, $f\in C_b(\mathbb{R})$ and $f_n\to f$ pointwise on $\mathbb{R}$. Give necessary and sufficient conditions for (f_n) such that $f\in\mathrm{AP}(\mathbb{R})$.

5.2 Give a nontrivial illustration of condition (B) of Proposition 4.3.
5.3 Is it true that for all $\lambda_k \to \infty$ and the sequence (f_n) of Example 4.9 one has $f_n \not\to 0$ quasiuniformly on $\mathbb{R}$?
5.4 Give nontrivial examples of equi-almost periodic families in AP($\mathbb{R}$), which are not relatively norm compact.
5.5 Find a Bohr net not converging in $\mathbb{R}$.
5.6 Does there exist a strictly monotone sequence $(a_n) \subset \mathbb{R}$ such that $a_n \to a$ in $\mathbb{R}$ and $e^{ia_n(.)} \to e^{ia(.)}$ weakly in AP($\mathbb{R}$)?

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Application of the absolute Euler method to some series related to Fourier series and its conjugate series

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Abstract. We study the absolute Euler summability problem of some series associated with Fourier series and its conjugate series generalizing some known results in the literature. Also, it is shown that absolute Euler summability of rth derived Fourier series and rth derived conjugate series can be ensured under local conditions.

Keywords. Absolute Euler summability; Fourier series; summability factor; local property.

1. Introduction

1.1. DEFINITION 1

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of partial sums $\{U_n\}$ and let

$$v_n = \frac{1}{(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} u_k, \quad q > 0, \tag{1}$$

If $\sum v_n$ is convergent, we say that the series $\sum u_n$ of the sequence $\{U_n\}$ is summable (E, q), $q > 0$ ([7], [11]) and in short we write

$$\sum u_n (\text{or } \{U_n\}) \in (E, q), \quad q > 0$$

If $\sum v_n$ is absolutely convergent, we say that the series $\sum u_n$ or the sequence $\{U_n\}$ is absolutely summable (E, q) and in short we write

$$\sum u_n (\text{or } \{U_n\}) \in |E, q|, \quad q > 0.$$

1.2. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. We assume without loss of generality that the constant term in the Fourier series of $f(t)$ is zero. Thus

$$\int_{-\pi}^{\pi} f(t) \mathrm{d}t = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{2}$$

The series conjugate to the Fourier series of f at $t = x$ is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{3}$$

The rth derived series of (2) at $t=x$ and rth derived series of (3) are respectively

$$\sum_{n=1}^{\infty}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r A_n(t)\right]_{t=x} \equiv \sum_{n=1}^{\infty} A_{n,r}(x) \tag{4}$$

and

$$\sum_{n=1}^{\infty}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r B_n(t)\right]_{t=x} \equiv \sum_{n=1}^{\infty} B_{n,r}(x). \tag{5}$$

We write

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2S\},$$

where S is a function of x. In particular if f is a continuous function at $t=x$ then S is taken to be $f(x)$.

$$\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}$$

$$g(t)=\phi(t)\left(\log\frac{2\pi}{t}\right)^{1-\delta}$$

$$h(t)=\psi(t)\left(\log\frac{2\pi}{t}\right)^{-\delta}$$

$$\Phi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\mathrm{d}u, \quad \alpha>0$$

$$\Phi_0(t)=\phi(t)$$

$$\phi_\alpha(t)=(\alpha+1)t^{-\alpha}\Phi_\alpha(t), \quad \alpha\geqslant 0$$

$$P(t)=\sum_{i=0}^{n-1}\frac{\theta_i t^i}{i!},$$

where θ_i for $i=0,1,\ldots,(r-1)$ are arbitrary.

$$g^*(t)=\frac{\{f(x+t)-P(t)\}+(-1)^r\{f(x-t)-P(-t)\}}{2t^r}$$

$$h^*(t)=\frac{\{f(x+t)-P(t)\}-(-1)^r\{f(x-t)-P(-t)\}}{2t^r}.$$

1.3. In 1968, Mohanty and Mohapatra [11] began the study of absolute Euler summability of Fourier series and its conjugate series in proving the following theorems:

Theorem A. *For* $0<\delta<1$

$$\phi(t)\log\frac{1}{t}\in BV(0,\delta)\Rightarrow \sum A_n(x)\in|E,q|, \quad q>0.$$

Theorem B. *For* $0<\delta<1$

$$\psi(+0)=0 \textit{ and } \int_0^\delta \log\frac{1}{t}|\mathrm{d}\psi(t)|<\infty\Rightarrow \sum B_n(x)\in|E,q|, \quad q>0.$$

The above two theorems were also independently obtained by Kwee [9] in 1972. Further Kwee [9] proved

Theorem C. *The condition $\phi(t)\log(1/t)\in BV(0,\delta)$, $0<\delta<1$ in Theorem A cannot be replaced by the weaker assumption:*

$$\phi(t)\left(\log\frac{1}{t}\right)^{\eta}\in BV(0,\delta),\quad 0<\delta<1 \text{ for any } \eta<1.$$

Kwee [9] first studied the absolute Euler summability of derived Fourier series and his result reads as follows:

Theorem D ([9], Theorem 6). If $\psi(+0)=0$ *and*

$$\int_0^{\delta} u^{-2}|d\psi(u)|<\infty \quad \text{for } 0<\delta<1 \tag{6}$$

then $\sum_{n=1}^{\infty} nB_n(x)\in|E,q|$, $q>0$. *Further* (6) *cannot be replaced by the weaker assumption*

$$\int_0^{\delta} u^{-\eta}|d\psi(u)|<\infty \quad \text{for any } \eta<2.$$

Subsequently several authors such as Chandra [4], Chandra and Dikshit [5] have obtained number of results on the absolute Euler summability of Fourier series, conjugate series, first derived Fourier series and first derived conjugate series. Chandra and Dikshit [5] have obtained the following result which is an improvement of an earlier work due to Kwee (Theorem D of the present paper).

Theorem E ([5], Theorem 4(i)). *Let* $0<\delta<\pi$, $q>0$. *Then*

$$\frac{\psi(t)}{t^2}\in BV(0,\delta)\Rightarrow \sum_{n=1}^{\infty} nB_n(x)\in|E,q|.$$

Concerning the absolute Euler summability factors of Fourier series, Tripathy [15] and Chandra [4] have independently proved the following.

Theorem F. $\phi(t)\in BV(0,\pi)\Rightarrow \sum_{n=1}^{\infty}\dfrac{A_n(x)}{\log(n+1)}\in|E,q|,\quad q>0.$

Also Tripathy [16] proved the following

Theorem G. *If* $\phi_\alpha(t)\in BV(0,\pi)$, $0<\alpha<3/2$ *then*

$$\sum n^{-\lambda}A_n(x)\in|E,q|,\quad q>0$$

where

$$\lambda>\max(\alpha-1/2,\ 1/2).$$

Ray and Patra [13] improved upon Theorem G by proving the following:

Theorem H.

$$\phi_\alpha(t)\in BV(0,\delta)0<\delta<\pi\Rightarrow \sum_{n=1}^{\infty}\frac{A_n(x)}{n^{\alpha/2}}\in|E,q|,\quad q>0 \quad \text{for } 0<\alpha\leqslant 2.$$

Concerning the absolute Euler summability factors of conjugate series Chandra ([4], Theorem 3, 4) proved the following theorems:

Theorem I.

$$\psi(+0)=0 \text{ and } \psi(t)\in BV(0,\pi) \Rightarrow \sum_{n=1}^{\infty} \frac{B_n(x)}{\log(n+1)} \in |E,q|, \quad q>0.$$

Theorem J.

$$\psi(t)\in BV(0,\pi) \text{ and } \frac{\psi(t)}{t}\in L(0,\pi) \Rightarrow \sum_{n=1}^{\infty} \frac{B_n(x)}{\log(n+1)} \in |E,q|, \quad q>0.$$

2. Theorems

2.1. The purpose of the present work is manifold as detailed below from (I) to (IV).

(I) The conditions taken on the respective generating functions in proving Theorem F. Theorem I and Theorem J are non-local in character. We wish to show that $|E,q|, q>0$ summability of

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{\{\log(n+1)\}^{\delta}}, \quad 0\leqslant \delta \leqslant 1 \tag{7}$$

and

$$\sum_{n=1}^{\infty} \frac{B_n(x)}{\{\log(n+1)\}^{\delta}}, \quad 0\leqslant \delta \leqslant 1 \tag{8}$$

can be ensured by imposing local conditions on the respective generating functions. To be more exact, we prove

Theorem 1. *Let $0<c<1$ and $0\leqslant\delta\leqslant 1$. If $g(t)\in BV(0,c)$ then the series* (7) *is summable* $|E,q|, q>0$.

Theorem 2. *Let $0<c<1$ and $0\leqslant\delta\leqslant 1$. If $h(t)\log(2\pi/t)\in BV(0,c)$ and $h(t)/t\in L(0,c)$ then the series* (8) *is summable* $|E,q|, q>0$.

(II) We wish to point out that Theorem I of Chandra [4] is not true. In this context, we prove

Theorem 3. *There exists a function $f(t)$ of the class L such that*

$$\psi(t)\log\log\frac{k}{t}\in BV(0,\pi), \quad \text{where } k>e\pi$$

but the series $\sum_{n=1}^{\infty} B_n(x)/\log(n+1)$ is not summable by any regular summability method and a fortiori not summable $|E,q|, q>0$.

We observe that $\psi(+0)=0$ and $\psi(t)\in BV(0,\pi)$ whenever $\psi(t)\log\log(k/t)\in BV(0,\pi)$ and hence Theorem I turns out to be false by an appeal to Theorem 3. The defect in the proof of Theorem I can be explained as follows.

We shall adopt a technique already developed by Kuttner (see Chandra [3]). We know that (see [4], p. 1015) when $\psi(+0)=0=\psi(\pi)$

$$B_m(x)=\begin{cases}\dfrac{2}{m\pi}\displaystyle\int_0^\pi \cos mt\,\mathrm{d}\psi(t) & (9)\\[2ex] \dfrac{2}{m\pi}\displaystyle\int_0^\pi (\cos mt-1)\,\mathrm{d}\psi(t). & (10)\end{cases}$$

Kuttner [3] while remarking on a paper [2] pointed out that if E_1 and E_2 are two disjoint intervals (or finite sums of intervals) whose union is $[0,\pi]$, it is not in general true that

$$B_m(x)=\frac{2}{m\pi}\int_{E_1}\cos mt\,\mathrm{d}\psi(t)+\frac{2}{m\pi}\int_{E_2}(\cos mt-1)\,\mathrm{d}\psi(t). \tag{11}$$

On p. 1016 (Chandra [4]) after splitting the sum

$$\sum=\sum_{n=1}^{\infty}\left|\sum_{m=1}^{n} v_{m-1}^{p}(n-1)y_mB_m(x)\right|\equiv\sum_{n=1}^{T_1}+\sum_{n=T_1+1}^{\infty}$$

where $v_{m-1}^{p}(n-1)=(1+p)^{-n}\binom{n}{m}p^{n-m}$, $y_m=[\log(m+1)]^{-1}$ and $T_1=[2\pi/t]$. Chandra has used (10) and (9) respectively in the sums $\sum_{n=1}^{T_1}$ and $\sum_{n=T_1+1}^{\infty}$ while expressing $B_m(x)$ as an integral. By doing so the integrand $(\cos mt-1)$ has been taken when $n\leqslant 2\pi/t$ (i.e., $0<t\leqslant 2\pi/n$) and $\cos mt$ when $n>2\pi/t$ (i.e., $t>2\pi/n$) while expressing $B_m(x)$ as an integral. This is equivalent to assuming (11) with $E_1=(2\pi/n,\pi)$ and $E_2=[0,2\pi/n]$ and hence the proof of Theorem I is erroneous.

(III) It is known that ([7], p. 364) the method (E,q) is not Fourier effective, i.e., the Fourier series of a continuous function need not be summable (E,q), $q>0$. We may therefore ask about $|E,q|$, $q>0$ summability factors of Fourier series, that is to say, a sequence $\{\lambda_n\}$ to be obtained such that

$$\sum\lambda_nA_n(x)\in|E,q|,\quad q>0$$

whenever $f(x)$ is continuous at $t=x$. In this connection, we prove

Theorem 4. *If*

$$\int_0^t|\phi(u)|\,\mathrm{d}u=O(t) \tag{12}$$

(in particular when $\phi(t)/t\in L(0,\pi)$) then $\sum_{n=1}^{\infty}\dfrac{A_n(x)}{n^{(1/2)+\varepsilon}}$ is summable $|E,q|$, $q>0$. Clearly the mentioned series is summable $|E,q|$, $q>0$ almost everywhere.

Theorem 5. *If*

$$\frac{\phi(t)}{t}\in L(0,c),\quad 0<c<1 \tag{13}$$

then

$$\sum_{n=1}^{\infty}\frac{A_n(x)}{n^{1/2}}\in|E,q|,\quad q>0.$$

It is known that [6], if (12) holds then $\sum_{n=1}^{\infty} \frac{A_n(x)}{\{\log(n+1)\}^{(3/2)+\varepsilon}}$ is summable $|C, 1|$ for every $\varepsilon > 0$. It was shown by Wang [14] that the factor $\{\log(n+1)\}^{-(3/2)-\varepsilon}$ cannot be replaced by $\{\log(n+1)\}^{-3/2}$. In Theorem 5 the factor $1/n^{1/2}$ is best possible in the sense that it cannot be replaced by $1/n^{(1/2)-\varepsilon}$ for $0 < \varepsilon \leqslant 1/2$. In this connection, we prove

Theorem 6. *There exists a function $f(t)$ of the class L such that $\phi(t)/t \in L(0, \pi)$ but the series $\sum_{n=1}^{\infty} (A_n(x)/n^{(1/2)-\varepsilon})$ is not summable $|E, q|$, $q > 0$ for $0 < \varepsilon < 1/2$.*

It is worth mentioning that Theorem 6 still holds when $\varepsilon = 1/2$ by a result of Kwee ([9], Theorem 3) who proves that the condition $\phi(t)/t^{\eta} \in L(0, \pi)$ where $\eta < 2$, does not ensure that $\sum A_n(x)$ is summable $|E, q|$, $q > 0$. As (13) implies (12), we may remark that the factor $1/(n^{(1/2)-\varepsilon})$ in Theorem 4 cannot be replaced by $1/(n^{(1/2)-\varepsilon})$, $0 < \varepsilon \leqslant 1/2$. It is natural to ask whether Theorem 4 remains valid when $\varepsilon = 0$. As we have not been able to answer this question it remains as an open problem.

(IV) It is known that ([9], also see Theorems D and E of the present paper) $|E, q|$, $q > 0$ summability of first derived Fourier series and first derived conjugate series hold under local conditions. We prove

Theorem 7. *The absolute Euler summability of rth derived Fourier series* (4) *is a local property of its generating function.*

Theorem 8. *The absolute Euler summability of rth derived conjugate series* (5) *is a local property of its generating function.*

3. Notations, order estimates and lemmas

3.1 *Notations.* Let

$$\tau = [t^{-1}],\ T = [t^{-2}],\ N = \left[\frac{n+1}{q+1}\right]$$

$$V(n, \nu) = (1+q)^{-n} \binom{n}{\nu} q^{n-\nu},\ \nu \leqslant n$$

$$S_\nu(t) = \sum_{k=1}^{\nu} V(n, k) \cos kt,\ \nu \leqslant n$$

$$\bar{S}_\nu(t) = \sum_{k=1}^{\nu} V(n, k) \sin kt,\ \nu \leqslant n$$

$$\rho(t) = \frac{(1+q^2+2q\cos t)^{1/2}}{1+q}, \qquad \Phi = \tan^{-1}\frac{\sin t}{q+\cos t}$$

Let $\{\lambda_n\}$ be a non-increasing sequence of positive numbers such that

(i) $\{n^i \lambda_n\}$ is increasing for some non-negative integer i.
(ii) $\{n^j \Delta\lambda_n\}$ is increasing for some non-negative integer j,

where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

$$g_c(n, \lambda, t) = \sum_{\nu=1}^{n} V(n, \nu) \lambda_\nu \cos \nu t$$

$$g_s(n,\lambda,t)=\sum_{\nu=1}^{n} V(n,\nu)\lambda_\nu \sin\nu t$$

$$\xi(n,t)=\int_0^t \frac{g_s(n,n^{-1}(\log(n+1))^{-\delta},u)}{u(\log(2\pi/u))^{2-\delta}}\,du$$

$$\eta(n,t)=\int_t^c \frac{g_s(n,n^{-1}(\log(n+1))^{-\delta},u)}{u(\log(2\pi/u))^{2-\delta}}\,du$$

$$\xi^*(n,t)=\int_0^t \left(\log\frac{2\pi}{u}\right)^{\delta} g_s(n,(\log(n+1))^{-\delta},u)\,du$$

$$\eta^*(n,t)=\int_t^c \left(\log\frac{2\pi}{u}\right)^{\delta} g_s(n,(\log(n+1))^{-\delta},u)\,du$$

3.2. *Estimates.* We need the following estimates:

Clearly

$$\sum_{m=0}^{n} V(n,m)=1 \quad\text{and}\quad V(n,m)=\frac{(m+1)(q+1)}{n+1}V(n+1,m+1) \tag{14}$$

$$V(n,m)=O\{V(n,N)\}=O(n^{-1/2}),\quad m\leqslant n \tag{15}$$

$$S_n(t)=\begin{cases} O\{\rho^n(t)\}+\left\{\left(\dfrac{q}{1+q}\right)^n\right\} & (16)\\ O(t^{-1}n^{-1/2}) & (17)\end{cases}$$

$$\bar{S}_n(t)=\begin{cases} O\{\rho^n(t)\} & (18)\\ O(t^{-1}n^{-1/2}) & (19)\end{cases}$$

$$g_c(n,\lambda,t)=\begin{cases} O(\lambda_n) & (20)\\ O(t^{-1}\Delta\lambda_n)+O(n^{-1/2}t^{-1}\lambda_n) & (21)\\ O(t^{-1}\Delta\lambda_n)+O(\lambda_n\rho^n(t))+O\left\{\left(\dfrac{q}{q+1}\right)^n\lambda_n\right\} & (22)\end{cases}$$

$$g_s(n,\lambda,t)=\begin{cases} O(nt\lambda_n) & (23)\\ O(t^{-1}\Delta\lambda_n)+O(n^{-1/2}t^{-1}\lambda_n) & (24)\\ O(t^{-1}\Delta\lambda_n)+O(\lambda_n\rho^n(t)) & (25)\end{cases}$$

$$g_c(n,n^{-(1/2)+\varepsilon},t)=S_n(t)(N+1)^{-(1/2)+\varepsilon}+O\{t^{-1}n^{-(3/2)+\varepsilon}\} \tag{26}$$

$$\xi(n,t)=O\left\{t\left(\log\frac{2\pi}{t}\right)^{\delta-2}(\log n)^{-\delta}\right\} \tag{27}$$

$$\begin{aligned}\eta(n,t)=&\,O\left\{t^{-2}\left(\log\frac{2\pi}{t}\right)^{\delta-2}n^{-3}(\log n)^{-\delta}\right\}\\ &+O\left\{\rho^n(t)t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}n^{-2}(\log n)^{-\delta}\right\}\\ &+O\left\{\left(\frac{q}{q+1}\right)^n t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}n^{-2}(\log n)^{-\delta}\right\}\end{aligned} \tag{28}$$

$$\xi^*(n,t) = O\left\{t^2\left(\log\frac{2\pi}{t}\right)^{\delta} n(\log n)^{-\delta}\right\} \tag{29}$$

$$\frac{1}{1-\rho(t)} = O(t^{-2}), \quad 0 < t \leqslant \pi \tag{30}$$

$$\frac{1}{1-\rho(t)} > 4t^{-2}, \quad 0 < t \leqslant \pi \tag{31}$$

It is known that ([7], p. 214) $V(n,m)$ attains its maximum value when $m = N$ and $V(n,N) = O(n^{-1/2})$ and hence (15) follows.

Proof of (16) *and* (18). Taking $\rho(t)$ and Φ as in § 3.1, we get

$$\begin{aligned}
S_n(t) + i\bar{S}_n(t) &= \sum_{v=0}^{n} V(n,v)e^{ivt} - \left(\frac{q}{q+1}\right)^n \\
&= \left(\frac{q+e^{it}}{q+1}\right)^n - \left(\frac{q}{q+1}\right)^n \\
&= \rho^n(t)(\cos n\Phi + i\sin n\Phi) - \left(\frac{q}{q+1}\right)^n,
\end{aligned} \tag{32}$$

from which (16) and (18) follow at once.

Proof of (17) *and* (19). As $V(n,k)$ attains its maximum value when $k = N$, we get

$$\begin{aligned}
|S_n(t)| &\leqslant V(n,N) \max_{1<L,L'<n} \left|\sum_{L}^{L'} \cos kt\right| \\
&= O\{V(n,N)t^{-1}\} \\
&= O\{t^{-1}n^{-1/2}\}
\end{aligned}$$

using (15). Similarly (19) can be derived.

Proof of (20), (21) *and* (22). There exists a non-negative integer i such that $\{n^i\lambda_n\}$ is increasing. After this choice of i, we have by using (14)

$$\begin{aligned}
g_c(n,\lambda,t) &= \sum_{k=1}^{n} V(n,k)\lambda_k \cos kt \\
&= \frac{(q+1)^i}{(n+1)(n+2)\dots(n+i)} \sum_{k=1}^{n} V(n+i,k+i)(k+1)(k+2)\dots \\
&\qquad\qquad (k+i)\lambda_k \cos kt \\
&= O(\lambda_n) \sum_{k=1}^{n} V(n+i,k+i), \text{ as } \{k^i\lambda_k\} \text{ is increasing} \\
&= O(\lambda_n) \sum_{k=-i}^{n} V(n+i,k+i) \\
&= O(\lambda_n),
\end{aligned}$$

as the last sum is unity. By Abel's method of partial summation

$$
\begin{aligned}
g_c(n,\lambda,t) &= \sum_{\nu=1}^{n} V(n,\nu)\lambda_\nu \cos \nu t \\
&= \sum_{\nu=1}^{n-1} \Delta\lambda_\nu S_\nu(t) + \lambda_n S_n(t) \\
&= \sum_{\nu=1}^{N} \Delta\lambda_\nu S_\nu(t) + \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu S_\nu(t) + \lambda_n S_n(t) \\
&= \sum_{\nu=1}^{N} \Delta\lambda_\nu S_\nu(t) + \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \left[S_n(t) - \sum_{k=\nu+1}^{n} V(n,k)\cos kt \right] + \lambda_n S_n(t) \\
&= \sum_{\nu=1}^{N} \Delta\lambda_\nu S_\nu(t) + S_n(t)(\lambda_{N+1} - \lambda_n) \\
&\quad - \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \sum_{k=\nu+1}^{n} V(n,k)\cos kt + \lambda_n S_n(t) \\
&= \sum_{\nu=1}^{N} \Delta\lambda_\nu S_\nu(t) + \lambda_{N+1} S_n(t) - \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \sum_{k=\nu+1}^{n} V(n,k)\cos kt \\
&= T_1 + T_2 - T_3.
\end{aligned} \tag{33}
$$

As $V(n,k)$ is monotonic increasing for $k \leqslant N$, we get

$$
\begin{aligned}
|T_1| &\leqslant \sum_{\nu=1}^{N} \Delta\lambda_\nu \left| \sum_{k=1}^{\nu} V(n,k)\cos kt \right| \\
&\leqslant \sum_{\nu=1}^{N} \Delta\lambda_\nu \, V(n,\nu) \max_{1<L,L'<\nu} \left| \sum_{L}^{L'} \cos kt \right| \\
&= O(t^{-1}) \sum_{\nu=1}^{N} \Delta\lambda_\nu \, V(n,\nu) \\
&= O(t^{-1}\Delta\lambda_n),
\end{aligned}
$$

by employing argument similar to those used in proving (20). By (16) and (17)

$$
T_2 = \begin{cases} O\{\rho^n(t)\lambda_n\} + O\left\{\left(\dfrac{q}{q+1}\right)^n \lambda_n\right\} \\ O\{\lambda_n t^{-1} n^{-1/2}\}. \end{cases}
$$

As $V(n,k)$ is monotonic decreasing for $k > N$, we have

$$
\begin{aligned}
|T_3| &\leqslant \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \, V(n,\nu) \max_{\nu+1<M,M'<n} \left| \sum_{M}^{M'} \cos kt \right| \\
&= O(t^{-1}) \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \, V(n,\nu) \\
&= O(t^{-1}) \sum_{\nu=1}^{n} \Delta\lambda_\nu \, V(n,\nu) = O(t^{-1}\Delta\lambda_n),
\end{aligned}
$$

using the technique employed in proving (20). Now using the estimates for T_1, T_2 and T_3 in (33) we obtain (21) and (22).

Proof of (23), (24) *and* (25). We have

$$g_s(n,\lambda,t) = \sum_{k=1}^{n} V(n,k)\lambda_k \sin kt$$
$$= O\left(t\sum_{k=1}^{n} V(n,k)k\,\lambda_k\right) = O(tn\,\lambda_n),$$

using the technique employed in proving (20). The proof of (24) and (25) are similar to those of (21) and (22).

Proof of (26). Taking $\lambda_n = n^{-(1/2)+\varepsilon}$ in the proof of (21) and (22) we get

$$T_1 = O(t^{-1}n^{-(3/2)+\varepsilon}),\ T_2 = \frac{S_n(t)}{(N+1)^{(1/2)-\varepsilon}},\ T_3 = O(t^{-1}n^{-(3/2)+\varepsilon})$$

and hence using these results in (33) we obtain

$$g_c(n, n^{-(1/2)+\varepsilon}, t) = T_1 + T_2 - T_3$$
$$= \frac{S_n(t)}{(N+1)^{(1/2)-\varepsilon}} + O(t^{-1}n^{-(3/2)+\varepsilon}).$$

Proof of (27). Using (23), we get

$$\xi(n,t) = \int_0^t \frac{g_s(n,n^{-1}(\log(n+1))^{-\delta},u)}{u(\log(2\pi/u))^{2-\delta}}\mathrm{d}u$$
$$= O\left\{\int_0^t \frac{\mathrm{d}u}{(\log n)^{\delta}(\log(2\pi/u))^{2-\delta}}\right\} = O\left\{t(\log n)^{-\delta}\left(\log\frac{2\pi}{t}\right)^{\delta-2}\right\}.$$

Proof of (28). By mean value theorem for some $t < t' = t'(n) < c$

$$\eta(n,t) = \int_t^c \frac{g_s(n,n^{-1}(\log(n+1))^{-\delta},u)}{u(\log(2\pi/u))^{2-\delta}}\mathrm{d}u$$
$$= t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}\int_t^{t'} g_s(n,n^{-1}(\log(n+1))^{-\delta},u)\mathrm{d}u$$
$$= t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}\{g_c(n,n^{-2}(\log(n+1))^{-\delta},t)$$
$$- g_c(n,n^{-2}(\log(n+1))^{-\delta},t')\}.$$

Now using (22) we get (28) at once since $t'^{-1} < t^{-1}$ and $\rho(t') < \rho(t)$.

Proof of (29). Using (23), we get

$$\xi^*(n,t) = \int_0^t \left(\log\frac{2\pi}{u}\right)^{\delta} g_s(n,(\log(n+1))^{-\delta},u)\mathrm{d}u$$

$$= O\left\{n(\log n)^{-\delta} \int_0^t u\left(\log\frac{2\pi}{u}\right)^{\delta} \mathrm{d}u\right\}$$

$$= O\left\{t^2\left(\log\frac{2\pi}{t}\right)^{\delta} n(\log n)^{-\delta}\right\}.$$

Proof of (30) *and* (31). Using the expression for $\rho(t)$ as given in §3.1, we get

$$[1-\rho^2(t)]^{-1} = \left[1 - \frac{1+2q\cos t + q^2}{(1+q)^2}\right]^{-1}$$

$$= \frac{(1+q)^2}{2q(1-\cos t)} = \frac{(1+q)^2}{4q\sin^2(1/2)t} \tag{34}$$

from which it follows that

$$\frac{(1+q)^2}{q}t^{-2} \leqslant \frac{1}{1-\rho^2(t)} \leqslant \frac{\pi^2(1+q)^2}{4q}t^{-2}, \quad 0<t\leqslant\pi \tag{35}$$

as $t/\pi \leqslant \sin(1/2)t \leqslant t/2$ whenever $0<t\leqslant\pi$. Writing $1/(1-\rho(t)) = (1+\rho(t))/(1-\rho^2(t))$ and making use of the fact that $1<1+\rho(t)<2$ for all $0<t\leqslant\pi$, we obtain

$$\frac{(1+q)^2}{q}t^{-2} < \frac{1}{1-\rho(t)} < \frac{\pi^2(1+q)^2}{2q}t^{-2}, \quad 0<t\leqslant\pi. \tag{36}$$

Now (30) and (31) follow at once from (36) as q is a fixed positive constant and $(1+q)^2/q \geqslant 4$ for all $q>0$.

3.3. *Lemmas.* We need the following lemmas:

Lemma 1 [10]. *If* $\eta>0$, *then necessary and sufficient conditions that* (i) $h(t)\log(2\pi/t)$ *should be of bounded variation in* $(0,\eta)$ *and* (ii) $h(t)/t$ *should be integrable L in* $(0,\eta)$ *are that*

$$\int_0^\eta \log\frac{2\pi}{t}|\mathrm{d}h(t)| < \infty \text{ and } h(+0)=0.$$

Lemma 2. (i) $\displaystyle\int_0^c (\log(2\pi/t))^\beta \sin nt\,\mathrm{d}t \sim \frac{(\log n)^\beta}{n}$ *for all* $\beta>0$.

(ii) $\displaystyle\int_0^c \left(\log\frac{2\pi}{t}\right)^\beta \frac{\sin nt}{t}\mathrm{d}t \sim \frac{\pi}{2}(\log n)^\beta$ *for all* β.

The proof is similar (merely replace π by c) to the proof of Lemma 3(i) of [12].

Lemma 3. $\displaystyle\int_0^\pi \frac{\sin nt}{\log\log(k/t)}\mathrm{d}t = -\frac{(-1)^n}{n\log\log(k/\pi)} + \theta_n$

where

$$\theta_n \sim 1/(n\log\log n).$$

We omit the proof of Lemma 3 as it can be proved using arguments similar to those used in proving Lemma 3 of [12].

Lemma 4.

$$\text{(i)}\ \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r S_n(t) = \begin{cases} -\left(\dfrac{q}{1+q}\right)^n + O(\rho^n(t)), & \textit{for } r=0 \\ O(n^r\rho^{n-r}(t)), & \textit{for } r=1,2,3,\ldots \end{cases}$$

$$\text{(ii)}\ \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \bar{S}_n(t) = O(n^r\rho^{n-r}(t)). \qquad \textit{for } r=0,1,2,3,\ldots.$$

Proof. Put

$$w = S_n(t) + i\bar{S}_n(t) + \left(\frac{q}{q+1}\right)^n, \quad u = q + e^{it}$$

$$w_r = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r w, \quad u_r = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r u.$$

Using (32) we get

$$w = \left(\frac{q+e^{it}}{q+1}\right)^n = \rho^n(t)e^{in\Phi} = O(\rho^n(t)).$$

Clearly

$$w_1 = \frac{ni}{q+1}\left(\frac{q+e^{it}}{q+1}\right)^{n-1} e^{it} = O(n\,\rho^{n-1}(t)). \tag{37}$$

Let us assume that

$$w_r = O\{n^r\rho^{n-r}(t)\} \quad \text{for } r=0,1,2,\ldots,k. \tag{38}$$

From (32), we get

$$w_1 u = ni\,we^{it}. \tag{39}$$

Differentiating k times both sides of (39) with respect to t, we obtain

$$w_{k+1}u + \sum_{\nu=1}^{k}\binom{k}{\nu} w_{k+1-\nu}u_\nu = ni\sum_{\nu=0}^{k}\binom{k}{\nu} w_{k-\nu}(i)^\nu e^{it}.$$

Using (37), we get

$$w_{k+1}u = O\left(\sum_{\nu=1}^{k}\binom{k}{\nu} n^{k+1-\nu}\rho^{n-k-1+\nu}(t)\right) + O\left(n\sum_{\nu=0}^{k}\binom{k}{\nu} n^{k-\nu}\rho^{n-k+\nu}(t)\right)$$
$$= O(n^{k+1}\rho^{n-k}(t))$$

as the first term in each of the above sums dominates over the remaining terms. Since $|u| = (1+q)\rho(t)$, we have

$$w_{k+1} = O(n^{k+1}\rho^{n-k-1}(t)).$$

which is in conjunction with (38) gives

$$w_r = O(n^r\rho^{n-r}(t)) \quad \text{for } r=0,1,2,3,\ldots,(k+1).$$

Hence by induction principle

$$w_r = O(n^r\rho^{n-r}(t)) \quad \text{for } r=0,1,2,\ldots.$$

This completes the proof of the lemma.

Lemma 5. *For* $r = 1, 2, 3, \ldots$

$$\sum n^r(-1)^n \in |E, q|, \quad q > 0.$$

Proof. Case (I) Let r be odd, i.e., $r = 2m + 1$, $m = 0, 1, 2, \ldots$. Clearly $n^r(-1)^n = [(\mathrm{d}/\mathrm{d}t)^{2m+1} \sin nt]_{t=\pi}$.
So the series $\sum n^r(-1)^n \in |E, q|$

$$\Leftrightarrow \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} V(n, k) \left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^{2m+1} \sin kt \right]_{t=\pi} \right| < \infty$$

$$\Leftrightarrow \sum_{n=1}^{\infty} \left| \left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^{2m+1} \bar{S}_n(t) \right]_{t=\pi} \right| < \infty. \tag{40}$$

By Lemma 4

$$\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^{2m+1} \bar{S}_n(t) \right]_{t=\pi} = O(n^{2m+1} \rho^{n-2m-1}(\pi)) = O\left(n^r \left(\frac{|1-q|}{1+q} \right)^{n-r} \right),$$

and hence (40) follows at once.

Case (II). Let r be even, i.e., $r = 2m$, $m = 1, 2, 3, \ldots$. Clearly

$$n^r(-1)^n = \left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^{2m} \cos nt \right]_{t=\pi}$$

So the series $\sum n^r(-1)^n \in |E, q|$

$$\Leftrightarrow \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} V(n, k) \left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^{2m} \cos kt \right]_{t=\pi} \right| < \infty$$

$$\Leftrightarrow \sum_{n=1}^{\infty} \left| \left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^{2m} S_n(t) \right]_{t=\pi} \right| < \infty. \tag{41}$$

By Lemma 4

$$\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^{2m} S_n(t) \right]_{t=\pi} = O(n^{2m} \rho^{n-2m}(\pi)) = O\left(n^r \left(\frac{|1-q|}{1+q} \right)^{n} \right),$$

which ensures the validity of (41).

This completes the proof of the lemma.

Lemma 6. (Bosanquet and Kestelman [1]). *Suppose that $f_n(x)$ is measurable in (a, b) where $b - a < \infty$, for $n = 1, 2, 3 \ldots$. Then a necessary and sufficient condition that for every $\lambda(x) \in L(a, b)$ the function $f_n(x)\lambda(x)$ should be $L(a, b)$ and*

$$\sum_{n=1}^{\infty} \left| \int_a^b \lambda(x) f_n(x) \mathrm{d}x \right| \leqslant K$$

is that

$$\sum_{n=1}^{\infty} |f_n(x)| \leqslant K$$

for almost every $x \in (a, b)$, where K is an absolute constant.

4. Proof of theorem 1

4.1. *Proof of theorem* 1. We have for $n \geqslant 1$

$$\begin{aligned}\frac{\pi}{2}A_n(x) &= \int_0^c g(t)\frac{\cos nt}{(\log(2\pi/t))^{1-\delta}}\mathrm{d}t + \int_c^\pi \phi(t)\cos nt\,\mathrm{d}t\\ &= \alpha_n + \beta_n. \end{aligned} \tag{42}$$

The series $\sum_{n=1}^\infty \frac{\beta_n}{(\log(n+1))^\delta} \in |E, q|$

$$\Leftrightarrow \Sigma_1 = \sum_{n=1}^\infty \left| \sum_{k=1}^n V(n,k)\int_c^\pi \frac{\phi(t)\cos kt}{(\log(k+1))^\delta}\mathrm{d}t\right| < \infty$$

Now

$$\begin{aligned}\Sigma_1 &\leqslant \int_c^\pi |\phi(t)| \sum_{n=1}^\infty \left|\sum_{k=1}^n V(n,k)\frac{\cos kt}{(\log(k+1))^\delta}\right| \mathrm{d}t\\ &= \int_c^\pi |\phi(t)| \sum_{n=1}^\infty |g_c(n, \log(n+1))^{-\delta}, t)|\mathrm{d}t. \end{aligned}\tag{43}$$

In the case $\delta > 0$, using (22) and (30)

$$\begin{aligned}\sum_{n=1}^\infty |g_c(n, \log(n+1))^{-\delta}, t)| &= O\left(t^{-1}\sum_{n=2}^\infty \frac{1}{n(\log n)^{\delta+1}}\right) + O\left(\sum_{n=2}^\infty \frac{\rho^n(t)}{(\log n)^\delta}\right)\\ &\quad + O\left(\sum_{n=2}^\infty \left(\frac{q}{q+1}\right)^n (\log n)^{-\delta}\right)\\ &= O(t^{-1}) + O\left(\frac{1}{1-\rho(t)}\right) + O(1) = O(t^{-2}).\end{aligned}$$

For $\delta = 0$, using (16) and (30)

$$\begin{aligned}\sum_{n=1}^\infty \left|g_c(n, (\log(n+1))^{-\delta}, t)\right| &= O\left(\sum_{n=1}^\infty \rho^n(t)\right) + O\left(\sum_{n=1}^\infty \left(\frac{q}{q+1}\right)^n\right)\\ &= O\left(\frac{1}{1-\rho(t)}\right) + O(1) = O(t^{-2}).\end{aligned}$$

Using these results in (43), we get

$$\Sigma_1 = O\left(\int_c^\pi \frac{|\phi(t)|}{t^2}\mathrm{d}t\right) = O(1),$$

which ensures that $\sum_{n=1}^\infty \frac{\beta_n}{(\log(n+1))^\delta} \in |E, q|$.

Next, integrating by parts

$$\begin{aligned}\alpha_n &= \frac{g_c}{(\log(2\pi/c))^{1-\delta}}\frac{\sin nc}{n} - \frac{1-\delta}{n}g(c)\int_0^c \frac{\sin nt}{t(\log(2\pi/t))^{2-\delta}}\mathrm{d}t\\ &\quad -\frac{1}{n}\int_0^c \frac{\sin nt}{(\log(2\pi/t))^{1-\delta}}\mathrm{d}g(t) + \frac{1-\delta}{n}\int_0^c\left(\int_0^t \frac{\sin nu}{u(\log(2\pi/u))^{2-\delta}}\mathrm{d}u\right)\mathrm{d}g(t)\\ &= \alpha_n^{(1)} - \alpha_n^{(2)} - \alpha_n^{(3)} + \alpha_n^{(4)}.\end{aligned}\tag{44}$$

When $\delta = 1$, the terms $\alpha_n^{(2)}$ and $\alpha_n^{(4)}$ drop out and hence it is enough to study the absolute Euler summability of

$$\sum \frac{\alpha_n^{(1)}}{(\log(n+1))^\delta} \quad \text{and} \quad \sum \frac{\alpha_n^{(3)}}{(\log(n+1))^\delta} \quad \text{for } 0 \leqslant \delta \leqslant 1$$

and that of

$$\sum \frac{\alpha_n^{(2)}}{(\log(n+1))^\delta} \quad \text{and} \quad \sum \frac{\alpha_n^{(4)}}{(\log(n+1))^\delta} \quad \text{for } 0 \leqslant \delta < 1.$$

For $0 \leqslant \delta \leqslant 1, \sum \frac{\alpha_n^{(1)}}{(\log(n+1))^\delta} \in |E, q|$ if and only if

$$\sum_{n=1}^{\infty} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, c)| < \infty. \tag{45}$$

By (24), $g_s(n, n^{-1}(\log(n+1))^{-\delta}, c) = O(n^{-3/2}(\log n)^{-\delta})$ and hence (45) follows. The series $\sum \frac{\alpha_n^{(3)}}{\{\log(n+1)\}^\delta} \in |E, q|$, if and only if

$$\sum_{n=1}^{\infty} \left| \int_0^c \frac{dg(t)\, g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)}{(\log(2\pi/t))^{1-\delta}} \right| < \infty.$$

Since $\int_0^c |dg(t)|$ is finite, it remains to show that

$$\sum_{n=1}^{\infty} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)| = O\left\{\left(\log \frac{2\pi}{t}\right)^{1-\delta}\right\} \quad \text{in } 0 < t < c. \tag{46}$$

Using (23), (25) and (30)

$$\begin{aligned} \sum_{n=1}^{\infty} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)| &= \sum_{n \leqslant \tau} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)| \\ &\quad + \sum_{n > \tau} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)| \\ &= O\left(t \sum_{n \leqslant \tau} (\log n)^{-\delta}\right) + O\left(\sum_{n > \tau} \frac{\rho^n(t)}{n(\log n)^\delta}\right) \\ &\quad + O\left(t^{-1} \sum_{n > \tau} \frac{1}{n^2(\log n)^\delta}\right) \\ &= O\left\{\left(\log \frac{2\pi}{t}\right)^{1-\delta}\right\}, \end{aligned} \tag{47}$$

since

$$\sum_{n=1}^{\infty} \frac{\rho^n(t)}{n} = \log \frac{1}{1 - \rho(t)} = O\left(\log \frac{2\pi}{t}\right).$$

For $0 \leqslant \delta < 1$, by Lemma 2(ii)

$$\frac{\alpha_n^{(2)}}{(\log(n+1))^\delta} \sim (1 - \delta) g(c) \frac{\pi}{2} \frac{1}{n(\log n)^2}.$$

So the series $\sum \frac{\alpha_n^{(2)}}{(\log(n+1))^\delta}$ is absolutely convergent and hence *a fortiori* it is summable $|E,q|$.

For $0 \leqslant \delta < 1$ the series $\sum \frac{\alpha_n^{(4)}}{(\log(n+1))^\delta} \in |E,q|$, if and only if

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} \frac{V(n,k)}{k\{\log(k+1)\}^\delta} \int_0^c \mathrm{d}g(t) \int_0^t \frac{\sin ku}{u(\log(2\pi/u))^{2-\delta}} \mathrm{d}u \right| < \infty$$

i.e.,

$$\sum_{n=1}^{\infty} \left| \int_0^c \mathrm{d}g(t)\, \xi(n,t) \right| < \infty. \tag{48}$$

Since $\int_0^c |\mathrm{d}g(t)|$ is finite for the validity of (48) it is enough to show that in $0 < t < c$

$$\sum_{n=1}^{\infty} |\xi(n,t)| = O(1). \tag{49}$$

Using (27)

$$\sum_{n \leqslant \tau} |\xi(n,t)| = O\left\{ t \left(\log \frac{2\pi}{t} \right)^{\delta - 2} \sum_{n \leqslant \tau} (\log n)^{-\delta} \right\} = O\left(\left(\log \frac{2\pi}{t} \right)^{-2} \right). \tag{50}$$

Now

$$\sum_{n > \tau} |\xi(n,t)| = \sum_{n > \tau} |\xi(n,c) - \eta(n,t)|$$

$$\leqslant \sum_{n > \tau} |\xi(n,c) + \sum_{n > \tau} |\eta(n,t)|. \tag{51}$$

By simple verification, we have

$$\sum \frac{\alpha_n^{(2)}}{(\log(n+1))^\delta} \in |E,q| \text{ if and only if } \sum |\xi(n,c)| < \infty.$$

The series $\sum \frac{\alpha_n^{(2)}}{\{\log(n+1)\}^\delta}$ is known to be summable $|E,q|$ and hence

$$\sum_{n > \tau} |\xi(n,c)| \leqslant \sum_{n=1}^{\infty} |\xi(n,c)| = O(1). \tag{52}$$

Using (28), we have

$$\begin{aligned}
\sum_{n > \tau} |\eta(n,t)| &= O\left(t^{-2} \left(\log \frac{2\pi}{t} \right)^{\delta - 2} \sum_{n > \tau} n^{-3} (\log n)^{-\delta} \right) \\
&\quad + O\left(t^{-1} \left(\log \frac{2\pi}{t} \right)^{\delta - 2} \sum_{n > \tau} \frac{\rho^n(t)}{n^2 (\log n)^\delta} \right) \\
&\quad + O\left\{ t^{-1} \left(\log \frac{2\pi}{t} \right)^{\delta - 2} \sum_{n > \tau} \left(\frac{q}{q+1} \right)^n n^{-2} (\log n)^{-\delta} \right\} \\
&= O\left\{ \left(\log \frac{2\pi}{t} \right)^{-2} \right\} + O\left\{ t^{-1} \left(\log \frac{2\pi}{t} \right)^{\delta - 2} \tau^{-2} (\log \tau)^{-\delta} \sum_{n=1}^{\infty} \rho^n(t) \right\}
\end{aligned}$$

$$+O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}\tau^{-2}(\log\tau)^{-\delta}\sum_{n=1}^{\infty}\left(\frac{q}{q+1}\right)^{n}\right\}$$

$$=O\left\{\left(\log\frac{2\pi}{t}\right)^{-2}\right\}, \tag{53}$$

as $\sum_{n=1}^{\infty}\rho^{n}(t)=1/(1-\rho(t))=O(t^{-2})$ by (30).

Now (49) follows immediately from (50), (51), (52) and (53). This completes the proof of Theorem 1.

5. Proof of theorem 2

5.1. By Lemma 1, Theorem 2 is equivalent to

Theorem 2(a). *Let* $0<c<1$, *If* $h(+0)=0$ *and* $\int_0^c |dh(t)|\log\frac{2\pi}{t}<\infty$ *then* $\sum\frac{B_n(x)}{\{\log(n+1)\}^{\delta}}\in|E,q|$, $q>0$ *for* $0\leqslant\delta\leqslant 1$.

Proof of theorem 2(*a*). For $n\geqslant 1$, we write

$$\frac{\pi}{2}\frac{B_n(x)}{\{\log(n+1)\}^{\delta}}=\frac{1}{\{\log(n+1)\}^{\delta}}\left[\int_0^c\psi(t)\sin nt\,dt+\int_c^{\pi}\psi(t)\sin nt\,dt\right]$$

$$=\alpha_n+\beta_n. \tag{54}$$

$\sum\beta_n\in|E,q|$, $q>0$ if and only if

$$\Sigma_1=\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}V(n,k)\int_c^{\pi}\frac{\psi(t)\sin kt}{\{\log(k+1)\}^{\delta}}dt\right|<\infty$$

Now

$$\Sigma_1\leqslant\int_c^{\pi}|\psi(t)|\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}V(n,k)\frac{\sin kt}{\{\log(k+1)\}^{\delta}}\right|dt$$

$$=\int_c^{\pi}|\psi(t)|\sum_{n=1}^{\infty}|g_s(n,(\log(n+1))^{-\delta},t)|\,dt \tag{55}$$

by (24) and (30)

$$\sum_{n=1}^{\infty}|g_s(n,(\log(n+1))^{-\delta},t)|=O\left\{t^{-1}\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{1+\delta}}\right\}+O\left\{\sum_{n=2}^{\infty}\frac{\rho^{n}(t)}{(\log n)^{\delta}}\right\}$$

$$=O(t^{-1})+O\left(\sum_{n=1}^{\infty}\rho^{n}(t)\right)=O(t^{-2}).$$

Using this estimate in (55), we get

$$\Sigma_1=O\left(\int_c^{\pi}\frac{|\psi(t)|}{t^2}dt\right)=O(1),$$

which ensures that $\sum\beta_n\in|E,q|$.

We have

$$\alpha_n = \int_0^c h(t)\left(\log\frac{2\pi}{t}\right)^\delta \sin nt\,\mathrm{d}t$$
$$= \int_0^c \mathrm{d}h(t)\int_t^c \left(\log\frac{2\pi}{u}\right)^\delta \sin nu\,\mathrm{d}u,$$

the integrated part vanishes as $h(+0)=0$ and $\int_t^c (\log(2\pi/u))^\delta \sin nu\,\mathrm{d}u = O(1)$. The series $\sum\alpha_n \in |E,q|$, if and only if

$$\Sigma_2 = \sum_{n=1}^{\infty}\left|\sum_{k=1}^{n} V(n,k)\int_0^c \mathrm{d}h(t)\int_t^c \left(\log\frac{2\pi}{u}\right)^\delta \frac{\sin ku}{\{\log(k+1)\}^\delta}\mathrm{d}u\right| < \infty. \tag{56}$$

Clearly

$$\Sigma_2 = \sum_{n=1}^{\infty}\left|\int_0^c \mathrm{d}h(t)\int_t^c \left(\log\frac{2\pi}{u}\right)^\delta g_s(n,(\log(n+1))^{-\delta},u)\mathrm{d}u\right|$$
$$= \sum_{n=1}^{\infty}\left|\int_0^c \mathrm{d}h(t)\,\eta^*(n,t)\right|$$
$$\leqslant \int_0^c |\mathrm{d}h(t)| \sum_{n=1}^{\infty} |\eta^*(n,t)|.$$

Since $\int_0^c |\mathrm{d}h(t)|\log(2\pi/t)$ is finite for the validity of (56) it is enough to show that in $0<t<c$

$$\sum_{n=1}^{\infty} |\eta^*(n,t)| = O\left(\log\frac{2\pi}{t}\right). \tag{57}$$

By Lemma 2(i), we get

$$\xi^*(n,c) = \sum_{k=1}^{n} V(n,k)\{\log(k+1)\}^{-\delta}\int_0^c \left(\log\frac{2\pi}{t}\right)^\delta \sin ku\,\mathrm{d}u$$
$$= O\left(\sum_{k=1}^{n}\frac{V(n,k)}{k}\right) = O(n^{-1}), \tag{58}$$

using the technique used in proving (20).

Now by (58) and (29)

$$\sum_{n\leqslant\tau} |\eta^*(n,t)| = \sum_{n\leqslant\tau} |\xi^*(n,c) - \xi^*(n,t)|$$
$$\leqslant \sum_{n\leqslant\tau} |\xi^*(n,c)| + \sum_{n\leqslant\tau} |\xi^*(n,t)|$$
$$= O\left(\sum_{n\leqslant\tau}\frac{1}{n}\right) + O\left\{t^2\left(\log\frac{2\pi}{t}\right)^\delta \sum_{n\leqslant\tau}\frac{n}{(\log n)^\delta}\right\}$$
$$= O\left(\log\frac{2\pi}{t}\right). \tag{59}$$

Now for some $t' = t'(n)$ with $t < t' < t'(n) < c$, we have

$$\eta^*(n,t) = \int_t^c \left(\log\frac{2\pi}{u}\right)^{\delta} g_s(n, (\log(n+1))^{-\delta}, u)\mathrm{d}u$$
$$= \left(\log\frac{2\pi}{t}\right)^{\delta} \{g_c(n, n^{-1}(\log(n+1))^{-\delta}, t) - g_c(n, n^{-1}(\log(n+1))^{-\delta}, t')\}.$$

Using (22) for the estimates of $g_c(n, n^{-1}(\log(n+1))^{-\delta}, t)$ and $g_c(n, n^{-1}(\log(n+1))^{-\delta}, t')$ and taking note of the fact that $t'^{-1} < t^{-1}$, we obtain

$$\eta^*(n,t) = O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta} n^{-2}(\log n)^{-\delta}\right\} + O\left\{\left(\log\frac{2\pi}{t}\right)^{\delta} n^{-1}(\log n)^{-\delta}\rho^n(t)\right\}$$
$$+ O\left\{\left(\frac{q}{q+1}\right)^n \left(\log\frac{2\pi}{t}\right)^{\delta} n^{-1}(\log n)^{-\delta}\right\}$$
$$+ O\left\{\left(\log\frac{2\pi}{t}\right)^{\delta} n^{-1}(\log n)^{-\delta}\rho^n(t')\right\}. \tag{60}$$

Using (60), we get

$$\sum_{n>\tau} |\eta^*(n,t)| = O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta} \sum_{n>\tau} n^{-2}(\log n)^{-\delta}\right\}$$
$$+ O\left\{\left(\log\frac{2\pi}{t}\right)^{\delta} \sum_{n>\tau} \frac{\rho^n(t)}{n(\log n)^{\delta}}\right\}$$
$$+ O\left\{\left(\log\frac{2\pi}{t}\right)^{\delta} \sum_{n>\tau} \left(\frac{q}{q+1}\right)^n \frac{1}{n(\log n)^{\delta}}\right\}$$
$$+ O\left\{\left(\log\frac{2\pi}{t}\right)^{\delta} \sum_{n>\tau} \frac{\rho^n(t')}{n(\log n)^{\delta}}\right\}$$
$$= O(1) + O\left(\log\frac{1}{1-\rho(t)}\right) + O\left(\log\frac{1}{1-\rho(t')}\right)$$
$$= O\left(\log\frac{2\pi}{t}\right), \tag{61}$$

since $(1-\rho(t))^{-1} = O(t^{-2})$, $(1-\rho(t'))^{-1} = O(t'^{-2})$ and $t'^{-1} < t^{-1}$. Now (57) follows immediately from (59) and (61) and this completes the proof of Theorem 2(a).

6. Proof of theorem 3

6.1. *Proof of theorem* 3. Let ψ be odd and 2π-periodic function defined by

$$\psi(t) = \begin{cases} \dfrac{\pi}{2}\left(\log\log\dfrac{k}{t}\right)^{-1}, & 0 < t < \pi \\ 0, & t \in \{0, \pi\} \end{cases}$$

where $k > \pi e$. By Lemma 3

$$B_n(x) = \int_0^{\pi} \frac{\sin nt}{\log\log(k/t)}\mathrm{d}t = -\frac{(-1)^n}{n\log\log(k/\pi)} + 0_n,$$

where

$$\theta_n \sim \frac{1}{n \log\log n}.$$

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\log(n+1)} \in |E,q|$, if and only if

$$\sum_{n=1}^{\infty} |g_c(n, n^{-1}(\log(n+1))^{-1}, \pi)| = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} V(n,k) \frac{\cos k\pi}{k\log(k+1)} \right| < \infty \tag{62}$$

Using (21), we have $g_c(n, n^{-1}(\log(n+1))^{-1}, \pi) = O(n^{-3/2}(\log n)^{-1})$ and hence (62) follows at once.

Lastly $\sum \frac{\theta_n}{\log(n+1)}$ is not summable by any totally regular method as

$$\frac{\theta_n}{\log(n+1)} \sim \frac{1}{n \log n \log\log n}.$$

This completes the proof of Theorem 3.

7. Proof of theorems 4, 5 and 6

7.1 *Proof of theorem* 4. For $n \geqslant 1$, we write

$$\begin{aligned} \frac{\pi}{2} A_n(x) &= \int_0^{1/n} \phi(t) \cos nt \, dt + \int_{1/n}^{\pi} \phi(t) \cos nt \, dt \\ &= \alpha_n + \beta_n. \end{aligned}$$

$\sum \frac{\alpha_n}{n^{(1/2)+\varepsilon}} \in |E,q|$, if and only if

$$\sum_{n=1}^{\infty} \left| \int_0^{1/n} \phi(t) g_c(n, n^{-(1/2)-\varepsilon}, t) dt \right| < \infty.$$

Using (20), we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \left| \int_0^{1/n} \phi(t) g_c(n, n^{-(1/2)-\varepsilon}, t) dt \right| \\ &= O\left(\sum_{n=1}^{\infty} n^{-(1/2)-\varepsilon} \int_0^{1/n} |\phi(t)| dt \right) \\ &= O\left(\sum_{n=1}^{\infty} n^{-(3/2)-\varepsilon} \right) = O(1), \end{aligned}$$

which ensures that $\sum \alpha_n/(n^{(1/2)+\varepsilon}) \in |E,q|$.

Next $\sum \beta_n/(n^{(1/2)+\varepsilon}) \in |E,q|$, if and only if

$$\sum_{n=1}^{\infty} \left| \int_{1/n}^{\pi} \phi(t) g_c(n, n^{-(1/2)-\varepsilon}, t) dt \right| < \infty. \tag{63}$$

Using (21), we get

$$\sum_{n=1}^{\infty}\left|\int_{1/n}^{\pi}\Phi(t)g_c(n,n^{-(1/2)-\varepsilon},t)\mathrm{d}t\right| = O\left(\sum_{n=1}^{\infty} n^{-1-\varepsilon}\int_{1/n}^{\pi}\frac{|\phi(t)|}{t}\mathrm{d}t\right)$$

$$= O\left(\sum_{n=1}^{\infty} n^{-1-\varepsilon}\log n\right) = O(1).$$

This completes the proof of Theorem 4.

7.2. *Proof of theorem* 5. For $n \geqslant 1$, we have

$$\frac{\pi}{2}A_n(x) = \int_0^{\pi}\phi(t)\cos nt\,\mathrm{d}t.$$

The series $\sum \frac{A_n(x)}{n^{1/2}} \in |E,q|$, if and only if

$$\sum_{n=1}^{\infty}\left|\int_0^{\pi}\phi(t)g_c(n,n^{-1/2},t)\mathrm{d}t\right| < \infty. \tag{64}$$

Since $\phi(t)/t \in L(0,\pi)$, for the validity of (64) it suffices to show that in $0 < t < \pi$

$$\sum_{n=1}^{\infty}|g_c(n,n^{-1/2},t)| = O(t^{-1}). \tag{65}$$

Using (20) and (22), we get for $T = [t^{-2}]$

$$\sum_{n=1}^{\infty}|g_c(n,n^{-1/2},t)| = \sum_{n\leqslant T}|g_c(n,n^{-1/2},t) + \sum_{n>T}|g_c(n,n^{-1/2},t)|$$

$$= O\left(\sum_{n\leqslant T} n^{-1/2}\right) + O\left(t^{-1}\sum_{n>T} n^{-3/2}\right)$$

$$+ O\left(\sum_{n>T}\frac{\rho^n(t)}{\sqrt{n}}\right) + O\left(\sum_{n>T}\left(\frac{q}{q+1}\right)^n n^{-1/2}\right)$$

$$= O(t^{-1})$$

which ensure (65) and this completes the proof of the theorem.

7.3. *Proof of theorem* 6. The series

$$\sum \frac{A_n(x)}{n^{(1/2)-\varepsilon}} \in |E,q|$$

if and only if

$$\sum_{n=1}^{\infty}\left|\int_0^{\pi}\phi(t)\,g_c(n,n^{-1/2+\varepsilon},t)\mathrm{d}t\right| < \infty. \tag{66}$$

By an appeal to Lemma 6, we note that necessary and sufficient condition for (66) to hold whenever $O(t)/t \in L(0,\pi)$ is that

$$\operatorname{ess}\overline{bd}\sum_n t|g_c(n,n^{-(1/2)+\varepsilon},t)| \leqslant K \quad \text{for } 0 < t \leqslant \pi \tag{67}$$

where K is an absolute constant.

Thus for the proof of our theorem, it suffices to show that

$$t \sum_{n=1}^{\infty} |g_c(n, n^{-(1/2)+\varepsilon}, t)| \to +\infty \text{ at } t \to 0+ \tag{68}$$

From (26), we get

$$\begin{aligned} tg_c(n, n^{-(1/2)+\varepsilon}, t) &= \frac{tS_n}{(N+1)^{(1/2)-\varepsilon}} + O(n^{-(3/2)+\varepsilon}) \\ &= \frac{t}{(N+1)^{(1/2)-\varepsilon}}\left[\rho^n(t)\cos n\phi - \left(\frac{q}{q+1}\right)^n\right] + O(n^{-(3/2)+\varepsilon}) \\ &= \frac{t\rho^n(t)\cos n\phi}{(N+1)^{(1/2)-\varepsilon}} + O\left\{tn^{\varepsilon-1/2}\left(\frac{q}{q+1}\right)^n\right\} + O(n^{-(3/2)+\varepsilon}). \end{aligned}$$

As the series $\sum n^{\varepsilon-1/2}(q/q+1)^n$ and $\sum n^{-(3/2)+\varepsilon}$, $0<\varepsilon<1/2$ are both convergent, we need only show that

$$\sum{}^* = t \sum_{n=1}^{\infty} \frac{\rho^n(t)|\cos n\phi|}{(N+1)^{(1/2)-\varepsilon}} \to +\infty \text{ as } t \to 0+ \tag{69}$$

We have

$$\begin{aligned} \sum{}^* &\geqslant t \sum_{n=1}^{\infty} \frac{\rho^n(t)\cos^2 n\phi}{n^{(1/2)-\varepsilon}} \\ &= \frac{t}{2}\sum_{n=1}^{\infty} \frac{\rho^n(t)}{n^{(1/2)-\varepsilon}} + \frac{t}{2}\sum_{n=1}^{\infty} \frac{\rho^n(t)\cos 2n\phi}{n^{(1/2)-\varepsilon}} \\ &= \sum{}_1^* + \sum{}_2^*. \end{aligned} \tag{70}$$

As $\dfrac{\rho^n(t)}{n^{(1/2)-\varepsilon}}$ is a monotonic decreasing function of n for fixed t

$$\begin{aligned} \sum{}_2^* &\leqslant \frac{t\rho(t)}{2} \max_{1<M,M'<\infty} \left|\sum_{M}^{M'} \cos 2n\phi\right| \\ &\leqslant \frac{t\rho(t)}{2\sin\phi} = O(1) \end{aligned} \tag{71}$$

since $\sin t = (1+q)\rho(t)\sin\phi$.

Using (31), we obtain as $t \to 0+$

$$\sum{}_1^* \sim \frac{t}{2}\frac{\Gamma((1/2)+\varepsilon)}{(1-\rho(t))^{(1/2)+\varepsilon}} > Ct^{-2\varepsilon}$$

where C is some positive constant. Therefore

$$\sum{}_1^* \to +\infty \text{ as } t \to 0+. \tag{72}$$

Now (69) follows immediately from (70), (71) and (72) and this completes the proof of the theorem.

8. Proof of theorems 7 and 8

8.1. *Proof of theorem* 7. We assume that $g^*(t)\in L(0,\pi)$.

For $r\geqslant 1$, we have ([8], Theorem 1)

$$A_{n,r}(x)=\frac{2}{\pi}(-1)^r\int_0^\pi \frac{1}{2}\{f(x+t)+(-1)^r f(x-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r\cos nt\,\mathrm{d}t$$

$$=\frac{2}{\pi}(-1)^r\int_0^\pi \frac{1}{2}\{P(t)+(-1)^r P(-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r\cos nt\,\mathrm{d}t$$

$$+\frac{2}{\pi r!}(-1)^r\int_0^\pi g^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r\cos nt\,\mathrm{d}t$$

$$=\alpha_n+\beta_n. \tag{73}$$

We first proceed to show that $\sum\alpha_n\in|E,q|, q>0$.

Case I. Let r be odd, i.e., $r=2m+1$ $(m=0,1,2,\ldots)$, then

$$\alpha_n=-\frac{2}{\pi}\int_0^\pi \frac{1}{2}\{P(t)-P(-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m+1}\cos nt\,\mathrm{d}t$$

$$=\frac{2}{\pi}(-1)^m n^{2m+1}\sum_{\mu=1}^{m}\frac{\theta_{2\mu-1}}{(2\mu-1)!}\int_0^\pi t^{2\mu-1}\sin nt\,\mathrm{d}t$$

$$=\frac{2}{\pi}(-1)^m n^{2m+1}\sum_{\mu=1}^{m}\frac{\theta_{2\mu-1}}{(2\mu-1)!}(-1)^n\sum_{\rho=1}^{\mu}(-1)^\rho$$

$$\left(\frac{(2\mu-1)!}{(2\mu-\rho)!}\pi^{2\mu-2\rho+1}n^{-2\rho+1}\right)$$

$$=\frac{2}{\pi}(-1)^n\sum_{\rho=1}^{m}(-1)^{m+\rho}n^{2m-2\rho+2}\sum_{\mu=\rho}^{m}\frac{\theta_{2\mu-1}}{(2\mu-\rho)!}\pi^{2\mu-2\rho+1}. \tag{74}$$

Case II. Let r be even, i.e., $r=2m$ $(m=1,2,\ldots)$, then

$$\alpha_n=\frac{2}{\pi}\int_0^\pi \frac{1}{2}\left\{P(t)+P(-t)\right\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m}\cos nt\,\mathrm{d}t$$

$$=\frac{2}{\pi}(-1)^m n^{2m}\sum_{\mu=1}^{m-1}\frac{\theta_{2\mu}}{(2\mu)!}\int_0^\pi t^{2\mu}\cos nt\,\mathrm{d}t$$

$$=\frac{2}{\pi}(-1)^m n^{2m}\sum_{\mu=1}^{m-1}\frac{\theta_{2\mu}}{(2\mu)!}(-1)^n\sum_{\rho=1}^{\mu}(-1)^{\rho-1}\frac{(2\mu)!}{(2\mu-2\rho+1)!}\pi^{2\mu-2\rho+1}n^{-2\rho}$$

$$=\frac{2}{\pi}(-1)^n\sum_{\rho=1}^{m-1}(-1)^{m+\rho-1}n^{2m-2\rho}\sum_{\mu=\rho}^{m-1}\frac{\theta_{2\mu}}{(2\mu-2\rho+1)!}\pi^{2\mu-2\rho+1}. \tag{75}$$

Taking α_n as given in (74) and (75), we see that by Lemma 5 the series $\sum\alpha_n\in|E,q|, q>0$. Next, we write for any positive constant c, however small

$$\beta_n=\frac{2}{\pi r!}(-1)^r\left\{\int_0^c+\int_c^\pi\right\}g^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r\cos nt\,\mathrm{d}t$$

$$=\beta_{n,1}+\beta_{n,2}.$$

The absolute Euler summability of rth derived Fourier series becomes a local property of its generating function, if

$$g^*(t)\in L(0,\pi)\Rightarrow \sum_{n=1}^{\infty} \beta_{n,2}\in|E,q|,\quad q>0. \tag{76}$$

Now $\sum\beta_{n,2}\in|E,q|$, if and only if

$$\sum \equiv \sum_{n=1}^{\infty}\left|\sum_{k=1}^{n} V(n,k)\int_c^\pi g^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \cos kt\,\mathrm{d}t\right| < \infty.$$

We have

$$\begin{aligned}
\sum &= \sum_{n=1}^{\infty}\left|\int_c^\pi g^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r S_n(t)\mathrm{d}t\right|\\
&= O\left\{\int_c^\pi |g^*(t)|t^r\mathrm{d}t\sum_{n=1}^{\infty} n^r\rho^{(n-r)}(t)\right\},\text{ by Lemma 4}\\
&= O\left\{\int_c^\pi |g^*(t)|t^r\frac{1}{(1-\rho(t))^{r+1}}\mathrm{d}t\right\}\\
&= O\left\{\int_c^\pi \frac{|g^*(t)|}{t^{r+2}}\mathrm{d}t\right\} = O(1),
\end{aligned}$$

since $g^*\in L(0,\pi)$ and this completes the proof of the theorem.

8.2. *Proof of theorem* 8. We assume that $h^*(t)\in L(0,\pi)$.
For $r\geqslant 1$ ([8], Theorem 2)

$$\begin{aligned}
B_{n,r}(x) &= \frac{2}{\pi}\int_0^\pi (-1)^r\frac{1}{2}\{f(x+t)-(-1)^r f(x-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \sin nt\,\mathrm{d}t\\
&= \frac{2}{\pi}(-1)^r\int_0^\pi \frac{1}{2}\{P(t)-(-1)^r P(-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \sin nt\,\mathrm{d}t\\
&\quad + \frac{2}{\pi}(-1)^r\int_0^\pi h^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \sin nt\,\mathrm{d}t\\
&= \alpha_n+\beta_n.
\end{aligned}\tag{77}$$

We claim that $\sum\alpha_n\in|E,q|$, $q>0$.

Case I. Let $r=2m+1$ $(m=0,1,2,\ldots)$, then

$$\begin{aligned}
\alpha_n &= -\frac{2}{\pi}\int_0^\pi \frac{1}{2}\{P(t)+P(-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m+1} \sin nt\,\mathrm{d}t\\
&= \frac{2}{\pi}(-1)^{m+1}\sum_{\mu=1}^{m}\frac{\theta_{2\mu}}{(2\mu)!}\int_0^\pi t^{2\mu}n^{2m+1}\cos nt\,\mathrm{d}t\\
&= \frac{2}{\pi}(-1)^{m+1}\sum_{\mu=1}^{m}\frac{\theta_{2\mu}}{(2\mu)!}(-1)^n n^{2m+1}\sum_{\nu=1}^{\mu}\frac{(-1)^{\nu-1}(2\mu)!}{(2\mu-2\nu+1)!}\frac{\pi^{2\mu-2\nu+1}}{n^{2\nu}}\\
&= \frac{2}{\pi}(-1)^n\sum_{\nu=1}^{n}(-1)^{m+\nu}n^{2m-2\nu+1}\sum_{\mu=\nu}^{m}\frac{\theta_{2\mu}}{(2\mu-2\nu+1)!}\pi^{2\mu-2\nu+1}.
\end{aligned}\tag{78}$$

Case II. Let $r = 2m$ $(m = 1, 2, 3, \ldots)$, then

$$\begin{aligned}\alpha_n &= \frac{2}{\pi}\int_0^\pi \frac{1}{2}(P(t) - P(-t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m} \sin nt\,\mathrm{d}t \\ &= \frac{2}{\pi}(-1)^m n^{2m} \sum_{\mu=1}^{m} \frac{\theta_{2\mu-1}}{(2\mu-1)!}\int_0^\pi t^{2\mu-1} \sin nt\,\mathrm{d}t \\ &= \frac{2}{\pi}(-1)^m n^{2m} \sum_{\mu=1}^{m} \frac{\theta_{2\mu-1}}{(2\mu-1)!}(-1)^n \sum_{\nu=1}^{\mu} \frac{(2\mu-1)!}{(2\mu-2\nu+1)!}\frac{\pi^{2\mu-2\nu+1}}{n^{2\nu-1}} \\ &= \frac{2}{\pi}(-1)^n \sum_{\nu=1}^{m} (-1)^{m+\nu+1} n^{2m-2\nu+1} \sum_{\mu=1}^{m} \frac{\theta_{2\mu-1}}{(2\mu-2\nu+1)!}\pi^{2\mu-2\nu+1}. \qquad (79)\end{aligned}$$

We observe that for both odd and even n the series $\sum \alpha_n \in |E, q|$, $q > 0$ by Lemma 4.

Next, we write for $0 < c < \pi$

$$\begin{aligned}\beta_n &= \frac{2}{\pi r!}(-1)^r\left(\int_0^c + \int_c^\pi\right) h^*(t) t^r \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \sin nt\,\mathrm{d}t \\ &= \beta_{n.1} + \beta_{n.2}.\end{aligned}$$

Using the technique similar to those used in the proof of Theorem 7, we can show that

$$h^*(t) \in L(0, \pi) \Rightarrow \sum \beta_{n.2} \in |E, q|, \quad q > 0.$$

This shows that absolute Euler summability of rth derived conjugate series is a local property of its generating function.

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Asymptotic behaviour of certain zero-balanced hypergeometric series

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Abstract. In this paper an attempt has been made to give a very simple method of extending certain results of Ramanujan, Evans and Stanton on obtaining the asymptotic behaviour of a class of zero-balanced hypergeometric series. A more recent result of Saigo and Srivastava has also been used to obtain a Ramanujan type of result for a partial sum of a zero-balanced ${}_4F_3$ (1) and similar other partial series of higher order.

Keywords. Asymptotic behaviour; zero-balanced hypergeometric series; Kampé de Feriét function.

1. Introduction

A number of results exist in the literature which express the sum of the first n terms of an ordinary hypergeometric series, with unit argument, in terms of an infinite series of the type ${}_3F_2$. The earliest of such results are due to Hill and Whipple [6, 7, 8] but new interest was aroused in them by a theorem due to Ramanujan [2; p. 92 (10.4.1)]. Various generalizations of Ramanujan's result have been given from time to time by Whipple [2; p. 94 (10.4.4)], Hodgkinson [2, p. 94 (10.4.5)], Bailey [2, p. 93 (10.4.3)], Agarwal [1, p. 442 (2.5)] and others.

Ramanujan [9 Ch. XI Theorem 1] stated an asymptotic formula as $m \to \infty$ for the mth partial sums of a zero-balanced (i.e. when $d + e = a + b + c$) hypergeometric series ${}_3F_2\left[\begin{matrix} a, b, c; \\ d, e \end{matrix} \; 1\right]$. Evans and Stanton [5, p. 1017 (4)] have given a very elegant proof of this asymptotic formula and extended his method to obtain a similar result for a zero-balanced basic hypergeometric series ${}_3\Phi_2$. Evans [4, p. 556 (Theorem 22)] has also given a proof for the asymptotic formula for a zero-balanced ${}_5F_4$ series stated by Ramanujan [9; Entry 6] alongwith several other particular results. It would be interesting if one could extend Ramanujan's results to series of higher order series also.

More recently Saigo and Srivastava [10] have studied the behaviour of a zero-balanced hypergeometric series ${}_pF_{p-1}$ near the boundary point $z = 1$ of its region of convergence. They have investigated the behaviour by considering a transformation connecting a ${}_pF_{p-1}(x)$ with another ${}_{p-1}F_{p-2}(x)$ and a Kampê de Feriêt type of series $F^{0:p-1;3}_{1:p-3;1}(x, 1)$. They have shown that special cases of their result yield the behaviour of zero-balanced ${}_3F_2(1)$ and ${}_4F_3(1)$ series.

Our approach in the present paper is markedly different. Our objective is to obtain a Ramanujan type of result [9; Chapter XI, Theorem I and Chapter X, Entry 6] for a partial sum of a zero-balanced ${}_4F_3$ (1) and similar other partial series of higher order.

In our analysis, incidently we get a direct generalization of the results of Ramanujan, Evans and Stanton [5, Theorem 3]. The present method has the advantage of giving on explicit expression for the order terms also [see (4.2), (4.3), (7.1)] which are not available as such in the results of Saigo and Srivastava.

By repeated iteration of our result one can lead to the behaviour of a zero-balanced ${}_{3r+3}F_{3r+2}$ (1) and ${}_{3r+5}F_{3r+4}$ (1) series.

In § 7 of paper the partial-sum method is applied to a transformation between two ${}_pF_{p-1}$'s and a Kampê de Feriêt function given by Saigo and Srivastava [10] to derive the behaviour of a zero-balanced ${}_pF_{p-1}$ (1).

2. Definitions and Notations

Let $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$, $n>0$ $(\alpha)_0 = 1$.

An ordinary hypergeometric series is then defined as

$$
{}_{p+1}F_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p+1};\\ \rho_1,\rho_2,\ldots,\rho_p\end{matrix}\; Z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_{p+1})_n Z^n}{(\rho_1)_n(\rho_2)_n\cdots(\rho_p)_n n!}. \tag{2.1}
$$

The series converges when $|Z|<1$ and also when $Z=1$, provided that $\mathrm{Re}[\Sigma(\rho_p)-\Sigma(\alpha_{p+1})]>0$. The series (2.1) shall be called a zero-balanced series if $\Sigma(\rho_p)=\Sigma(\alpha_{p+1})$.

By

$$
{}_{p+1}F_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p+1};\\ \rho_1,\rho_2,\ldots,\rho_p\end{matrix}\; Z\right]_m
$$

we shall denote the partial sum

$$
\sum_{n=0}^{m} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_{p+1})_n Z^n}{(\rho_1)_n(\rho_2)_n\cdots(\rho_p)_n n!}.
$$

A Kampê de Feriêt series is defined as

$$
F^{0:p-1;2}_{1:p-3;0}\left[\begin{matrix}\text{———}:(\alpha_{p-1});\beta_{p-2}-\alpha_p,\beta_{p-1}-\alpha_p;\\ \beta_{p-2}+\beta_{p-1}-\alpha_p:(\beta_{p-3});\text{———};\end{matrix}\; x,y\right]
$$

$$
= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{[(\alpha_{p-1})]_n(\beta_{p-2}-\alpha_p)_m(\beta_{p-1}-\alpha_p)_m x^n y^m}{(\beta_{p-2}+\beta_{p-1}-\alpha_p)_{n+m}[(\beta_{p-3})]_n n!\,m!} (p \geqq 3) \tag{2.2}
$$

where, for convergence, $\max\{|x|,|y|\}<1$.

We shall use the following two results of Ramanujan [9; Chapter XI Theorem I and Chapter X Entry 6; See also [3] Theorem I, p. 70 and Entry 6, p. 12] stated without proof. The asymptotic formula for a zero-balanced hypergeometric series ${}_3F_2$ as:

If $a+b+c=d+e$ and $\mathrm{Ri}(c)>0$, then for

$$
L = \sum_{k=0}^{\infty}\left\{\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c+k)}{\Gamma(d+k)\Gamma(e+k)\Gamma(l+k)} - \frac{1}{k+1}\right\} \tag{2.3}
$$

$$
= -2\gamma - \Psi(a) - \Psi(b) + \sum_{k=1}^{\infty}\frac{(d-c)_k(e-c)_k}{(a)_k(b)_k k}, \tag{2.4}
$$

with γ, the Euler's constant and $\Psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ we have, as $m\to\infty$

$$
\sum_{k=0}^{m-1}\frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k k!} \sim \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)}\{\log m + L + \gamma\} + O\left(\frac{1}{m}\right), \tag{2.5}
$$

where the implied constant depends on a, b, c, d, e but not on m.

The asymptotic formula for a zero-balanced hypergeometric series ${}_5F_4$ as:
If $a+b+c\notin\{0,-1,-2,\ldots\}$, then as the integer m tends to ∞,

$$\frac{\Gamma(a+b+c)\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(b+c)\Gamma(a+c)\Gamma(a+b)}\,{}_5F_4\left[\begin{matrix}\frac{a+b+c+1}{2}, & a+b+c-1, & a,b,c;\\ \frac{a+b+c-1}{2}, & b+c, & a+c, & a+b\end{matrix}\right]_m$$

$$\sim 2\log m-\gamma-\Psi(a)-\Psi(b)-\Psi(c)+O\left(\frac{\log m}{m}\right). \tag{2.6}$$

3. A general transformation theorem for partial hypergeometric series

We begin this section by proving the following transformation Theorem:
We have for arbitrary complex numbers $a, b, c, (a_s)$ and (p_s), that

$${}_{s+3}F_{s+2}\left[\begin{matrix}a,b,c, & a_1,a_2,\ldots,a_s; & \\ 1+a-b, & 1+a-c, & p_1,p_2,\ldots,p_s\end{matrix}\; z\right]_m$$

$$=\sum_{r=0}^{m}\frac{(-4)^r(1+a+b-c)_r\left(\frac{a}{2}\right)_r\left(\frac{a+1}{2}\right)_r(a_1)_r(a_2)_r\cdots(a_s)_r z^r}{r!(1+a-b)_r(1+a-c)_r(p_1)_r(p_2)_r\cdots(p_s)_r}$$

$$\times\,{}_{s+1}F_s\left[\begin{matrix}a+2r, & a_1+r, & a_2+r,\ldots,a_s+r;\\ p_1+r, & p_2+r,\ldots,p_s+r\end{matrix}\; z\right]_{(m-r)}. \tag{3.1}$$

Proof of the Theorem. Saalschütz's theorem is given by

$${}_3F_2\left[\begin{matrix}1+a-b-c, & a+n, & -n;\\ 1+a-b, & 1+a-c\end{matrix}\; 1\right]=\frac{(b)_n(c)_n}{(1+a-b)_n(1+a-c)_n}. \tag{3.2}$$

Now

$${}_{s+3}F_{s+2}\left[\begin{matrix}a,b,c, & a_1,a_2,\ldots,a_s; & \\ 1+a-b, & 1+a-c, & p_1,p_2,\ldots,p_s\end{matrix}\; z\right]_m$$

$$=\sum_{n=0}^{m}\frac{(a)_n(a_1)_n(a_2)_n\cdots(a_s)_n z^n(b)_n(c)_n}{(p_1)_n(p_2)_n\cdots(p_s)_n n!(1+a-b)_n(1+a-c)_n}$$

$$=\sum_{n=0}^{m}\frac{(a)_n(a_1)_n(a_2)_n\cdots(a_s)_n z^n}{(p_1)_n(p_2)_n\cdots(p_s)_n n!}\sum_{r=0}^{n}\frac{(1+a-b-c)_r(a+n)_r(-n)_r}{(1+a-b)_r(1+a-c)_r r!}$$

by (3.2).

Changing the order of summation, we get

$$=\sum_{r=0}^{m}\sum_{n=r}^{m}\frac{(a)_{n+r}(a_1)_n(a_2)_n\cdots(a_s)_n(1+a-b-c)_r(-n)_r z^n}{(p_1)_n(p_2)_n\cdots(p_s)_n(1+a-b)_r(1+a-c)_r n!r!}.$$

Putting $n=r+t$, we get

$$=\sum_{r=0}^{m}\sum_{t=0}^{m-r}\frac{(a)_{2r+t}(a_1)_{r+t}(a_2)_{r+t}\cdots(a_s)_{r+t}(1+a-b-c)_r(-r-t)_r z^{r+t}}{(p_1)_{r+t}(p_2)_{r+t}\cdots(p_s)_{r+t}(1+a-b)_r(1+a-c)_r r+t!\,r!}. \tag{3.3}$$

Writing the t-series as a F-series on the r.h.s., we get (3.1), after some simplification.

It may be remarked that we could obtain (3.1) directly [for $z=1$] from Bailey's known transformation Theorem [2, p. 24 (4.3.1)].

4. The asymptotic behaviour of certain zero-balanced partial hypergeometric series

We begin this section by proving the following two asymptotic formulae for certain zero-balanced hypergeometric series:

(A) For k, a fixed positive integer $<m$ and if $a_1+a_2+a_3+k=p_2+p_3$, $Rl(a_3+k)>0$, then as $m\to\infty$

$$
{}_6F_5\left[\begin{matrix} b+c-k-1, & b, & c, & a_1, & a_2, & a_3; \\ b-k, & c-k, & b+c-1, & p_2, & p_3 \end{matrix}\right]_m
$$
$$
\sim\frac{(-1)^k(2-b-c)_k\Gamma(p_2)\Gamma(p_3)}{(1-b)_k(1-c)_k\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\{\log(m-k+1)+L'+\gamma\}+O\left(\frac{1}{m-k+1}\right), \tag{4.1}
$$

where γ is Euler's constant and

$$
L'=-2\gamma-\Psi(a_1+k)-\Psi(a_2+k)+\sum_{s=1}^{\infty}\frac{(p_2-a_3)_s(p_3-a_3)_s}{(a_1+k)_s(a_2+k)_s s}
$$
$$
+\frac{(-1)^k(1-b)_k(1-c)_k\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{(2-b-c)_k\Gamma(p_2)\Gamma(p_3)}
$$
$$
\times\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\frac{b+c-k-1}{2}\right)_r\left(\frac{b+c-k}{2}\right)_r(a_1)_r(a_2)_r(a_3)_r}{r!(b-k)_r(c-k)_r(b+c-1)_r(p_2)_r(p_3)_r}
$$
$$
\times{}_4F_3\left[\begin{matrix} b+c-k-1+2r, & a_1+r, & a_2+r, & a_3+r; \\ b+c-1+r, & p_2+r, & p_3+r \end{matrix}\right]. \tag{4.2}
$$

(B) If $a+b+c+3k\notin\{0,-1,-2,\ldots\}$, then as the integer m tends to ∞, we have

$$
{}_8F_7\left[\begin{matrix} b_1+c_1-k-1, \; b_1, \; c_1, \; \frac{a+b+c+k+1}{2}, \; a+b+c+2k-1, \; a,b,c; \\ b_1-k, \; c_1-k, \; b_1+c_1-1, \; \frac{a+b+c+k-1}{2}, \; b+c+k, \; a+c+k, \; a+b+k \end{matrix}\right]_m
$$
$$
\sim\frac{(-1)^{k+1}(a+b+c-1)_{3k+1}(2-b_1-c_1)_k\Gamma(b+c+k)\Gamma(a+c+k)\Gamma(a+b+k)}{(1-a-b-c-k)\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+b+c+3k)(1-b_1)_k(1-c_1)_k(a+b+c-1)_{2k}}
$$
$$
\times\{2\log(m-k)-\gamma-\Psi(a+k)-\Psi(b+k)-\Psi(c+k)\}
$$
$$
+\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\frac{b_1+c_1-k-1}{2}\right)_r\left(\frac{b_1+c_1-k}{2}\right)_r\left(\frac{a+b+c+k+1}{2}\right)_r(a+b+c+2k-1)_r}{r!(b_1-k)_r(c_1-k)_r(b_1+c_1-1)_r\left(\frac{a+b+c+k-1}{2}\right)_r(b+c+k)_r(a+c+k)_r}
$$
$$
\times\frac{(a)_r(b)_r(c)_r}{(a+b+k)_r}
$$

$$
{}_6F_5\left[\begin{matrix} b_1+c_1-k-1+2r, \dfrac{a+b+c+k+1}{2}+r, a+b+c-1+2k+r, a+r, b+r, c+r; \\ b_1+c_1-1+r, \dfrac{a+b+c+k-1}{2}+r, b+c+k+r, a+c+k+r, a+b+k+r \end{matrix}\right] + O\left(\frac{\log(m-k)}{(m-k)}\right), \tag{4.3}
$$

where γ is Euler's constant, k is a fixed positive integer $< m$.

Proof of (4.1). Taking $s = 3$ in (3.1) and putting $a = b + c - k - 1$, $z = 1$, $p_1 = b + c - 1$, where k is a fixed positive integer $< m$, we get

$$
{}_6F_5\left[\begin{matrix} b+c-k-1, & b, & c, & a_1, a_2, a_3; \\ b-k, & c-k, & b+c-1, & p_2, & p_3 \end{matrix}\right]_m
$$

$$
= \sum_{r=0}^{k-1} \frac{(-4)^r(-k)_r \left(\dfrac{b+c-k-1}{2}\right)_r \left(\dfrac{b+c-k}{2}\right)_r (a_1)_r(a_2)_r(a_3)_r}{r!\,(b-k)_r(c-k)_r(b+c-1)_r(p_2)_r(p_3)_r}
$$

$$
\times {}_4F_3\left[\begin{matrix} b+c-k-1+2r, & a_1+r, & a_2+r, & a_3+r; \\ b+c-1+r, & p_2+r, & p_3+r \end{matrix}\right]_{(m-r)}
$$

$$
+ \frac{(-4)^k(-k)_k \left(\dfrac{b+c-k-1}{2}\right)_k \left(\dfrac{b+c-k}{2}\right)_k (a_1)_k(a_2)_k(a_3)_k}{k!\,(b-k)_k(c-k)_k(b+c-1)_k(p_2)_k(p_3)_k}
$$

$$
\times {}_3F_2\left[\begin{matrix} a_1+k, & a_2+k, & a_3+k; \\ p_2+k, & p_3+k \end{matrix}\right]_{(m-k)}.
$$

Let us now suppose that $a_1 + a_2 + a_3 + k = p_2 + p_3$, with $\mathrm{Rl}(a_3 + k) > 0$. Now apply Ramanujan's asymptotic formula (2.5) on r.h.s. of the above equation, to get for $m \to \infty$,

$$
{}_6F_5\left[\begin{matrix} b+c-k-1, & b, & c, & a_1, & a_2, & a_3; \\ b-k, & c-k, & b+c-1, & p_2, & p_3 \end{matrix}\right]_m
$$

$$
\sim \sum_{r=0}^{k-1} \frac{(-4)^r(-k)_r \left(\dfrac{b+c-k-1}{2}\right)_r \left(\dfrac{b+c-k}{2}\right)_r (a_1)_r(a_2)_r(a_3)_r}{r!\,(b-k)_r(c-k)_r(b+c-1)_r(p_2)_r(p_3)_r}
$$

$$
\times {}_4F_3\left[\begin{matrix} b+c-k-1+2r, & a_1+r, & a_2+r, & a_3+r; \\ b+c-1+r, & p_2+r, & p_3+r \end{matrix}\right]_{(m-r)}
$$

$$
+ \frac{(+4)^k \left(\dfrac{b+c-k-1}{2}\right)_k \left(\dfrac{b+c-k}{2}\right)_k (a_1)_k(a_2)_k(a_3)_k}{(b-k)_k(c-k)_k(b+c-1)_k(p_2)_k(p_3)_k}
$$

$$
\times \left[\frac{\Gamma(p_2+k)\Gamma(p_3+k)}{\Gamma(a_1+k)\Gamma(a_2+k)\Gamma(a_3+k)}\{\log(m-k+1) + L' + \gamma\} + O\left(\frac{1}{m-k+1}\right)\right].
$$

Simplifying we get (4.1).

Again applying the asymptotic formula (4.1) on r.h.s. in (3.1), for $s = 5$, $z = 1$ we get the asymptotic behaviour of a zero-balanced ${}_9F_8$ series. Similarly, by repeating this process we can get asymptotic behaviour of a zero-balanced ${}_{3r+3}F_{3r+2}$ series.

Proof of (4.3). Proceeding exactly as in the case of (4.1), we get (4.3).

Again applying the asymptotic formula (4.3) on r.h.s. in (3.1), we can get the asymptotic behaviour of a zero-balanced ${}_{11}F_{10}$ series. Similarly, by repeating this process, we can get the asymptotic behaviour of a zero-balanced ${}_{3r+5}F_{3r+4}$ series.

5. Special cases

(i) Putting $k = 1, a_2 = b - 1$ and $c \to \infty$ in (4.1), we get back Ramanujan's result (2.5) for a zero-balanced ${}_3F_2$.

(ii) Putting $a_2 = b - k$ in (4.1), we get

For k a fixed positive integer $< m$ and if $a_1 + b + a_3 = p_2 + p_3$, $R1(a_3 + k) > 0$, then as $m \to \infty$

$$
{}_5F_4\left[\begin{matrix} b+c-k-1, & b, & c, & a_1, & a_3; \\ c-k, & b+c-1, & p_2, & p_3 & \end{matrix}\right]_m
$$
$$
\sim \frac{(-1)^k(2-b-c)_k\Gamma(p_2)\Gamma(p_3)}{(1-b)_k(1-c)_k\Gamma(a_1)\Gamma(b-k)\Gamma(a_3)}\{\log(m-k+1)+L''+\gamma\}
$$
$$
+O\left(\frac{1}{m-k+1}\right), \tag{5.1}
$$

where γ is Euler's constant and

$$
L'' = -2\gamma - \Psi(a_1+k) - \Psi(b) + \sum_{s=1}^{\infty}\frac{(p_2-a_3)_s(p_3-a_3)_s}{(a_1+k)_s(b)_s\, s}
$$
$$
+\frac{(-1)^k(1-b)_k(1-c)_k\Gamma(a_1)\Gamma(b-k)\Gamma(a_3)}{(2-b-c)_k\Gamma(p_2)\Gamma(p_3)}
$$
$$
\times\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\frac{b+c-k-1}{2}\right)_r\left(\frac{b+c-k}{2}\right)_r(a_1)_r(a_3)_r}{r!(c-k)_r(b+c-1)_r(p_2)_r(p_3)_r}
$$
$$
\times{}_4F_3\left[\begin{matrix} b+c-k-1+2r, & a_1+r, & b-k+r, & a_3+r; \\ b+c-1+r, & p_2+r, & p_3+r & \end{matrix}\right]. \tag{5.2}
$$

(iii) Putting $a_1 = c - k$ in (5.1), we get the asymptotic behaviour of a zero-balanced series ${}_4F_3$.

(iv) Putting $c = b_1 - k$ in (4.3), we get

If $a + b + b_1 + 2k \notin \{0, -1, -2, \ldots\}$, then for the integer m tending to ∞

$$
{}_7F_6\left[\begin{matrix} b_1+c_1-k-1, & b_1, & c_1, & \frac{1}{2}(a+b+b_1+1), & a+b+b_1+k-1, & a, & b; \\ c_1-k, & b_1+c_1-1, & \frac{1}{2}(a+b+b_1-1), & b+b_1, & a+b_1, & a+b+k & \end{matrix}\right]_m
$$

$$\sim \frac{(-1)^k(2-b_1-c_1)_k\Gamma(b+b_1)\Gamma(a+b_1)\Gamma(a+b+k)}{\Gamma(a)\Gamma(b)\Gamma(b_1-k)\Gamma(a+b+b_1)(1-b_1)_k(1-c_1)_k(a+b+b_1-1)_k}$$

$$\times \{2\log(m-k)-\gamma-\Psi(a+k)-\Psi(b+k)-\Psi(b_1)\}$$

$$+\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\dfrac{b_1+c_1-k-1}{2}\right)_r\left(\dfrac{b_1+c_1-k}{2}\right)_r\left(\dfrac{a+b+b_1+1}{2}\right)_r}{r!(c_1-k)_r(b_1+c_1-1)_r\left(\dfrac{a+b+b_1-1}{2}\right)_r}$$

$$\times\frac{(a+b+b_1+k-1)_r(a)_r(b)_r}{(b+b_1)_r(a+b_1)_r(a+b+k)_r}$$

$${}_6F_5\left[\begin{matrix} b_1+c_1-k-1+2r, & \dfrac{a+b+b_1+1}{2}+r, & a+b+b_1-1+k+r, & a+r, & b+r, & b_1-k+r; \\ b_1+c_1-1+r, & \dfrac{a+b+b_1-1}{2}+r, & b+b_1+r, & a+b_1+r, & a+b+k+r & \end{matrix}\right]$$

$$+O\left(\frac{\log(m-k)}{(m-k)}\right), \tag{5.3}$$

where γ is Euler's constant, k is fixed positive integer $<m$.

(v) Putting $b=c_1-k$ in (5.3), we get the asymptotic behaviour of a zero-balanced series ${}_6F_5$.

(vi) Putting $b=c_1-k$, $a=b_1+c_1-1$ in (5.3), we get asymptotic behaviour of a zero-balanced series ${}_5F_4$ different from Evan's ${}_5F_4$ function.

6. Certain special cases of (3.1)

In this section we shall discuss certain particular cases of (3.1) to obtain transformations for special types of partial sums of hypergeometric series.

(i) Putting $c=-m$ in Vandermonde's theorem [11, p. 28 (1.7.7)], we get

$${}_1F_0\left[\begin{matrix} a; \\ - \end{matrix}\right]_m = \frac{(1+a)_m}{m!}. \tag{6.1}$$

Taking $s=0$, $z=1$ in (3.1) and then summing r.h.s. of (3.1) by (6.1), we get

$${}_3F_2\left[\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c & \end{matrix}\right]_m = \frac{(1+a)_m}{m!}$$

$${}_4F_3\left[\begin{matrix} 1+a-b-c, & \dfrac{a}{2}, & 1+a+m, & -m; \\ 1+a-b, & 1+a-c, & 1+\dfrac{a}{2} & \end{matrix}\right]_m \tag{6.2}$$

Again applying (6.2) on r.h.s. of (3.1), we get the following transformation for a new partial sum of a well-poised ${}_5F_4$ series:

$${}_5F_4\left[\begin{matrix} a, & b, & c, & a_1, & a_2; \\ 1+a-b, & 1+a-c, & 1+a-a_1, & 1+a-a_2 & \end{matrix}\right]_m$$

$$=\frac{(1+a)_m}{m!}\sum_{r=0}^{m}\frac{(1+a-b-c)_r(a/2)_r(a_1)_r(a_2)_r(1+a+m)_r(-m)_r}{r!(1+a-b)_r(1+a-c)_r(1+a/2)_r(1+a-a_1)_r(1+a-a_2)_r}$$

$$\times\, {}_4F_3\left[\begin{matrix}1+a-a_1-a_2, & a/2+r, & 1+a+m+r, & -m+r;\\ 1+a-a_1+r, & 1+a-a_2+r, & 1+a/2+r & \end{matrix}\right]_{(m-r)}. \tag{6.3}$$

Similarly, by repeating this process we can get transformations for partial sums of series of higher orders.

(ii) Putting $1+a-c=-m$ in Whipple's transformation between a terminating well-poised ${}_7F_6$ and a terminating Saalschützian ${}_4F_3$ [see 2(4.3.4)], we get

$${}_5F_4\left[\begin{matrix}a, & 1+\frac{a}{2}, & b, & d, & e;\\ \frac{a}{2}, & 1+a-b, & 1+a-d, & 1+a-e & \end{matrix}\right]_m$$

$$=\frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m}\;{}_3F_2\left[\begin{matrix}-b-m, & d, e;\\ 1+a-b, & d+e-a-m\end{matrix}\right]_m. \tag{6.4}$$

Taking $s=4$ and putting $a_1=1+(a/2)$, $a_2=d$, $a_3=e$, $a_4=f$, $z=1$, $p_1=a/2$, $p_2=1+a-d$, $p_3=1+a-e$, $p_4=1+a-f$ in (3.1) and then transforming r.h.s. of (3.1) by (6.4), we get

$${}_7F_6\left[\begin{matrix}a, & 1+\frac{a}{2}, & b, & c, & d, & e, & f;\\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f & \end{matrix}\right]_m$$

$$=\frac{(1+a)_m(1+a-e-f)_m}{(1+a-e)_m(1+a-f)_m}$$

$$\times\sum_{r=0}^{m}\frac{(1+a-b-c)_r(d)_r(e)_r(f)_r(1+a+m)_r}{r!(1+a-b)_r(1+a-c)_r(1+a-d)_r(e+f-a-m)_r}$$

$$\times\, {}_3F_2\left[\begin{matrix}-d-m, & e+r, & f+r;\\ 1+a-d+r, & e+f-a-m+r & \end{matrix}\right]_{(m-r)}. \tag{6.5}$$

Applying (6.5), again on r.h.s. in (3.1), we get the following transformation for partial sum of a ${}_9F_8$ series

$${}_9F_8\left[\begin{matrix}a, & 1+\frac{a}{2}, & b, & c, & a_2, & a_3, & a_4, & a_5, & a_6;\\ 1+a-b, & 1+a-c, & \frac{a}{2}, & 1+a-a_2, & 1+a-a_3, & 1+a-a_4, & 1+a-a_5, & 1+a-a_6 & \end{matrix}\right]_m$$

$$=\frac{(1+a)_m(1+a-a_5-a_6)_m}{(1+a-a_5)_m(1+a-a_6)_m}$$

$$\times \sum_{r=0}^{m} \frac{(1+a-b-c)_r (a_2)_r (a_3)_r (a_4)_r (a_5)_r (a_6)_r}{r!(1+a-b)_r (1+a-c)_r (1+a-a_2)_r (1+a-a_3)_r (1+a-a_4)_r} \times \frac{(1+a+m)_r}{(a_5+a_6-a-m)_r}$$

$$\times \sum_{k=0}^{m-r} \frac{(1+a-a_2-a_3)_k (a_4+r)_k (a_5+r)_k (a_6+r)_k (1+a+m+r)_k}{k!(1+a-a_2+r)_k (1+a-a_3+r)_k (1+a-a_4+r)_k (a_5+a_6-a-m+r)_k}$$

$$\times {}_3F_2\left[\begin{matrix} -a_4-m, & a_5+r+k, & a_6+r+k; \\ 1+a-a_4+r+k, & a_5+a_6-a-m+r+k \end{matrix}\right]_{(m-r-k)}. \tag{6.6}$$

Similarly, by repeating this process we get transformations for partial sums of series of higher orders.

(iii) Putting $1+a-c=-m$, in Whipple's transformation [2; (4.5.1)] we get a transformation for the partial sum of a nearly-poised, ${}_3F_2$ in terms of a partial Saalschützian ${}_4F_3$, namely

$${}_3F_2\left[\begin{matrix} a, & b, & 1+a+m; \\ 1+a-b, & w & \end{matrix}\right]_m$$

$$= \frac{(w-a)_m}{(w)_m} \, {}_4F_3\left[\begin{matrix} 1+a-w, & \frac{a}{2}, & \frac{1+a}{2}, & -b-m; \\ 1+a-b, & \frac{1+a-w-m}{2}, & 1+\frac{a-w-m}{2} & \end{matrix}\right]_m. \tag{6.7}$$

Taking $s=2$ and putting $a_1=d, a_2=1+a+m, p_1=1+a-d, p_2=w, z=1$, in (3.1) and then transforming the r.h.s. of (3.1) by (6.7), we get a transformation for a partial nearly-poised ${}_5F_4$ series

$${}_5F_4\left[\begin{matrix} a, b, c, d, & 1+a+m; & & \\ 1+a-b, & 1+a-c, & 1+a-d, & w \end{matrix}\right]_m = \frac{(w-a)_m}{(w_m)}$$

$$\times \sum_{r=0}^{m} \frac{(1+a-b-c)_r \left(\frac{a}{2}\right)_r \left(\frac{a+1}{2}\right)_r (d)_r (1+a+m)_r (1+a-w)_r}{r!(1+a-b)_r (1+a-c)_r (1+a-d)_r \left(\frac{1+a-w-m}{2}\right)_r \left(\frac{2+a-w-m}{2}\right)_r}$$

$$\times {}_4F_3\left[\begin{matrix} 1+a-w+r, & \frac{a}{2}+r, & \frac{1+a}{2}+r, & -d-m; \\ 1+a-d+r, & \frac{1+a-w-m}{2}+r, & 1+\frac{a-w-m}{2}+r & \end{matrix}\right]_{(m-r)}. \tag{6.8}$$

Applying (6.8) again on r.h.s. in (3.1) we get the following transformation for the partial sum of ${}_7F_6$ series

$${}_7F_6\left[\begin{matrix} a, b, c, & a_1, a_2, a_3, & 1+a+m; & & \\ 1+a-b, & 1+a-c, & 1+a-a_1, & 1+a-a_2, & 1+a-a_3, p_4 \end{matrix}\right]_m = \frac{(p_4-a)_m}{(p_4)_m}$$

$$\times \sum_{r=0}^{m} \frac{4^r (1+a-b-c)_r \left(\frac{a}{2}\right)_r \left(\frac{a+1}{2}\right)_r (a_1)_r (a_2)_r (a_3)_r (1+a+m)_r (1-p_4+a)_r}{r!(1+a-b)_r (1+a-c)_r (1+a-a_1)_r (1+a-a_2)_r (1+a-a_3)_r (1-p_4+a-m)_{2r}}$$

$$\times \sum_{k=0}^{m-r} \frac{(1+a-a_1-a_2)_k\left(\frac{a}{2}+r\right)_k\left(\frac{a+2r+1}{2}\right)_k(a_3+r)_k(1+a+m+r)_k(1+a+r-p_4)_k}{r!(1+a-a_1+r)_k(1+a-a_2+r)_k(1+a-a_3+r)_k}$$

$$\times\left(\frac{1+a-p_4-m+2r}{2}\right)_k\left(\frac{2+a-p_4-m+2r}{2}\right)_k$$

$${}_4F_3\begin{bmatrix} 1+a-p_4+r+k, & \frac{a}{2}+r+k, & \frac{1+a}{2}+r+k, & -a_3-m; \\ 1+a-a_3+r+k, & \frac{1+a-p_4-m+2r}{2}+k, & 1+\frac{a-p_4-m+2r}{2}+k & \end{bmatrix}_{(m-r-k)}. \quad (6.9)$$

Similarly by repeating this process we get transformations for partial sums of series of higher orders.

(iv) Putting $1+a-c=-m$ in a transformation due to Bailey giving a nearly-poised ${}_5F_4$ in terms of Saalschützian ${}_5F_4$ [2, (4.5.2)], we get

$${}_4F_3\begin{bmatrix} a, & 1+\frac{a}{2}, & b, & 1+a+m; \\ \frac{a}{2}, & 1+a-b, & w & \end{bmatrix}_m = \frac{(w-a-1-m)(w-a)_{m-1}}{(w)_m}$$

$$\times\,{}_4F_3\begin{bmatrix} 1+\frac{a}{2}, & \frac{1}{2}+\frac{a}{2}, & -b-m, & 1+a-w; \\ \frac{3+a-w-m}{2}, & 1+\frac{a-w-m}{2}, & 1+a-b & \end{bmatrix}_m. \quad (6.10)$$

Taking $s=3$ and putting $a_1=1+(a/2)$, $a_2=d$, $a_3=1+a+m$, $z=1$, $p_1=(a/2)$, $p_2=1+a-d$, $p_3=w$ in (3.1) and then transforming r.h.s. of (3.1) by (6.10), we get

$${}_6F_5\begin{bmatrix} a, & 1+\frac{a}{2}, & b, & c, & d, & 1+a+m; \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & w & \end{bmatrix}_m$$

$$=\frac{(w-a-1-m)(w-a)_{m-1}}{(w)_m}$$

$$\sum_{r=0}^{m} \frac{(1+a-b-c)_r\left(\frac{a+1}{2}\right)_r\left(1+\frac{a}{2}\right)_r(d)_r(1+a+m)_r(1+a-w)_r}{r!(1+a-b)_r(1+a-c)_r(1+a-d)_r\left(\frac{2+a-w-m}{2}\right)_r\left(\frac{3+a-w-m}{2}\right)_r}$$

$$\times\,{}_4F_3\begin{bmatrix} 1+\frac{a}{2}+r, & \frac{1}{2}+\frac{a}{2}+r, & -d-m, & 1+a-w+r; \\ \frac{3+a-w-m}{2}+r, & 1+\frac{a-w-m}{2}+r, & 1+a-d+r & \end{bmatrix}_{(m-r)}. \quad (6.11)$$

Applying (6.11) again on r.h.s. in (3.1), we get the following transformation for

a partial sum of ${}_8F_7$ series

$$
{}_8F_7\left[\begin{matrix} a,\ b,\ c,\ 1+\frac{a}{2},\ a_2,\ a_3,\ a_4,\ 1+a+m; \\ 1+a-b,\ 1+a-c,\ \frac{a}{2},\ 1+a-a_2,\ 1+a-a_3,\ 1+a-a_4,\ p_5 \end{matrix}\right]_m
$$

$$
=\frac{(p_5-a-m-1)(p_5-a)_{m-1}}{(p_5)_m}
$$

$$
\times\sum_{r=0}^{m}\frac{4^r(1+a-b-c)_r\left(\frac{a+1}{2}\right)_r\left(1+\frac{a}{2}\right)_r(a_2)_r(a_3)_r(a_4)_r(1+a+m)_r(1-p_5+a)_r}{r!(1+a-b)_r(1+a-c)_r(1+a-a_2)_r(1+a-a_3)_r(1+a-a_4)_r(2+a-p_5-m)_{2r}}
$$

$$
\times\sum_{k=0}^{m-r}\frac{(1+a-a_2-a_3)_r\left(\frac{1+a+2r}{2}\right)_k\left(1+\frac{a+2r}{2}\right)_k(a_4+r)_k(1+a+m+r)_k}{k!(1+a-a_2+r)_k(1+a-a_3+r)_k(1+a-a_4+r)_k\left(\frac{2+a+2r-p_5-m}{2}\right)_k}
$$

$$
\times\frac{(1+a-p_5+r)_k}{\left(\frac{3+a-p_5-m+2r}{2}\right)_k}
$$

$$
\times\,{}_4F_3\left[\begin{matrix} 1+\frac{a}{2}+r+k,\ \frac{1}{2}+\frac{a}{2}+r+k,\ -a_4-m,\ 1+a-p_5+r+k; \\ \frac{3+a+2r-p_5-m}{2}+k,\ \frac{2+a+2r-p_5-m}{2}+k,\ 1+a-a_4+r+k \end{matrix}\right]_{(m-r-k)}
$$

Similarly, by repeating this process we get transformations for partial sums of series of higher orders.

7. The asymptotic behaviour of zero-balanced ${}_4F_3$ [1] and ${}_6F_5$ [1]

We begin this section by proving the following two asymptotic formulae for certain zero-balanced hypergeometric series:

(A) If $\alpha_1+\alpha_2+\alpha_3+\alpha_5=\beta_1+\beta_3+\beta_4$, $\mathrm{Rl}(\alpha_3)>0$, $\mathrm{Rl}(\alpha_5)>0$ then as $m\to\infty$

$$
{}_4F_3\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_5; & 1\\ \beta_1,\beta_3,\beta_4 & \end{matrix}\right]_{m-1}\sim\frac{\Gamma(\beta_1)\Gamma(\beta_3)\Gamma(\beta_4)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_5)}
$$

$$
\{\log m+L'''+\gamma\}+O\left(\frac{1}{m}\right), \tag{7.1}
$$

where γ is Euler's constant and

$$
L'''=-2\gamma-\Psi(\alpha_1)-\Psi(\alpha_2)+\sum_{k=1}^{\infty}\frac{(\beta_1-\alpha_3)_k(\beta_3+\beta_4-\alpha_3-\alpha_5)_k}{(\alpha_1)_k(\alpha_2)_k\,k}
$$

$$
+\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)(\beta_3-\alpha_5)(\beta_4-\alpha_5)}{\Gamma(\beta_1)\Gamma(\beta_3+\beta_4-\alpha_5+1)}
$$

$$\times F^{0:3;3}_{1:1;1}\left[\begin{matrix} \text{———}:\alpha_1,\alpha_2,\alpha_3;\beta_3-\alpha_5+1,\beta_4-\alpha_5+1,1; \\ \beta_3+\beta_4-\alpha_5+1:\beta_1;\quad 2\end{matrix}\ 1,1\right] \tag{7.2}$$

(**B**) If $a+b+c\notin\{0,-1,-2,\ldots\}$ and $\mathrm{Rl}(\beta_5+\beta_6-a-b)>0$, then as the integer m tends to ∞,

$${}_6F_5\left[\begin{matrix}\frac{a+b+c+1}{2}, & a+b+c-1, & a,b,c,\beta_5+\beta_6-a-b; \\ \frac{a+b+c-1}{2}, & b+c, \quad a+c, & \beta_5, \quad \beta_6\end{matrix}\right]_m$$

$$\sim\frac{\Gamma(b+c)\Gamma(a+c)\Gamma(\beta_5)\Gamma(\beta_6)}{\Gamma(a+b+c)\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(\beta_5+\beta_6-a-b)}$$

$$\times\{2\log m-\gamma-\Psi(a)-\Psi(b)-\Psi(c)\}+\frac{(a+b-\beta_6)(a+b-\beta_5)\Gamma(\beta_5)\Gamma(\beta_6)}{\Gamma(\beta_5+\beta_6-a-b)\Gamma(a+b+1)}$$

$$\times F^{0:5;3}_{1:3;1}\left[\begin{matrix}\text{———}:\frac{a+b+c+1}{2}, a+b+c-1,\ a,b,c;a+b-\beta_6,\ a+b-\beta_5,1; \\ a+b+1:\frac{a+b+c-1}{2},\ b+c,\ a+c;\quad 2\end{matrix}\ 1,1\right]$$

$$+O\left(\frac{\log m}{m}\right). \tag{7.3}$$

Proof. We have that [See 10; p. 73 (7)]

$${}_pF_{p-1}\left[\begin{matrix}(\alpha_p); \\ (\beta_{p-1})\end{matrix}\ x\right]=\frac{\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p)}$$

$$\times{}_{p-1}F_{p-2}\left[\begin{matrix}(\alpha_{p-1}); \\ (\beta_{p-3}),\beta_{p-2}+\beta_{p-1}-\alpha_p\end{matrix}\ x\right]$$

$$+\frac{(\beta_{p-2}-\alpha_p)(\beta_{p-1}-\alpha_p)\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p+1)}$$

$$\times F^{0:p-1;3}_{1:p-3;1}\left[\begin{matrix}\text{———}:(\alpha_{p-1});\beta_{p-2}-\alpha_p+1,\beta_{p-1}-\alpha_p+1,1; \\ \beta_{p-2}+\beta_{p-1}-\alpha_p+1:(\beta_{p-3});\quad 2\end{matrix}\ x,1\right].$$

$$(p\geqq 3;\quad \mathrm{Re}(\alpha_p)>0) \tag{7.4}$$

One easily gets

$${}_{p-1}F_{p-2}\left[\begin{matrix}(\alpha_{p-2}),\alpha_p; \\ (\beta_{p-4}),\beta_{p-2},\beta_{p-1}\end{matrix}\ x\right]_{m-1}=\frac{\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p)}$$

$$\times{}_{p-2}F_{p-3}\left[\begin{matrix}(\alpha_{p-2}); \\ (\beta_{p-4}),\beta_{p-2}+\beta_{p-1}-\alpha_p\end{matrix}\ x\right]_{m-1}$$

$$+\frac{(\beta_{p-2}-\alpha_p)(\beta_{p-1}-\alpha_p)\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p+1)}$$

$$\times\sum_{k=0}^{m-1}\sum_{n=0}^{\infty}\frac{[(\alpha_{p-2})]_k(\beta_{p-2}-\alpha_p+1)_n(\beta_{p-1}-\alpha_p+1)_n x^k}{(\beta_{p-2}+\beta_{p-1}-\alpha_p+1)_{k+n}[(\beta_{p-4})]_k(2)_n k!}$$

$$\mathrm{Re}(\alpha_p)>0 \tag{7.5}$$

by taking $\alpha_{p-1} = -m+1$ and $\beta_{p-3} = -m+1$, respectively.

Taking $p=5$, $x=1$ and applying Ramanujan's asymptotic formula (2.5) on r.h.s. of the above equation, we get (7.1).

Repeating this process for $p = 6, 7, \ldots$ we can get the asymptotic behaviour of a zero-balanced ${}_rF_{r-1}$ (1) series.

Proof of (7.3). Proceeding exactly as in the case of (7.1), we can generalize (2.6) also.

8. Concluding remarks

Proceeding exactly as in the case of (4.1) and (7.1) we can obtain the generalization of a result of Evans and Stanton [5, Theorem 3, p. 1011, (1.11)] namely

If $a+b+c=d+e$ and Re $(c)>0$, then as $u \to 1$ with $0<u<1$

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)}\ {}_3F_2\left[\begin{matrix} a,\ b,\ c; \\ d,\ e \end{matrix}\ u\right] = -\log(1-u) + L + O(1-u)\log(1-u)) \tag{8.1}$$

where L is defined by (2.4).

On particularising parameters in this generalization, we get Saigo and Srivastava's result [10; p. 74, 75 (10), (11), (12)].

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On unified fractional integral operators

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Abstract. The present paper is in continuation to our recent paper [6] in these proceedings. Therein, three composition formulae for a general class of fractional integral operators had been established. In this paper, we develop the Mellin transforms and their inversions, the Mellin convolutions, the associated Parseval–Goldstein theorem and the images of the multivariable H-function together with applications for these operators. In all, seven theorems and two corollaries (involving the Konhauser biorthogonal polynomials and the Jacobi polynomials) have been established in this paper. On account of the most general nature of the polynomials $S_n^m[x]$ and the multivariable H-function whose product form the kernels of our operators, a large number of (new and known) interesting results involving simpler polynomials and special functions (involving one or more variables) obtained by several authors and hitherto lying scattered in the literature follow as special cases of our findings. We give here exact references to the results (in essence) of seven research papers which follow as simple special cases of our theorems.

Keywords. Fractional integral operator; general class of polynomials; multivariable H-function; Mellin transform; Mellin convolution.

1. Introduction

We shall study in this paper the fractional integral operators defined by means of the following equations

$$R_x^{\eta,\alpha}[f(x)]$$

$$= R^{\eta,\alpha;m,n,\mu,\nu;N,P,Q,M',N',P',Q',\dots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\dots,u_r,v_r}_{x;e;z_1,\dots,z_r,a_j,\alpha'_j,\dots,\alpha_j^{(r)},b_j,\beta'_j,\dots,\beta_j^{(r)};c'_j,\gamma'_j,d'_j,\delta'_j,\dots,c_j^{(r)},\gamma_j^{(r)};d_j^{(r)};\delta_j^{(r)}}[f(x)]$$

$$= x^{-\eta-\alpha-1}\int_0^x t^\eta (x-t)^\alpha S_n^m\left[e\left(\frac{t}{x}\right)^\mu\left(1-\frac{t}{x}\right)^\nu\right] H\left[z_1\left(\frac{t}{x}\right)^{u_1}\left(1-\frac{t}{x}\right)^{v_1},\dots,\right.$$

$$\left. z_r\left(\frac{t}{x}\right)^{u_r}\left(1-\frac{t}{x}\right)^{v_r}\right] f(t)\mathrm{d}t \qquad (1)$$

$$W_x^{\eta,\alpha}[f(x)]$$

$$= W^{\eta,\alpha;m,n,\mu,\nu;N,P,Q,M',N',P',Q',\dots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\dots,u_r,v_r}_{x;e;z_1,\dots,z_r,a_j,\alpha'_j,\dots,\alpha_j^{(r)},b_j,\beta'_j,\dots,\beta_j^{(r)};c'_j,\gamma'_j,d'_j,\delta'_j,\dots,c_j^{(r)},\gamma_j^{(r)};d_j^{(r)};\delta_j^{(r)}}[f(x)]$$

$$= x^\eta \int_x^\infty t^{-\eta-\alpha-1}(t-x)^\alpha S_n^m\left[e\left(\frac{x}{t}\right)^\mu\left(1-\frac{x}{t}\right)^\nu\right] H\left[z_1\left(\frac{x}{t}\right)^{u_1}\left(1-\frac{x}{t}\right)^{v_1},\dots,\right.$$

$$\left. z_r\left(\frac{x}{t}\right)^{u_r}\left(1-\frac{x}{t}\right)^{v_r}\right] f(t)\mathrm{d}t. \qquad (2)$$

Here $S_n^m[x]$ denotes the general class of polynomials introduced by Srivastava

[12, p. 1, eq. (1)]

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \quad n = 0, 1, 2, \ldots, \tag{3}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}(n, k \geqslant 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others [16, pp. 158–161].

The H-function of r complex variables $z_1, \ldots, z_r$ [15] occurring in the paper will be represented in the following form [14, p. 251, eq. (C.1)]

$$H[z_1, \ldots, z_r] = H^{0,N:M',N';\ldots;M^{(r)},N^{(r)}}_{P,Q:P',Q';\ldots;P^{(r)},Q^{(r)}} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,P}: \\ (b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q}: \end{matrix} \right.$$
$$\left. \begin{matrix} (c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,P^{(r)}} \\ (d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,Q^{(r)}} \end{matrix} \right]. \tag{4}$$

The defining integral and other details about this function can be found in the references given above. It is assumed throughout the present work that this function always satisfies the appropriate existence and convergence conditions of its defining integral [14, pp. 252–253, eqs (C.4–C.6)].

To be specific, we shall assume throughout this paper that

$$f(x) = \begin{cases} O(|x|^\gamma), & |x| \to 0 \\ O(|x|^\delta e^{-\lambda|x|}), & |x| \to \infty. \end{cases}$$

It is easy to verify that the operator defined by (1) exists if

(i) The quantities $\mu, v, u_1, v_1, \ldots, u_r, v_r$ are all positive (some of them may decrease to zero provided that the resulting operator has a meaning).

(ii) $\mathrm{Re}(\alpha) + \sum_{i=1}^{r} v_i \min_{1 \leq j \leq M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})] + 1 > 0.$

(iii) $\mathrm{Re}(\eta + \gamma) + \sum_{i=1}^{r} u_i \min_{1 \leq j \leq M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})] + 1 > 0.$

and the operator defined by (2) exists if $\mathrm{Re}(\lambda) > 0$ or $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Re}(\eta - \delta) + \sum_{i=1}^{r} u_i \min_{1 \leq j \leq M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})] > 0$, and the set of conditions (i) and (ii) specified for the existence of the operator (1) are satisfied.

2. The Mellin transforms and the inversion formulae

Theorem 1. *If* $M\{f(x);s\}$, $M\{R_x^{\eta,\alpha}[f(x)];s\}$ *exist*, $\mathrm{Re}(1+\alpha) > 0$, $\mathrm{Re}(1+\eta-s) > 0$ *and the conditions of the existence of the operator* $R_x^{\eta,\alpha}[f(x)]$ *are satisfied, then*

$$M\{R_x^{\eta,\alpha}[f(x)];s\} = \phi_1(s) M\{f(x);s\} \tag{5}$$

where

$$\phi_1(s) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^k H^{o,N+2:*}_{P+2,Q+1:*} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right|$$

$$\left. \begin{matrix} (s-\eta-\mu k; u_1, \ldots, u_r), (-\alpha-\nu k; v_1, \ldots, v_r), (a_j; \alpha'_j, \ldots, \alpha_j^{(r)})_{1,P} : * \\ (-1+s-\eta-\alpha-(\mu+\nu)k; u_1+v_1, \ldots, u_r+v_r), (b_j; \beta'_j, \ldots, \beta_j^{(r)})_{1,Q} : * \end{matrix} \right] \tag{6}$$

the asterisk $(*)$ *in* (6) *indicates that the parameters at these places are the same as the parameters of the H-function of r complex variables occurring in* (4) *and* $M\{f(x);s\}$ *stands for the well known Mellin transform of the function* $f(x)$ *defined by the following equation*

$$F(s) = M\{f(x);s\} = \int_0^\infty x^{s-1} f(x) \mathrm{d}x. \tag{7}$$

Proof. From (7) and (1), $M\{R_x^{\eta,\alpha}[f(x)];s\} = \Delta$ (say) takes the following form

$$\Delta = \int_0^\infty x^{s-1} \left\{ x^{-\eta-\alpha-1} \int_0^x t^\eta (x-t)^\alpha S_n^m \left[e\left(\frac{t}{x}\right)^\mu \left(1-\frac{t}{x}\right)^\nu \right] H\left[z_1 \left(\frac{t}{x}\right)^{u_1} \left(1-\frac{t}{x}\right)^{v_1}, \ldots, z_r \left(\frac{t}{x}\right)^{u_r} \left(1-\frac{t}{x}\right)^{v_r} \right] f(t) \mathrm{d}t \right\} \mathrm{d}x.$$

On changing the order of integration in the above equation (which is permissible under the conditions stated) we get

$$\Delta = \int_0^\infty t^\eta f(t) \left\{ \int_t^\infty x^{s-\eta-\alpha-2} (x-t)^\alpha S_n^m \left[e\left(\frac{t}{x}\right)^\mu \left(1-\frac{t}{x}\right)^\nu \right] H\left[z_1 \left(\frac{t}{x}\right)^{u_1} \left(1-\frac{t}{x}\right)^{v_1}, \ldots, z_r \left(\frac{t}{x}\right)^{u_r} \left(1-\frac{t}{x}\right)^{v_r} \right] \mathrm{d}x \right\} \mathrm{d}t. \tag{8}$$

Now we express the general class of polynomials involved in the above equation in the series form and the multivariable H-function in terms of its well known Mellin-Barnes contour integral and interchange the order of summation and integration in the result thus obtained. The equation given by (8) now yields the following result after a little simplification

$$\Delta = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^k \int_0^\infty t^\eta f(t) \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi_1(\xi_1) \cdots \phi_r(\xi_r) \psi(\xi_1, \ldots, \xi_r) z_1^{\xi_1} \cdots z_r^{\xi_r} \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_r \left\{ \int_t^\infty x^{s-\eta-\alpha-2-(\mu+\nu)k-(u_1+v_1)\xi_1-\cdots-(u_r+v_r)\xi_r} (x-t)^{\alpha+\nu k+v_1\xi_1+\cdots+v_r\xi_r} \mathrm{d}x \right\} t^{\mu k+u_1\xi_1+\cdots+u_r\xi_r} \mathrm{d}t. \tag{9}$$

Evaluating the x-integral occurring in (9) with the help of a known result [2, p. 201, eq. (6)] and reinterpreting the resulting multiple Mellin–Barnes contour integral so obtained in terms of the H-function of r variables, we easily arrive at the desired Theorem.

If in the left-hand side of (5) we take $n=0$ (the polynomial S_0^m will reduce to $A_{0,0}$ which can be taken to be unity without loss of generality) and put $N=P=Q=0$, $M^{(i)}=Q^{(i)}=1$, $N^{(i)}=P^{(i)}=0$, $d_1^{(i)}=0$, $\delta_1^{(i)}=1$, $u_i=v_i=0$ and let $z_i \to 0$ $(i=2,\ldots,r)$, the

H-function of r variables occurring therein reduces to the H-function of Fox [3] and the above Theorem reduces to a theorem which is in essence same as that obtained by Saxena and Kumbhat [11, p. 3, eq. (3.3)].

Theorem 2. *If* $M\{f(x);s\}$, $M\{W_x^{\eta,\alpha}[f(x)];s\}$ *exist,* $\mathrm{Re}(s+\eta)>0$, $\mathrm{Re}(1+\alpha)>0$ *and the conditions of the existence of the operator* $W_x^{\eta,\alpha}[f(x)]$ *are satisfied, then*

$$M\{W_x^{\eta,\alpha}[f(x)];s\} = \phi_2(s)M\{f(x);s\} \tag{10}$$

where

$$\phi_2(s) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^k H^{o,N+2:*}_{P+2,Q+1:*}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\right| \begin{array}{l} (1-s-\eta-\mu k;u_1,\ldots,u_r),(-\alpha-\nu k;v_1,\ldots,v_r),(a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,P}:* \\ (-s-\eta-\alpha-(\mu+\nu)k;u_1+v_1,\ldots,u_r+v_r),(b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q}:* \end{array}\left.\vphantom{\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}}\right] \tag{11}$$

the asterisk (*) *in* (11) *indicates that the parameters at these places are the same as the parameters of the H-function of r complex variables occurring in* (4).

Proof. If we follow the lines of proof as given in Theorem 1 and make use of another well known result [2, p. 185, eq. (7)] we easily obtain the Theorem 2.

Again, if in the left-hand side of (10) we take $n=0$ and reduce the multivariable H-function occurring therein to the Fox's H-function in the manner explained earlier, the above Theorem reduces to the other similar theorem obtained by Saxena and Kumbhat [11, p. 4, eq. (3.6)].

It may be remarked that the Theorems 1 and 2 given above are also generalizations of the theorems obtained by Saxena [10, p. 289, eqs (6), (8)] and Kalla [7, pp. 270–271, eqs (34), (37)] for the operators studied by them.

On using the well known Mellin inversion theorem in (5) and (10) in succession, we arrive at the following interesting Theorems

Theorem 3.

$$\frac{1}{2}[f(t+0)+f(t-0)] = \frac{1}{(2\pi\omega)}\lim_{\tau\to\infty}\int_{c-\omega\tau}^{c+\omega\tau}\frac{t^{-s}}{\phi_1(s)}M\{R_x^{\eta,\alpha}[f(x)];s\}\,\mathrm{d}s \tag{12}$$

where $f(t)$ *is of bounded variation at the point* $t=x$ $(x>0)$, *the conditions stated with Theorem* 1 *are satisfied and* $\phi_1(s)$ *is defined by* (6).

Theorem 4.

$$\frac{1}{2}[f(t+0)+f(t-0)] = \frac{1}{(2\pi\omega)}\lim_{\tau\to\infty}\int_{c-\omega\tau}^{c+\omega\tau}\frac{t^{-s}}{\phi_2(s)}M\{W_x^{\eta,\alpha}[f(x)];s\}\,\mathrm{d}s \tag{13}$$

where $f(t)$ *is of bounded variation at the point* $t=x$ $(x>0)$, *the conditions stated with Theorem* 2 *are satisfied and* $\phi_2(s)$ *is as defined by* (11).

Also when $f(t)$ is continuous at $t=x$ $(x>0)$ then the left-hand sides of (12) and (13) are equal to $f(t)$.

If in (1) and (2) we take $n=0$ and put $N=P=Q=0$, $M^{(i)}=Q^{(i)}=1$, $N^{(i)}=P^{(i)}=0$, $d_1^{(i)}=0$, $\delta_1^{(i)}=1$, $u_i=v_i=0$ and let $z_i\to 0\,(i=2,\ldots,r)$, $u_1=0$, $v_1=1$ and further reduce

the H-function of Fox so obtained in terms of the generalized hypergeometric function in the usual way the Theorems 3 and 4 yield the inversion formulae which are in essence same as given by Goyal and Jain [4, p. 257, eqs (3.10), (3.11)].

3. The Mellin convolutions

From a well known theorem by Titchmarsh [17, p. 60, Th. 44] we know that if $f\varepsilon L(0,\infty)$, $g\varepsilon L(0,\infty)$, then $(f*g)\varepsilon L(0,\infty)$, where

$$(f*g)(x)=\int_0^\infty t^{-1}f\left(\frac{x}{t}\right)g(t)\mathrm{d}t. \tag{14}$$

Following the lines adopted by Buschman [1], we shall define a function $R^{\eta,\alpha}(x)$ as follows

$$R^{\eta,\alpha}(x)=x^{-\eta-\alpha-1}(x-1)^{\alpha}U(x-1)S_n^m\left[e\left(\frac{1}{x}\right)^{\mu}\left(\frac{x-1}{x}\right)^{\nu}\right]H\left[z_1\left(\frac{1}{x}\right)^{u_1}\left(\frac{x-1}{x}\right)^{v_1},\ldots,z_r\left(\frac{1}{x}\right)^{u_r}\left(\frac{x-1}{x}\right)^{v_r}\right] \tag{15}$$

where U denotes the well known unit step function. It can be easily verified that $R^{\eta,\alpha}(x)\varepsilon L(0,\infty)$ if

$$\mathrm{Re}(\alpha)+\sum_{i=1}^{r}v_i\min_{1\leqq j\leqq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0,$$

$$\mathrm{Re}(\eta)+\sum_{i=1}^{r}u_i\min_{1\leqq j\leqq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]>0.$$

We can represent the operator (1) as a convolution of the form (14). Indeed, we have

$$\begin{aligned}R_x^{\eta,\alpha}f(x)=\int_0^\infty t^{-1}&\left\{\left(\frac{x}{t}\right)^{-\eta-\alpha-1}\left(\frac{x}{t}-1\right)^{\alpha}U\left(\frac{x}{t}-1\right)S_n^m\left[e\frac{1}{\left(\frac{x}{t}\right)^{\mu}}\left(\frac{\frac{x}{t}-1}{\frac{x}{t}}\right)^{\nu}\right]\right.\\&\left.H\left[z_1\frac{1}{\left(\frac{x}{t}\right)^{u_1}}\left(\frac{\frac{x}{t}-1}{\frac{x}{t}}\right)^{v_1},\ldots,z_r\frac{1}{\left(\frac{x}{t}\right)^{u_r}}\left(\frac{\frac{x}{t}-1}{\frac{x}{t}}\right)^{v_r}\right]\right\}f(t)\mathrm{d}t\\&=(R^{\eta,\alpha}*f)(x)\quad\text{(with the help of (14)).}\end{aligned} \tag{16}$$

Again, if we define

$$W^{\eta,\alpha}(x)=x^{\eta}(1-x)^{\alpha}U(1-x)S_n^m[e(x)^{\mu}(1-x)^{\nu}]H[z_1(x)^{u_1}(1-x)^{v_1},\ldots,z_r(x)^{u_r}(1-x)^{v_r}]. \tag{17}$$

On proceeding in a manner as indicated above, we have

$$W_x^{\eta,\alpha}f(x)=(W^{\eta,\alpha}*f)(x). \tag{18}$$

Also, $W^{\eta,\alpha}(x)\varepsilon L(0,\infty)$ for

$$\mathrm{Re}(\eta)+\sum_{i=1}^{r} u_i \min_{1\leq j\leq M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0,$$

$$\mathrm{Re}(\alpha)+\sum_{i=1}^{r} v_i \min_{1\leq j\leq M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0.$$

Results given by (16) and (18) yield the corresponding results given by Buschman [1, pp. 99–101] if we reduce our operators to the simple operators studied by him.

4. An analogue of the Parseval–Goldstein theorem for the operators defined by (1) and (2)

Theorem 5. *If*

$$\phi_1(x)=R_x^{\eta,\alpha}[f_1(x)] \tag{19}$$

and

$$\phi_2(x)=W_x^{\eta,\alpha}[f_2(x)] \tag{20}$$

then

$$\int_0^\infty f_1(x)\phi_2(x)\mathrm{d}x=\int_0^\infty f_2(x)\phi_1(x)\mathrm{d}x \tag{21}$$

provided that the various integrals involved converge absolutely.

Proof. We have from (2), $\int_0^\infty f_1(x)\phi_2(x)\mathrm{d}x$

$$=\int_0^\infty f_1(x)\left\{x^\eta\int_0^\infty t^{-\eta-\alpha-1}(t-x)^\alpha S_n^m\left[e\left(\frac{x}{t}\right)^\mu\left(1-\frac{x}{t}\right)^\nu\right]\right.$$
$$\left. H\left[z_1\left(\frac{x}{t}\right)^{u_1}\left(1-\frac{x}{t}\right)^{v_1},\ldots,z_r\left(\frac{x}{t}\right)^{u_r}\left(1-\frac{x}{t}\right)^{v_r}\right]f_2(t)\mathrm{d}t\right\}\mathrm{d}x.$$

Changing the order of integration in the right-hand side of the above equation (which is permissible under the conditions stated), it takes the following form after a little simplification

$$=\int_0^\infty f_2(t)\left\{t^{-\eta-\alpha-1}\int_0^t x^\eta(t-x)^\alpha S_n^m\left[e\left(\frac{x}{t}\right)^\mu\left(1-\frac{x}{t}\right)^\nu\right]H\left[z_1\left(\frac{x}{t}\right)^{u_1}\left(1-\frac{x}{t}\right)^{v_1},\right.\right.$$
$$\left.\left.\ldots,z_r\left(\frac{x}{t}\right)^{u_r}\left(1-\frac{x}{t}\right)^{v_r}\right]f_1(x)\mathrm{d}x\right\}\mathrm{d}t.$$

On reinterpreting the x-integral given in the above expression with the help of (1) we arrive at the required Theorem.

The above Theorem is a generalization of a theorem obtained by Kalla [7, p. 271, eq. (38)] for his operators and provides an analogue of the Parseval–Goldstein theorem for the operators studied in this paper.

5. Images of the multivariable H-function in the operators of our study

(i) $$R_x^{\eta,\alpha}\{x^l H[z_{r+1}x^{u_{r+1}},\ldots,z_{r+s}x^{u_{r+s}}]\}$$

$$= x^{l} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^{k} H^{o,N+N_1+2\,:M',N';\ldots;M^{(r+s)},N^{(r+s)}}_{P+P_1+2,Q+Q_1+1\,:P',Q';\ldots;P^{(r+s)},Q^{(r+s)}} \left[\begin{array}{c} z_1 \\ \vdots \\ z_r \\ z_{r+1}x^{u_{r+1}} \\ \vdots \\ z_{r+s}x^{u_{r+s}} \end{array}\right|$$

$$(-l-\eta-\mu k; u_1,\ldots,u_{r+s}),\ \left(-\alpha-\nu k; v_1,\ldots,v_r,\frac{o,\ldots,o}{s}\right),$$

$$(-1-l-\eta-\alpha-(\mu+\nu)k\,;\,u_1+v_1,\ldots,u_r+v_r,u_{r+1},\ldots,u_{r+s}),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha^{(r)}_j,\frac{o,\ldots,o}{s}\right)_{1,N},\ \left(a'_j;\frac{o,\ldots,o}{r},\alpha^{(r+1)}_j,\ldots,\alpha^{(r+s)}_j\right)_{1,P_1},$$

$$\left(b_j;\beta'_j,\ldots,\beta^{(r)}_j,\frac{o,\ldots,o}{s}\right)_{1,Q},\ \left(b'_j;\frac{o,\ldots,o}{r},\beta^{(r+1)}_j,\ldots,\beta^{(r+s)}_j\right)_{1,Q_1},$$

$$\left.\left(a_j;\alpha'_j,\ldots,\alpha^{(r)}_j,\frac{o,\ldots,o}{s}\right)_{N+1,P} \begin{array}{l} :(c'_j,\gamma'_j)_{1,P'};\ldots;(c^{(r+s)}_j,\gamma^{(r+s)}_j)_{1,P^{(r+s)}} \\ :(d'_j,\delta'_j)_{1,Q'};\ldots;(d^{(r+s)}_j,\delta^{(r+s)}_j)_{1,Q^{(r+s)}} \end{array}\right] \tag{22}$$

where the function occurring on the right-hand side of (22) is the H-function of $r+s$ variables and the following conditions are satisfied.

The quantities $\mu, \nu, u_1, v_1, \ldots, u_r, v_r, u_{r+1}, \ldots, u_{r+s}$ are all positive (some of them may however decrease to zero provided that the resulting image has a meaning),

$$\mathrm{Re}(\eta+l) + \sum_{i=1}^{r+s} u_i \min_{1 \leqq j \leqq M^{(i)}} [\mathrm{Re}(d^{(i)}_j/\delta^{(i)}_j)] + 1 > 0 \quad \text{and}$$

$$\mathrm{Re}(\alpha) + \sum_{i=1}^{r} v_i \min_{1 \leqq j \leqq M^{(i)}} [\mathrm{Re}(d^{(i)}_j/\delta^{(i)}_j)] + 1 > 0.$$

Also the number occurring below the line at any place on the right-hand side of (22) and throughout the paper indicates the total number of zeros covered by it. Thus $(o,\ldots,o)/r$ would mean r zeros, and so on.

Proof. We have from (1), $R^{\eta,\alpha}_x\{x^l H[z_{r+1}x^{u_{r+1}},\ldots,z_{r+s}x^{u_{r+s}}]\}$

$$= x^{-\eta-\alpha-1} \int_0^x t^{\eta}(x-t)^{\alpha} S^m_n\left[e\left(\frac{t}{x}\right)^{\mu}\left(1-\frac{t}{x}\right)^{\nu}\right] H\left[z_1\left(\frac{t}{x}\right)^{u_1}\left(1-\frac{t}{x}\right)^{v_1},\ldots,\right.$$

$$\left. z_r\left(\frac{t}{x}\right)^{u_r}\left(1-\frac{t}{x}\right)^{v_r}\right]\{t^l H[z_{r+1}t^{u_{r+1}},\ldots,z_{r+s}t^{u_{r+s}}]\}\,dt. \tag{23}$$

Now expressing the general class of polynomials involved in the right-hand side of (23) in the series form given by (3) and both the multivariable H-functions in terms of their well known Mellin–Barnes contour integrals and interchanging the order of summation and integration in the result thus obtained (which is permissible under the conditions stated) it takes the following form after a little simplification

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^{k} \frac{1}{(2\pi\omega)^{r+s}} \int_{L_1} \cdots \int_{L_{r+s}} \phi_1(\xi_1)\cdots\phi_{r+s}(\xi_{r+s})$$

$$\psi(\xi_1,\ldots,\xi_r)\psi'(\xi_{r+1},\ldots,\xi_{r+s})z_1^{\xi_1}\cdots z_{r+s}^{\xi_{r+s}}\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{r+s}$$
$$\left\{x^{-\eta-\alpha-1}\int_0^x t^{l+\eta+\mu k+u_1\xi_1+\cdots+u_{r+s}\xi_{r+s}}(x-t)^{\alpha+\nu k+v_1\xi_1+\cdots+v_r\xi_r}\,\mathrm{d}t\right\}$$
$$x^{-(\mu+\nu)k-(u_1+v_1)\xi_1-\cdots-(u_r+v_r)\xi_r}. \tag{24}$$

Evaluating the t-integral occurring in (24) with the help of a known result [2, p. 185, eq. (7)] and reinterpreting the resulting multiple Mellin–Barnes contour integrals so obtained in terms of the H-function of $r+s$ variables, we easily arrive at the desired image (22).

(ii) $$W_x^{\eta,\alpha}\{x^l H[z_{r+s}x^{-u_{r+1}},\ldots,z_{r+s}x^{-u_{r+s}}]\}$$

$$= x^l\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,e^k H^{o,N+N_1+2}_{P+P_1+2,Q+Q_1+1}\begin{array}{l}:**\\:**\end{array}\left[\begin{array}{l} z_1\\ \vdots\\ z_r\\ z_{r+1}x^{-u_{r+1}}\\ \vdots\\ z_{r+s}x^{-u_{r+s}}\end{array}\right|$$

$$(1+l-\eta-\mu k;u_1,\ldots,u_{r+s}),\left(-\alpha-\nu k;v_1,\ldots,v_r,\frac{o,\ldots,o}{s}\right),$$
$$(l-\eta-\alpha-(\mu+\nu)k\,;u_1+v_1,\ldots,u_r+v_r,u_{r+1},\ldots,u_{r+s}),$$
$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N},\left(a'_j;\frac{o,\ldots,o}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1},$$
$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q},\left(b'_j;\frac{o,\ldots,o}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1},$$
$$\left.\left(a_j;\alpha',\ldots,\alpha^{(r)},\frac{o,\ldots,o}{s}\right)_{N+1,P}\begin{array}{l}:**\\:**\end{array}\right] \tag{25}$$

where the asterisks $(**)$ in (25) indicate that the parameters at these places are the same as the parameters of the H-function of $r+s$ complex variables occurring in (22) and the quantities $\mu,\nu,u_1,v_1,\ldots,u_r,v_r,u_{r+1},\ldots,u_{r+s}$ are all positive (some of them may however decrease to zero provided that the resulting image has a meaning),

$$\mathrm{Re}(\alpha)+\sum_{i=1}^{r}v_i\min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0\quad\text{and}$$

$$\mathrm{Re}(\eta-l)+\sum_{i=1}^{r+s}u_i\min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]>0.$$

Proof. If we follow the same method as given for obtaining the result given by (22) and use another well known result [2, p. 201, eq. (6)], we obtain the image (25).

Images given by (22) and (25) are generalizations of the results obtained by Gupta (R.) [5, p. 73, eqs (2.12), (2.13)].

6. Applications

Now we shall make use of the Theorem 5 and the images obtained earlier in establishing two further theorems.

Theorem 6. *If*

$$\phi(x) = R_x^{\eta,\alpha}[f(x)] \tag{26}$$

then

$$\int_0^\infty x^l H[z_{r+1}x^{-u_{r+1}};\ldots,z_{r+s}x^{-u_{r+s}}]\phi(x)\mathrm{d}x$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, e^k \int_0^\infty x^l H^{o,N+N_1+2}_{P+P_1+2,Q+Q_1+1} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \\ z_{r+1}x^{-u_{r+1}} \\ \vdots \\ z_{r+s}x^{-u_{r+s}} \end{matrix}\right|\begin{matrix} :** \\ :** \end{matrix}$$

$$(1+l-\eta-\mu k;u_1,\ldots,u_{r+s}),\left(-\alpha-vk;v_1,\ldots,v_r,\frac{o,\ldots,o}{s}\right),$$

$$(l-\eta-\alpha-(\mu+v)k\,;u_1+v_1,\ldots,u_r+v_r,u_{r+1},\ldots,u_{r+s}),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N},\left(a'_j;\frac{o,\ldots,o}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1},$$

$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q},\left(b'_j;\frac{o,\ldots,o}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1},$$

$$\left.\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{N+1,P}\begin{matrix} :** \\ :** \end{matrix}\right] f(x)\mathrm{d}x \tag{27}$$

where the asterisks (**) in (27) indicate that the parameters at these places are the same as the parameters of the H-function of $r+s$ complex variables occurring in (22) and the conditions of the existence of the operator $R_x^{\eta,\alpha}[f(x)]$ mentioned earlier are satisfied and the integrals occurring in (27) are absolutely convergent.

Proof. On substituting the results given by (25) and (26) in the analogue of the Parseval–Goldstein theorem given by (21), we easily arrive at the required Theorem after a little simplification.

Theorem 7. *If*

$$\phi(x) = W_x^{\eta,\alpha}[f(x)] \tag{28}$$

then

$$\int_0^\infty x^l H[z_{r+1}x^{u_{r+1}},\ldots,z_{r+s}x^{u_{r+s}}]\phi(x)\mathrm{d}x$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, e^k \int_0^\infty x^l H^{o,N+N_1+2}_{P+P_1+2,Q+Q_1+1} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \\ z_{r+1}x^{u_{r+1}} \\ \vdots \\ z_{r+s}x^{u_{r+s}} \end{matrix}\right|\begin{matrix} :** \\ :** \end{matrix}$$

$$(-l-\eta-\mu k;u_1,\ldots,u_{r+s}),\ \left(-\alpha-vk;v_1,\ldots,v_r,\frac{o,\ldots,o}{s}\right),$$

$$(-1-l-\eta-\alpha-(\mu+v)k\,;u_1+v_1,\ldots,u_r+v_r,u_{r+1},\ldots,u_{r+s}),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N},\ \left(a'_j;\frac{o,\ldots,o}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1},$$

$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q},\ \left(b'_j;\frac{o,\ldots,o}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1}$$

$$\left.\left(a_j;\alpha',\ldots,\alpha^{(r)},\frac{o,\ldots,o}{s}\right)_{N+1,P}\begin{matrix}:**\\:**\end{matrix}\right]f(x)\mathrm{d}x \tag{29}$$

where the asterisks (**) in (29) indicate that the parameters at these places are the same as the parameters of the H-function of $r+s$ complex variables occurring in (22) and the conditions of the existence of the operator $W_x^{\eta,\alpha}[f(x)]$ are satisfied and the integrals occurring in (29) are absolutely convergent.

Proof. On substituting the results given by (22) and (28) in (21), we get the required Theorem after a little simplification.

If we take $n=0$, $N=P=Q=0$, $M^{(i)}=Q^{(i)}=1$, $N^{(i)}=P^{(i)}=0$, $d_1^{(i)}=0$, $\delta_1^{(i)}=1$, $u_i=v_i=0$ and let $z_i\to 0$ $(i=1,\ldots,r)$ in Theorems 6 and 7, we easily obtain the results which are in essence same as those obtained by Mathur [9, p. 108, Th. 1; p. 112, Th. 2].

If we take $e=\mu=1$, $v=0$, $m=1$, $A_{n,k}=\Gamma(1+\zeta+\beta n)/n!\Gamma(1+\zeta+\beta k)$ in (26), the polynomial $S_n^1[t/x]$ reduces to the Konhauser biorthogonal polynomials $Z_n^\zeta[(t/x)^{1/\beta};\beta]$ [13, p. 225, eq. (3.23); 8, p. 304, eq. (5)] and if we further let $v_i=0$ $(i=1,\ldots,r)$ therein, Theorem 6 takes the following interesting form.

COROLLARY 1

If

$$\phi(x)=x^{-\eta-\alpha-1}\int_0^x t^\eta(x-t)^\alpha Z_n^\zeta\left[\left(\frac{t}{x}\right)^{1/\beta};\beta\right]H\left[z_1\left(\frac{t}{x}\right)^{u_1},\ldots,z_r\left(\frac{t}{x}\right)^{u_r}\right]f(t)\mathrm{d}t \tag{30}$$

then

$$\int_0^\infty x^l H[z_{r+1}x^{-u_{r+1}},\ldots,z_{r+s}x^{-u_{r+s}}]\phi(x)\mathrm{d}x$$

$$=\Gamma(1+\alpha)\sum_{k=0}^{n}\frac{(-n)_k\Gamma(1+\zeta+\beta n)}{k!\,n!\,\Gamma(1+\zeta+\beta k)}\int_0^\infty x^l H^{o,N+N_1+1}_{P+P_1+1,Q+Q_1+1}\begin{matrix}:**\\:**\end{matrix}$$

$$\left[\begin{matrix}z_1\\ \vdots\\ z_r\\ z_{r+1}x^{-u_{r+1}}\\ \vdots\\ z_{r+s}x^{-u_{r+s}}\end{matrix}\left|\begin{matrix}(1+l-\eta-k;u_1,\ldots,u_{r+s}),\ \left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N},\\ \\ (l-\eta-\alpha-k\,;u_1,\ldots,u_{r+s}),\ \left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix} \left(a'_j;\frac{0,\ldots,0}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1}, & \left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{s}\right)_{N+1,P} & :** \\ \left(b'_j;\frac{0,\ldots,0}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1} & & :** \end{matrix}\right] f(x)\mathrm{d}x \tag{31}$$

where the asterisks (∗∗) *in* (31) *indicate that the parameters at these places are the same as the parameters of the H-function of* $r+s$ *complex variables occurring in* (22) *and the conditions easily obtainable from Theorem 6 are satisfied.*

If we take $\beta = 1$, in (30), we get the corresponding result involving the Laguerre polynomials $L_n^{(\zeta)}[t/x]$.

If we take $e = \mu = 1$, $v = 0$, $m = 1$, $A_{n,k} = \binom{n+\zeta}{n}(\zeta+\beta+n+1)_k/(\zeta+1)_k$ in (26), the polynomial $S_n^1[t/x]$ reduces to the Jacobi polynomials [16, p. 159, eq. (1.6)] and if we further let $v_i = 0$ $(i = 1,\ldots,r)$ therein, we arrive at the following result.

COROLLARY 2

If

$$\phi(x) = x^{-\eta-\alpha-1}\int_0^x t^\eta(x-t)^\alpha P_n^{(\zeta,\beta)}\left[1-\frac{2t}{x}\right] H\left[z_1\left(\frac{t}{x}\right)^{u_1},\ldots,z_r\left(\frac{t}{x}\right)^{u_r}\right] f(t)\mathrm{d}t \tag{32}$$

then

$$\int_0^\infty x^l H[z_{r+1}x^{-u_{r+1}},\ldots,z_{r+s}x^{-u_{r+s}}]\phi(x)\mathrm{d}x$$

$$= \Gamma(1+\alpha)\sum_{k=0}^n \frac{(-n)_k}{k!}\binom{n+\zeta}{n}\frac{(\zeta+\beta+n+1)_k}{(\zeta+1)_k}\int_0^\infty x^l H^{0,N+N_1+1}_{P+P_1+1,Q+Q_1+1} {:** \atop :**}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \\ z_{r+1}x^{-u_{r+1}} \\ \vdots \\ z_{r+s}x^{-u_{r+s}} \end{matrix}\ \middle|\ \begin{matrix} (1+l-\eta-k;u_1,\ldots,u_{r+s}), & \left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{s}\right)_{1,N}, \\ (l-\eta-\alpha-k;u_1,\ldots,u_{r+s}), & \left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{0,\ldots,0}{s}\right)_{1,Q}, \end{matrix}\right.$$

$$\left.\begin{matrix} \left(a'_j;\frac{0,\ldots,0}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1}, & \left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{s}\right)_{N+1,P} & :** \\ \left(b'_j;\frac{0,\ldots,0}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1} & & :** \end{matrix}\right] f(x)\mathrm{d}x \tag{33}$$

where the asterisks (∗∗) *in* (33) *indicate that the parameters at these places are the same as the parameters of the H-function of* $r+s$ *complex variables occurring in* (22) *and the conditions easily obtainable from Theorem 6 are satisfied.*

Corollaries similar to those obtained above can also be obtained for Theorem 7 but we do not record them here explicitly on account of the triviality of the analysis involved. A number of other corollaries of Theorems 6 and 7 involving various simpler

functions and polynomials can also be obtained but we do not record them here for lack of space.

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On maxima-minima

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Abstract. In this paper we have shown that one can obtain a curve, with prescribed maxima-minima, as a graph of a polynomial function. The proof involves elementary topology.

Keywords. Maxima-minima.

We define an n-tuple $(b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ to be a maxima-minima sequence if $b_{2i} < b_{2i-1}$ and $b_{2i+1} > b_{2i}$ for all $i = 1, 2, \ldots, [n/2]$. Let $P(t)$ be an $(n+1)$ degree polynomial over $\mathbb{R}$ and let $P'(t)$ have n distinct real zeros $a_1, a_2, \ldots, a_n$ with $a_1 < a_2 < \cdots < a_n$. For such a polynomial P, we say that $\{P(a_1), P(a_2), \ldots, P(a_n)\}$ is the set of maxima-minima of P if P has maxima at a_{2i-1} and has minima at a_{2i} for all $i = 1, 2, \ldots, [n/2]$. Clearly, if $\{P(a_1), P(a_2), \ldots, P(a_n)\}$ is the set of maxima-minima of an $(n+1)$ degree polynomial P, then the n-tuple $(P(a_1), P(a_2), \ldots, P(a_n))$ is a maxima-minima sequence.

Theorem. *Let $b_1, b_2, \ldots, b_n$ be real numbers such that the n-tuple $(b_1, b_2, \ldots, b_n)$ is a maxima-minima sequence. Then there exists a real polynomial P of degree $(n+1)$ such that $\{b_1, b_2, \ldots, b_n\}$ is the set of maxima-minima of P.*

Towards proving this theorem, let us set the following:
Let

$$U = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid 0 < x_1 < x_2 < \cdots < x_n\}$$

$$V = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1 > 0, (y_1, \ldots, y_n) \text{ is a maxima-minima sequence}\}.$$

Since U and V are both intersection of certain open half spaces, they are open and connected subsets of $\mathbb{R}^n$. Also, boundaries ∂U of U and ∂V of V are given by

$$\partial U = \bigcup_{i=1}^{n} \bar{B}_i$$

and

$$\partial V = \bigcup_{i=1}^{n} \bar{C}_i,$$

where $\bar{B}_i$ and $\bar{C}_i$ are closures of B_i and C_i respectively.

$$B_i = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 < x_1 < \cdots < x_{i-1} = x_i < x_{i+1} < \cdots < x_n, x_0 = 0\}$$

and

$$C_i = \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \mid y_i = y_{i-1}, y_0 = 0, y_{2j} < y_{2j+1}, y_{2j-1} > y_{2j}, i \neq 2j\}.$$

In the definition of C_i for i odd $y_i > y_{i+1}$ and for i even $y_i < y_{i+1}$. For

$x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, define

$$g_x(s)=\prod_{i=1}^{n}(x_i-s).$$

Let $f_x(s)$ be the unique polynomial with $f_x(0)=0$ and $\partial f_s/\partial s(s)=g_x(s)$. Define

$$\Phi:\mathbb{R}^n\to\mathbb{R}^n$$

as

$$\Phi(x)=(f_x(x_1),f_x(x_2),\ldots,f_x(x_n)).$$

Since f'_x vanishes at $x_1,x_2,\ldots,x_n$, for any $x\in U, f'_x$ has n distinct real zeros, namely, $x_1,x_2,\ldots,x_n$ with $0<x_1<x_2<\cdots<x_n$. In particular f_x has maxima at $x_1,x_3,\ldots$ and has minima at $x_2,x_4,\ldots$, i.e., $(f_x(x_1),f_x(x_2),\ldots,f_x(x_n))$ is a maxima-minima sequence. From the definition of f_x, it is clear that $f_x(x_1)>0$. Thus $(f_x(x_1),f_x(x_2),\ldots,f_x(x_n))\in V$, i.e., $\Phi(x)\in U$, which implies that $\Phi(U)\subseteq V$. For $x=(x_1,x_2,\ldots,x_n)\in B_i$ we have $x_i=x_{i-1}$. Then $f_x(x_i)=f_x(x_{i-1})$ and for i odd $f_x(x_i)>f_x(x_{i+1})$; for i even $f_x(x_i)<f_x(x_{i+1})$. Thus, Φ maps each B_i to the corresponding C_i, for $i=1,2,\ldots,n$ and hence by continuity $\bar{B}_i$ to $\bar{C}_i$. Hence Φ maps ∂U to ∂V.

Let us denote the restriction of Φ at U by ϕ.

Proof of the theorem. It suffices to prove that the map $\phi:U\to V$ is surjective (where ϕ, U and V are described as above), and if $(b_1,b_2,\ldots,b_n)$ is a maxima-minima sequence, then by adding a large positive number b to each b_i, we can make b_1+b positive. Let $b_i+b=b'_i$. Then $(b'_1,b'_2,\ldots,b'_n)\in V$, hence by surjectivity of ϕ, there exists a real polynomial Q of degree $n+1$, such that $\{b'_1,b'_2,\ldots,b'_n\}$ is the set of maxima-minima of Q. Let $P(t)=Q(t)-b$. Then the zeros of P' are same as the zeros of Q'. Let $a_1,a_2,\ldots,a_n$ be the n distinct zeros of Q' with $0<a_1<a_2<\cdots<a_n$. Then $P(a_i)=Q(a_i)-b=b'_i-b=b_i$. Thus $\{b_1,b_2,\ldots,b_n\}$ is the set of maxima-minima of P.

Surjectivity of the map ϕ is proved in two steps designated Lemma 1 and Lemma 2 in the sequel.

Lemma 1. *The map $\phi:U\to V$ is an open map.*

Proof. We will show that the Jacobian of ϕ at x is non-zero for all $x\in U$ [1]. Let $D\phi(x)$ denote the Jacobian matrix of ϕ at x and let $J\phi(x)$ be the Jacobian of ϕ at x, i.e., $J\phi(x)=\det(D\phi(x))$. The (i,j)-th entry D_{ij} of the matrix $D\phi(x)$ is given by:

When $i\neq j$.

$$\begin{aligned}D_{ij}&=\frac{\partial f_x(x_i)}{\partial x_j}\\&=\frac{\partial}{\partial x_j}\left((-1)^n\frac{x_i^{n+1}}{n+1}+(-1)^{n-1}s_1\frac{x_i^n}{n}+\cdots+s_nx_i\right)\\&=(-1)^{n-1}\frac{x_i^n}{n}\frac{\partial s_1}{\partial x_j}+\cdots+x_i\frac{\partial s_n}{\partial x_j},\end{aligned}$$

where s_p denotes the p-th symmetric function in $x_1,x_2,\ldots,x_n$, and

$$D_{ii}=(-1)^nx_i^n+\cdots+s_n+(-1)^{n-1}\frac{x_i^n}{n}\frac{\partial s_1}{\partial x_i}+\cdots+x_i\frac{\partial s_n}{\partial x_i}$$

$$= (-1)^{n-1}\frac{x_i^n}{n}\frac{\partial s_1}{\partial x_i} + \cdots + x_i\frac{\partial s_n}{\partial x_i} + f'_x(x_i)$$

$$= (-1)^{n-1}\frac{x_i^n}{n}\frac{\partial s_1}{\partial x_i} + \cdots + x_i\frac{\partial s_n}{\partial x_i}, \quad \text{since } f'_x(x_i) = 0.$$

Therefore, in general

$$D_{ij} = \sum_{p=1}^{n} (-1)^{n-p}\frac{x_i^{n+1-p}}{(n+1-p)}\frac{\partial s_p}{\partial x_j}.$$

Notice that $D_{ij}(x) = 0$ for all j if $x_j = 0$, and that $D_{ik}(x) = D_{il}(x)$ for all i if $x_k = x_l$. Also from the definition of Φ, we have $\Phi(\sigma x) = \sigma\Phi(x)$ for any permutation matrix σ. Hence $J(\Phi)$ is a symmetric, homogeneous polynomial of degree n^2, which vanishes if some $x_i = 0$ or $x_k = x_l$ for some $k \neq l$. Thus $x_1 \cdot x_2 \cdot \cdots \cdot x_n \Pi_{i<j}(x_i - x_j)^2$ is a factor of $J(\Phi)$. Comparing the degree we deduce that $J(\Phi)$ is a constant multiple of $x_1 \cdot x_2 \cdot \cdots \cdot x_n \Pi_{i<j}(x_i - x_j)^2$. Thus $J(\Phi(x)) \neq 0$ for $x \in U$. To see that $J(\Phi)$ is not identically zero, it is enough to observe that $D_{ij}(x)$ is a homogeneous polynomial of degree n, linear in x_k, $k \neq i$ and is independent of x_j if $j \neq i$. Thus, the monomial $x_1 \cdot x_2^3 \cdot \cdots \cdot x_n^{2n-1}$ occurs in the expansion of $J(\Phi) = \det(D_{ij})$ precisely once: as the product $(x_1 x_2 \ldots x_n)(x_2^2 x_3 \ldots x_n)\ldots(x_{n-1}^{n-1}) \cdot x_n^n$ comes from the leading diagonal.

This proves that ϕ is an open map. □

Lemma 2. *The map $\phi: U \to V$ defined as in Lemma* 1 *is a proper and hence a closed map.*

Proof. First we prove that the map $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ is proper. Assume that $n > 1$, since for $n = 1$ polynomial maps are always proper. Now, observe that $\Phi^{-1}((0, 0, \ldots, 0)) = (0, 0, \ldots, 0)$. To see this, let $\bar{x} = (x_1, x_2, \ldots, x_n) \in \Phi^{-1}((0, 0, \ldots, 0))$. Assume that $\bar{x} \neq (0, 0, \ldots, 0)$. Note that if x_i is a root of $g_x(s)$ of multiplicity k and $f_x(x_i) = 0$ then x_i is a root of $f_x(s)$ of multiplicity $k + 1$. In our situation each x_i is a simple root of $g_x(s)$ and $f_x(x_i) = 0$. Thus each x_i is a double root of $f_x(s)$. This cannot happen for all i unless $\bar{x} = 0$. Since, in that case the degree of the polynomial $f_x(s)$ will be at least $2n > n + 1$. In particular $\Phi^{-1}((0, 0, \ldots, 0)) = (0, 0, \ldots, 0)$ even when Φ is regarded as $\Phi: \mathbb{C}^n \to \mathbb{C}^n$.

To prove that the map $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ is proper, it suffices to show that the inverse image of the unit ball is bounded. Identifying $\mathbb{R}^n \backslash (0, 0, \ldots, 0)$ with $S^{n-1} \times \mathbb{R}^+$, we can write any non-zero vector v as λv_1 where $\|v_1\| = 1$. Note that each component of $\Phi(\bar{x})$ is a homogeneous polynomial of degree $n + 1$. Thus we have

$$\begin{aligned} \|\Phi(v)\| &= \|\Phi(\lambda v_1)\| \\ &= \lambda^{n+1}\|\Phi(v_1)\| \\ &> \lambda^{n+1}\varepsilon \\ &= (\|\Phi(v)\|)^{n+1}\varepsilon, \end{aligned}$$

since $\Phi(v_1) \neq 0$ and lies on a compact set. Now for $\|\Phi(v)\| \leqslant 1$ we have

$$1 \geqslant \|\Phi(v)\| > \varepsilon\|\Phi(v)\|^{n+1}.$$

Thus, $\|\Phi(v)\|^{n+1} \leqslant 1/\varepsilon$. Hence the map $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ is proper.

Since the map $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ is proper and we have remarked that the map Φ maps boundary of U to the boundary of V, the restriction map $\Phi_U = \phi; U \to V$ is proper. This completes the proof of the lemma. □

Now, by Lemma 1, $\phi(U)$ is an open set in V and by Lemma 2 $\phi(U)$ is a closed set in V. Since V is connected, $\phi(U) = V$. Hence $\phi: U \to V$ is surjective. This completes the proof of the theorem. □

Remarks. (1) If $(b_1, \ldots, b_n) \in \mathbb{R}^n$ is such that $b_{2i-1} > 0$ for each $i = 1, \ldots, (n+1)/2$ when n is odd and $b_{2i} < 0$ for each $i = 1, \ldots, n/2$ when n is even; then there exists a polynomial $P(t)$ of degee $(n+1)$ of the form

$$P(t) = t(t - \alpha_1), \ldots, (t - \alpha_n)$$

for some $\alpha_1, \ldots, \alpha_n$, such that $\{b_1, \ldots, b_n\}$ is the set of maxima-minima for $P(t)$. Since a local homeomorphism which is a proper map is a covering projection [2], $\phi: U \to V$ is a covering map. As V is contractible, ϕ is a homeomorphism.

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Deriving Karmarkar's LP algorithm using angular projection matrix

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Abstract. Understanding Karmarkar's algorithm is both desirable and necessary for its efficient implementation, for further improvement and for carrying out complexity analysis. In this report an algorithm based on the concept of angular projection matrix, to solve linear programming problems is derived. Surprisingly, this algorithm coincides with the affine version of Karmarkar's algorithm.

Keywords. Karmarkar's algorithm; angular projection matrix.

1. Introduction

The quest for solution of general optimization problems led to applications of the differential calculus and to the development of calculus of variations. Optimization problems are of much interest because of their occurrence in numerous situations that arise in government, military, industry and in economic theory.

Classical optimization techniques have been successfully employed in solving some of the optimization problems that are encountered in the physical sciences and engineering but are not found to be amenable to many new and important optimization problems that have emerged in the field of economics.

In 1947 Dantzig had developed an algorithm known as simplex algorithm that can be applied to a special but very important class of optimization problems known as linear programming problems. But the simplex algorithm has exponential time complexity in its worst case. In 1979 Khachiyan had given a polynomial time algorithm [4] known as ellipsoid algorithm, for linear programming problems which is theoretically interesting but not superior to simplex algorithm in practice. More recently Karmarkar has given a polynomial time algorithm [2] based on projective transformation technique for linear programming problems that has not only theoretical interest, but is claimed to be superior even in practice. This result has brought about a resurgence of interest in linear programming.

Several papers have appeared since the inception of Karmarkar's algorithm in 1984. All these articles dealt with either variation, extension or application of Karmarkar's algorithm. But none of them, at least to the best of our knowledge, have attempted to unearth the sequence of ideas that led to Karmarkar's algorithm. This is important because there is a consensus among the researchers that many crucial details of the implementation of Karmarkar's algorithm are not available. In this report, a sequence of algorithms representing the sequence of ideas that led to Karmarkar's algorithm, starting with an algorithm that solves the trivial linear programming problem wherein only the objective function and positivity constraints are present, have been derived. Each algorithm is a modified version of the previous one in the sequence. Each

modification is based on the observations that are listed. The final algorithm in the sequence can be seen to coincide with the affine version of Karmarkar's algorithm [1]. Therefore, these algorithms will aid in understanding the implementation details of the Karmarkar's algorithm.

We address the trivial linear programming problem in §2 and propose an iterative algorithm to solve this problem. 'Angular' projection matrix for the general linear programming problem is discussed in §3. Special linear programming problem, considered in §4, can be seen to have commonalities with the problem considered by Karmarkar. Finally, the algorithm that we propose in §5 can be identified with the affine version of Karmarkar's algorithm.

2. Trivial problem

Consider the following problem

$$\begin{aligned} &\text{minimise} \quad c^t x \\ &\text{subject to} \quad x \geqslant 0. \end{aligned}$$

This problem is trivial because the solution can be easily obtained by setting those components of x to zero that correspond to positive components of c and other components of x to infinity. For example, if $c = [1 \; -1]^t$ then set $x_1 = 0$ and $x_2 = \infty$. We will then have either minimum or infimum depending on whether the solution is bounded or unbounded. We differ from the usual practice and propose an iterative method to solve the above problem. Given an initial feasible solution $x^0 > 0$ the iterative method converges to an optimal solution. At each iteration this method calculates a new direction vector, by 'angularly' projecting the cost vector. By 'angular' projection we shall mean projection that deviates from orthogonal projection by an amount which in our case is dependent on the distances of the currently computed point from the hyperplanes that make up the feasible region. Therefore angular projection matrix for the trivial case is computed using the distances from the axes to the currently computed point. The iterative method is given in the following algorithm. Graphical explanation of the algorithm is given in figure 1.

2.1 *Algorithm A1*

step–1 Select an initial feasible solution $x^0 > 0$.
step–2 Set $y = c$ and $k = 0$.
step–3 Compute

$$\lambda = \min \left\{ \frac{x_i^k}{y_i} : y_i > 0 \right\}$$

step–4 Compute

$$x^{k+1} = x^k - \varepsilon \lambda y$$

where $0 < \varepsilon < 1$.
step–5 If the optimum is achieved then stop.
step–6 Set $k = k + 1$, $D_k = \text{diag}\,(x_1^k x_2^k x_3^k \ldots x_n^k)$ is a diagonal matrix with $x_i^k, i = 1, 2, \ldots, n$ as diagonal entities.
step–7 Compute $y = D_k c$ and go to step 3

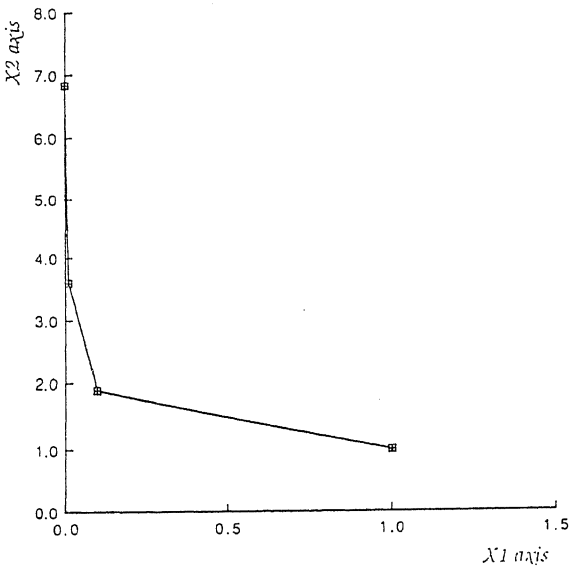

Figure 1. Algorithm A1 applied to example 1.

Initially the direction vector is set to the cost vector in step 2. Step 3 can be seen to compute the distance of the nearest axis from the point along the direction vector. Note that $-y$ is the actual direction vector because the cost function here is to be minimized. The new feasible solution is computed in step 4. If this feasible solution is an optimal solution then we stop. Otherwise in step 7, the direction vector is set to the angularly projected cost vector. Note that in the trivial case, D_k is an angular projection matrix. This fact is transparent from the following observations. The procedure is repeated by start computing λ. The algorithm is illustrated by the following numerical example.

$$\text{minimise} \quad x_1 - x_2$$

$$\text{subject to} \quad x_i \geqslant 0.$$

Note that $c^t = [1 \; -1]$. Let $x^0 = [1 \; 1]^t$ and $\varepsilon = 0{\cdot}9$. Then

$$y = [1 \; -1]^t, k = 0, \lambda = 1$$

$$x^1 = [1 \; 1]^t - 0{\cdot}9 * [1 \; -1]^t = [0{\cdot}1 \; 1{\cdot}9]^t$$

$$k = 1, D_1 = \text{diag}(0{\cdot}1 \; 1{\cdot}9)$$

$$y = [0{\cdot}1 \; -1{\cdot}9]^t, \lambda = 1$$

$$x^2 = [0{\cdot}1 \; 1{\cdot}9]^t - 0{\cdot}9 * [0{\cdot}1 \; -1{\cdot}9]^t = [0{\cdot}01 \; 3{\cdot}61]^t$$

$$k = 2, D_2 = \text{diag}(0{\cdot}01 \; 3{\cdot}61)$$

$$y = [0{\cdot}01 \; -3{\cdot}61]^t, \lambda = 1$$

$$x^3 = [0{\cdot}01 \; 3{\cdot}61]^t - 0{\cdot}9 * [0{\cdot}01 \; -3{\cdot}61]^t = [0{\cdot}001 \; 6{\cdot}859]^t$$

etc.

Observations

Let the general linear programming problem be in the following form

$$\begin{aligned} &\text{minimise} && c^t x \\ &\text{subject to} && Ax = b \\ &&& x \geqslant 0. \end{aligned}$$

Also, let

$$Q_k = I - \frac{(1 - x_1^k)}{\|e_1\|^2} e_1 e_1^t - \cdots - \frac{(1 - x_n^k)}{\|e_n\|^2} e_n e_n^t,$$

where e_i is a column vector with n components one in the ith position and zeros elsewhere, I is the identity matrix and $x^k = (x_1^k, \ldots, x_n^k)$ is a feasible solution to the trivial linear programming problem. Then

1. Q_k is an angular projection matrix that projects a given vector angularly onto the coordinate axes.
2. The coefficient matrix A is identically zero in the trivial case and hence $P = I_n = [e_1 e_2 \ldots e_n] = I$ can be interpreted as the null space of A.
3. $Q_k P = PD_k = D_k$ in the trivial case.
4. D_k is a positive definite matrix.
5. Conventionally, if the currently computed solution x^k is on a set of bounding hyperplanes, which we call the set of touching hyperplanes of the feasible region, we project the direction vector (the orthogonal projection of c onto the null space of A, denoted by c_p) orthogonally onto the intersection of the set of touching hyperplanes and proceed to compute the next solution [5]. Note that this conventional procedure takes into account only those hyperplanes at zero distance from the point but not the remaining hyperplanes and their distances. Whereas the proposed algorithm computes $Q_k Pc = PD_k c = D_k c$, angular projection of the direction vector; thereby giving weightage to all distances of the point from the hyperplanes. Therefore, if these observations are correctly translated to the general linear programming problem, it is reasonable to believe that the resulting algorithm will be computationally more efficient than the conventional algorithms. In our earlier paper [6] some of these observations were not correctly translated to the general case. Therefore the resulting algorithm inherited convergence problems, see [3].

3. Angular projection matrix for the general case

It appears that designing an algorithm for general linear programming problems parallel to algorithm A1 is fairly simple if we have a method to compute an angular projection matrix in the null space of A (the coefficient matrix of the constraint set $Ax = b$) which projects angularly a given vector and possesses the above observed properties. But computing an angular projection matrix Q that has these properties is equivalent to computing w_i's in the expression

$$Q = I - \frac{w_1}{P_{11}} P_1 P_1^t - \cdots - \frac{w_n}{P_{nn}} P_n P_n^t,$$

such that for some positive definite matrix $W, QP = PW$, where P is the orthogonal projection matrix that spans the null space of A. Note that $P_i^t P_i = P_{ii}$. P_{ii} is zero if the ith column of P, i.e. P_i, is zero. This happens if the ith coordinate axis is normal to the null space of A which is a special case and can be easily handled at the implementation level. Since P is symmetric and

$$\begin{aligned} QP &= \left[P_1 - \sum_{i=1}^{n}\left(w_i \frac{P_{1i}}{P_{ii}}\right)P_i P_2 - \sum_{i=1}^{n}\left(w_i \frac{P_{2i}}{P_{ii}}\right)P_i \ldots P_n - \sum_{i=1}^{n}\left(w_i \frac{P_{ni}}{P_{ii}}\right)P_i\right] \\ &= PW, \end{aligned}$$

it follows that

$$W = \begin{vmatrix} 1 - w_1 & -w_1\dfrac{P_{12}}{P_{11}} & \cdots & -w_1\dfrac{P_{1n}}{P_{11}} \\ -w_2\dfrac{P_{12}}{P_{22}} & 1 - w_2 & \cdots & -w_2\dfrac{P_{2n}}{P_{22}} \\ \vdots & \vdots & \vdots & \vdots \\ -w_n\dfrac{P_{1n}}{P_{nn}} & -w_n\dfrac{P_{2n}}{P_{nn}} & \cdots & 1 - w_n \end{vmatrix}.$$

We require W to be positive definite. So, first it should be symmetric. Therefore, w_i's must satisfy the following relations:

$$\frac{w_1}{P_{11}} = \frac{w_2}{P_{22}} = \cdots = \frac{w_n}{P_{nn}}. \tag{1}$$

Since diagonal dominance together with symmetry is a sufficient condition for positive definiteness, choose w_i so that they satisfy (1) and so that the choice results in a diagonal dominance of W. Assuming $|P_{12}| < P_{22}, |P_{13}| < P_{33}, \ldots$ (which may not be the case), the choice of w_i such that $\sum_{i=1}^{n} w_i = 1$, results W to be diagonally dominant and hence in positive definite matrix. But

$$\sum_{i=1}^{n} w_i = 1$$

and the relations given in (1) imply

$$w_1 = \frac{P_{11}}{\sum P_{ii}}, \ldots, w_n = \frac{P_{nn}}{\sum P_{ii}}.$$

Observations

1. These latter expressions resemble the projective transformation of Karmarkar.
2. Unlike D in the trivial case, W is not a function of the distances of the hyperplanes from the currently computed solution. Hence if the current solution is closer or farther from a hyperplane, this is not reflected in the W matrix as it does in the corresponding D matrix of the trivial case.

Since there is a breakdown in our line of thinking, let us take a fresh look at the angular projection matrix expression and write down the requirements. Writing the

angular projection matrix

$$Q = I - \frac{1 - w_1}{P_{11}} P_1 P_1^t - \cdots - \frac{1 - w_n}{P_{nn}} P_n P_n^t,$$

we have the following.

Requirements

1. w_i should be function of x_i.
2. If all P_i are parallel (that is, if $P_1 = k_j P_j$ for all j and for some constant k_j) then we have

$$Q = I - [n - (w_1 + w_2 + \cdots + w_n)] \frac{P_1 P_1^t}{P_{11}}.$$

So, $w_1 + w_2 + \cdots + w_n = n$. Otherwise, the algorithm that we design based on this angular projection matrix may trace the same path again and again without actually improving the solution.

4. Algorithm for the special linear programming problem

If we choose $w_i = x_i$ then according to our discussion, the following linear programming problem

$$\begin{aligned} \text{minimise} \quad & c^t x \\ \text{subject to} \quad & Ax = b \\ & \sum_{i=1}^{n} x_i = n \\ & x \geqslant 0 \end{aligned}$$

that has from the second requirement an extra constraint $\sum x_i = n$, can be solved by the following procedure, in which x^0 is an initial feasible solution. Note that the equation $\sum x_i = n$ and the inequalities $x_i \geqslant 0$ specify a simplex. So, the set of constraints $Ax = b$, $\sum x_i = n$, $x_i \geqslant 0$ specify a feasible region which is the intersection of the simplex and the space specified by $Ax = b$. Observe the commonality between this problem and the problem considered by Karmarkar.

4.1 *Procedure*

step–1 Set $A = \begin{bmatrix} A \\ e \end{bmatrix}, b = \begin{bmatrix} b \\ n \end{bmatrix}$, and $k = 0$, where $e = [11 \ldots 1]$.

step–2 Compute $P = I - A^+ A$ and $c_p = Pc$, where A^+ is the Moore–Penrose inverse of A.

step–3 Compute

$$\lambda = \min \left\{ \frac{x_i^k}{c_{p_i}} : c_{p_i} > 0 \right\}$$

and set $k = k + 1$.

step–4 Compute

$$x^k = x^{k-1} - \varepsilon \lambda c_p$$

where $0 < \varepsilon < 1$.

step–5 Compute

$$Q = I - \frac{(1 - x_1^k)}{P_{11}} P_1 P_1^t - \cdots - \frac{(1 - x_n^k)}{P_{nn}} P_n P_n^t.$$

step–6 Compute $c_p = QPc$ and go to step 3.

Projection matrix P whose columns span the null space of A and the initial direction vector c_p, which is the orthogonal projection of the cost vector c onto the null space of A, is computed in step 2. Step 3 computes the distance of the nearest bounding hyperplane of the convex polytope that specify the feasible region from the point along the direction vector. The new feasible solution is computed in step 4. Step 5 computes the angular projection matrix and in the last step new direction vector for the next iteration is computed by angularly projecting Pc in the null space of A.

But computational experience shows that this algorithm does not converge to an optimal solution. For example the algorithm when applied to the problem

$$\begin{aligned} \text{minimise} \quad & x_1 + 2x_2 + 3x_3 + 4x_4 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 4 \\ & x_i \geqslant 0 \end{aligned}$$

converges to [2·2322 1·2540 0·5138 0·0000] which is not an optimal solution. Optimal solution is [4 0 0 0].

However, the algorithm designed in the next section, based on this procedure, can easily be identified with the affine version of Karmarkar's algorithm and hence can solve the general linear programming problem.

5. Proposed algorithm

Assume that the procedure given in §4.1 converges to an optimal solution and hence can solve the previously mentioned special class of linear programming problems. Then the natural question that arises is the following. How can we extend this procedure to solve the general linear programming problems? A way around this problem can be found if we can find a transformation that converts a general linear programming problem to a special linear programming problem and vice-versa. A method that incorporates this idea is the following.

Let x^k denote the solution of the given linear programming problem, obtained at the kth iteration. Also, let $D_k = \mathrm{diag}(x_1^k x_2^k \ldots x_n^k)$ be a diagonal matrix. If $\|x^k - x^{k-1}\| < 2^{-L}$, where L is the word length, then x^k is an optimal solution. Otherwise, use the transformation $x = D_k y$ and append a new constraint $\sum_{i=1}^n y_i = n$ to obtain the special linear programming problem

$$\begin{aligned} \text{minimise} \quad & (D_k c)^t y \\ \text{subject to} \quad & AD_k y = b \\ & \sum_{i=1}^{n} y_i = n \\ & y \geqslant 0 \end{aligned}$$

for which $y^0 = (1 \cdots 1)$ is an initial feasible solution. Use the previous procedure to

obtain a solution y^1, which may not be optimal, of the special linear programming problem. Then compute $x^{k+1} = D_k y_1$. Increment k by one and repeat the process. Stepwise, the method is given below.

step–1 Compute an initial feasible solution x^0 of the general linear programming problem and set $k = 0$.
step–2 Set $D_k = \mathrm{diag}(x_1^k x_2^k \ldots x_n^k)$.
step–3 Set $A_d = \begin{bmatrix} AD_k \\ e \end{bmatrix}$, $j = 0$, and $y^j = [1 \cdots 1]$.
step–4 Compute $P = I - A_d^+ A_d$ and $c_p = PD_k c$.
step–5 Compute

$$\lambda = \min \left\{ \frac{y_i^j}{c_{p_i}} : c_{p_i} > 0 \right\}.$$

step–6 Compute

$$y^{j+1} = y^j - \varepsilon \lambda c_p$$

where $0 < \varepsilon < 1$ and set $j = j + 1$.

step–7 Compute

$$Q = I - \frac{(1 - y_1^j)}{P_{11}} P_1 P_1^t - \cdots - \frac{(1 - y_n^j)}{P_{nn}} P_n P_n^t.$$

step–8 Compute $c_p = QPD_k c$.
step–9 If $\|y^j - y^{j-1}\| \geqslant 2^{-L}$ go to step 5.
step–10 Compute $x^{k+1} = D_k y^j$ and set $k = k + 1$.
step–11 If $\|x^k - x^{k-1}\| \geqslant 2^{-L}$ go to step 2.

Steps 2, 3 and 4 transform the general linear programming problem to a special linear programming problem for which $y^0 = [1 \cdots 1]$ is an initial feasible solution, compute orthogonal projection matrix P that span the null space of A_d and compute initial direction vector c_p of the special linear programming problem. Steps 5 to 9 constitute the procedure described in § 4.1 to solve the special linear programming problem. Step 10 computes an improved solution of the general linear programming problem from that of the special linear programming problem.

Observations

1. Numerical calculations reveal that the algorithm converges to an optimal solution.
2. Repetition of steps 5 to 9 is not contributing to significant improvement of the solution.
3. Computational requirement of the steps 5 to 9 is of the order of n^3, n is the number of columns of A.

These observations motivate us to design the following algorithm. This new algorithm does away with angular projection matrix and related computation. It can be readily identified with the affine version of Karmarkar's algorithm [see § 1 of 1].

5.1 *The algorithm*

step–1 Compute an initial feasible solution x^0 of the given linear programming problem and set $k = 0$, $y_0 = e = [1 \ldots 1]$.

step–2 Set $D_k = \mathrm{diag}(x_1^k x_2^k \dots x_n^k)$.

step–3 Set $A_d = \begin{bmatrix} AD_k \\ e \end{bmatrix}$.

step–4 Compute $P = I - A_d^+ A_d$ and $c_p = PD_k c$,

step–5 Compute

$$\lambda = \min\left\{\frac{1}{c_{p_i}} : c_{p_i} > 0\right\}.$$

step–6 Compute

$$y^1 - y^0 - \varepsilon\lambda c_p$$

where $0 < \varepsilon < 1$.

step–7 Compute $x^{k+1} = D_k y^1$ and set $k = k + 1$.

step–8 If $\|x^k - x^{k-1}\| \geqslant 2^{-L}$ go to step 2.

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Continuity of homomorphism pairs into H^*-triple systems

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Abstract. We show that the separating subspaces for the component operators of a densely valued homomorphism pair into an H^*-triple system are contained in the annihilator ideal. Accordingly, the continuity of densely valued homomorphisms into H^*-algebras and H^*-triple systems with zero annihilator follows.

Keywords. Homomorphism pairs; triple system.

1. Introduction

From the celebrated theorem by Johnson [14] our knowledge of the continuity properties of homomorphisms onto Banach algebras is fairly complete and satisfactory, even in the non-associative context [19]. However, it is still not currently known if densely valued homomorphisms into semisimple associative Banach algebras are continuous, even when the domain algebra and the coming algebra are C^*-algebras. Rodríguez contributed in [20] to this old open problem stating the continuity of densely valued homomorphisms into (non-associative) H^*-algebras with zero annihilator (we note that, by [3; Lemma 34.9], an associative H^*-algebra has zero annihilator if and only if it is semisimple).

As it has been announced in the abstract, in the present paper our main goal is the study of the continuity properties of densely valued homomorphism pairs into an H^*-triple system.

Since the paper by Ambrose [1] introduced associative H^*- algebras and provided the corresponding structure theory, the frame of H^*-algebras has been extensively studied, even in the general non-associative context [11, 12] and successfully developed in several directions. Concerning results on automatic continuity in this context, the first author proved in [23] that the separating subspace for a derivation on an H^*-algebra is contained in the annihilator. The most important novelty of that paper was the introduction of techniques of central closeability in the treatment of automatic continuity problems. This line was successfully exploited in [20, 24, 25].

It should be pointed out the extensive treatment of some ternary H^*-structures [4–10, 16, 18, 25–27]. Specially outstanding are the Hilbert triple systems introduced by Kaup in [16], who used them to solve the classification problem for a certain type of manifolds. H^*-triple systems are a ternary version of the H^*-algebras introduced in [5] for the Jordan case and they are a generalization of the Hilbert triple systems of [16, 18].

We note that, in the papers cited above, a nice structure theory has been developed for H^*-algebras, Hilbert triple systems and H^*-triple systems. Actually two crucial results in the theory of H^*-triple systems will be used.

Theorem 1 [7]. *Every H^*-triple system W is an orthogonal sum*

$$W = \mathrm{Ann}(W) \oplus W_0$$

where $\mathrm{Ann}(W)$ *denotes the annihilator of W and W_0 is an H^*-triple system with zero annihilator.*

Theorem 2 [7]. *Every H^*-triple system with zero annihilator is the orthogonal sum of its minimal closed ideals which are topologically simple H^*-triple systems themselves.*

An extensive investigation about the continuity of derivation pairs on Hilbert triple systems was achieved in [25]. In that paper, the theorem on derivations on H^*-algebras was extended to Hilbert triple systems and central closeability was the fundamental tool. Now it is reasonable to extend the Rodríguez theorem on homomorphisms [20] to this ternary context.

2. Central closeability in triple systems

A linear space V over a field $\mathbb{K}$ endowed with a tri-linear triple product $[\cdots]: V \times V \times V \to V$ is said to be a *triple system.* A *normed triple system* is a real or complex triple system V whose underlying linear space is a normed space and the triple product is jointly continuous. We define *Banach triple systems* as those normed triple systems whose underlying normed space is a Banach space.

A real or complex triple system W is said to be an *H^*-triple system* when the underlying linear space is a Hilbert space endowed with an involutive mapping $*$, which is linear in the real case and conjugate-linear in the complex one, and satisfies $[xyz]^* = [x^* y^* z^*]$ and

$$([xyz]|w) = (x|[wz^* y^*]) = (y|[z^* w x^*]) = (z|[y^* x^* w])$$

for all $x, y, z, w \in W$. Here and subsequently $(\cdot|\cdot)$ denotes the inner product of the Hilbert space W. For a given continuous linear operator F on W we denote by $F^\bullet$ its adjoint operator.

It is easy to verify that, for a given H^*-triple system W, $\{x \in W : [xWW] = 0\} = \{x \in W : [WxW] = 0\} = \{x \in W : [WWx] = 0\}$. We write $\mathrm{Ann}(W)$ to denote the above set and it shall be called the *annihilator* of W.

Let V be a triple system. A subspace U of V is said to be a *subtriple* when $[UUU] \subset U$. If $x, y, z \in V$, then we denote by $L(x, y)$, $M(x, z)$, and $R(y, z)$ the linear operators of left, middle, and right multiplication on V defined by $L(x, y)(z) = M(x, z)(y) = R(y, z)(x) = [xyz]$, respectively. We define the multiplication algebra of V, denoted by $M(V)$, as the subalgebra of $L(V)$ (the algebra of all linear operators on V) generated by all multiplication operators on V. For a given subtriple U of V let us denote by $M_U(V)$ the subalgebra of $M(V)$ generated by the multiplication operators $L(x, y)$, $M(x, z)$, and $R(y, z)$ where $x, y, z \in U$. Note that, for a normed triple system V, every element in $M(V)$ is obviously a continuous linear operator on V. Furthermore, if U is a dense subtriple of V, then the elements of $M(U)$ are nothing but the restriction to U of those of $M_U(V)$. This fact will be used in the sequel without previous comment. Given elements x, y, and z in an H^*-triple system W, we have $L(x, y)^\bullet = L(y^*, x^*)$, $M(x, z)^\bullet = M(z^*, x^*)$, $R(y, z)^\bullet = R(z^*, y^*)$, which immediately implies the joint continuity of the triple product. Therefore every H^*-triple system is a Banach triple system.

Observe that, for a subtriple U of the H^*-triple system W, $(M_U(W))^* = M_{U^*}(W)$. Accordingly, $M(W)$ is a self-adjoint subalgebra of the algebra of all continuous linear operator on the Hilbert space W.

A subspace I of a triple system V is said to be an *ideal* if $[IVV] + [VIV] + [VVI] \subset I$. The definition of the quotient triple system is left to the imagination of the reader.

We say that a triple system V is *prime* if $[IVJ] = 0$ implies either $I = 0$ or $J = 0$, when I and J are ideals of V. A normed triple system V is called *topologically simple* if $[VVV] \neq 0$ and V has no non-trivial closed ideals.

PROPOSITION 1

Let V be a prime (respectively, topologically simple) normed triple system. If U is a dense subtriple of V, then U is prime (respectively, topologically simple).

Proof. It is easy to check that the closure $\bar{I}$ in V of every ideal I of U is an ideal in V and the result follows. □

Central closeability seems to be a powerful tool in the treatment of the automatic continuity problems into H^*-structures, whose use was started in an earlier work [23] about the continuity of derivations on H^*-algebras. Successively this technique has been used in the investigation of the continuity of densely valued homomorphisms into H^*-algebras [20], random derivations on H^*-algebras [24] and derivation pairs on Hilbert triple systems [25]. So it is not surprising that in order to investigate the continuity of homomorphisms we also apply techniques of central closeability. In the present paper several essential and supporting results in this direction are stated.

A *centralizer* on a triple system V is a linear map $f\colon V \to V$ commuting with all the elements $P \in M(V)$. The set of all the centralizers on V is called the *centroid* of V and we denote it as $Z(V)$. Note that V may be viewed as a $Z(V)$-module. A *partially defined centralizer* on V is a linear operator f from a suitable non-zero ideal I of V into V satisfying:

$$f([xyz]) = [f(x)yz], \quad f([yxz]) = [yf(x)z], \quad \text{and} \quad f([zyx]) = [zyf(x)]$$

for all $x \in I$ and $y, z \in V$.

In the sequel the triple systems V satisfying the following conditions will play a prominent role:

C1: Every partially defined centralizer on V can be extended to an everywhere defined centralizer on V.

C2: The centroid $Z(V)$ determines a finite extension of the ground field.

Actually, if in addition $Z(V)$ is the ground field, then the triple system V is said to be *centrally closed.*

Theorem 3. *Let W be a topologically simple H^*-triple system. Then W satisfies the conditions* C1 *and* C2. *Actually, $Z(W) = \mathbb{C}$ in the complex case and $Z(W)$ is either $\mathbb{R}$ or $\mathbb{C}$ in the real case.*

Proof. On account of [10; Theorem 3], if W is a complex H^*-triple system, then $Z(W) = \mathbb{C}I_V$ and following the pattern established in [25; Lemmas 1–3 and Theorem 2], it may be concluded that W is centrally closed.

Now we assume that W is a real H^*-triple system. In this case, from [10; Theorem 3], $Z(W)$ is either the real or complex field and C2 follows. If W is the realification of a topologically simple complex H^*-triple system, then a straightforward verification, taking into account what has already been proved, shows that the condition C1 is satisfied and actually $Z(W) = \mathbb{C}I_W$. Otherwise, according to [10; Proposition 6], the complexified $W_{\mathbb{C}}$ (see [10; Proposition 5]) is a complex topologically simple H^*-triple system. Partially defined centralizers on W could be complexified to get partially defined centralizers on $W_{\mathbb{C}}$ which can be extended to everywhere defined centralizers on $W_{\mathbb{C}}$. Therefore C1 follows. □

PROPOSITION 2

Let U be a dense subtriple of a prime normed triple system V satisfying C1. *Then every partially defined centralizer on U can be extended to a centralizer on V. Accordingly, if in addition V satisfies* C2, *then the partially defined centralizers on U, $C(U)$, are a finite extension of the ground field with* $\dim_{\mathbb{K}} C(U) \leqslant \dim_{\mathbb{K}} Z(V)$.

Proof. Let f be a partially defined centralizer on U. Let I be the set of those x in V such that $x = \lim f(x_n)$ for some sequence $\{x_n\}$ in the domain of f, denoted by $\mathrm{dom}(f)$, converging to zero. It follows easily that I is a closed ideal of V.

Now let $x \in I$, then $x = \lim f(x_n)$ for some sequence $\{x_n\}$ in $\mathrm{dom}(f)$ converging to zero. For all $y \in U$ and $z \in \mathrm{dom}(f)$ we have:

$$[xyz] = \lim f(x_n)yz] = \lim [f(x_n)yz] = \lim [x_n y f(z)] = 0.$$

Thus we have $[IU\,\mathrm{dom}(f)] = 0$ and so $[IV\overline{\mathrm{dom}(f)}] = 0$. By primeness $I = 0$ and this means that f is closeable as a partially defined operator from V into V. In the next we prove that the closure of f, say f^-, is a partially defined centralizer on V. We have to verify first that the domain, say J, of f^- is an ideal of V. Given $x \in J$ and $y, z \in V$ there exist sequences $\{x_n\}$ in $\mathrm{dom}(f)$ and $\{y_n\}$, $\{z_n\}$ in U converging to x, y and z respectively, and $f^-(x) = \lim f(x_n)$. On the other hand $\lim [x_n y_n z_n] = [xyz]$ and $\lim f([x_n y_n z_n]) = \lim [f(x_n) y_n z_n] = [f^-(x)yz]$, which shows that $[xyz] \in J$ and $f^-([xyz]) = [f^-(x)yz]$. In a similar way we obtain that $[yxz]$, $[zyx] \in J$, $f^-([yxz]) = [yf^-(x)z]$, and $f^-[zyx]) = [zyf^-(x)]$. Therefore f^- is a partially defined centralizer with domain J into V. By C1, f^- can be extended to an everywhere defined centralizer on V. □

From Theorem 3, Proposition 1 and the preceding result we obtain the following.

COROLLARY 1

Every dense subtriple U of a topologically simple H^*-triple system V with $Z(V) = \mathbb{K}$ is topologically simple and satisfies C1 *and* C2.

Following the pattern established in [23–25] our next goal consists of providing suitable sequences for the continuity problem. To this end we start obtaining an analogous result to [13; Theorem 3.1].

Lemma 1. *Let V be a triple system satisfying* C1. *Given $x_1, \ldots, x_n \in V$ $Z(V)$-linearly independent, there is $P \in M(V)$ such that $P(x_1) \neq 0$ and $P(x_k) = 0$ for $1 < k \leqslant n$.*

Proof. For $n=1$ the result is trivially true and we assume inductively that the result holds for n. Now we assume that for all $P \in M(V)$, if $P(x_k)=0$ for $k=2,\ldots,n+1$ then $P(x_1)=0$. Define $\mathscr{I}=\{P \in M(V): P(x_k)=0 \text{ for } k=2,\ldots,n\}$. Since obviously $\mathscr{I}$ is a left ideal of $M(V)$, $I=\{P(x_{n+1}): P \in \mathscr{I}\}$ is an ideal of V. Furthermore by induction assumption I is non-zero. The mapping $f: I \to V$ given by $P(x_{n+1}) \mapsto P(x_1)$ is well defined and is clearly a partially defined centralizer on V. So it can be extended to a centralizer on V which, for abbreviation, we continue to write it as f. Since $0=P(x_1)-f(P(x_{n+1}))=P(x_1-f(x_{n+1}))$ for each P in $\mathscr{I}$. By the induction assumption applied to $x_1-f(x_{n+1}), x_2, \ldots, x_n$ there exists $P \in M(V)$ such that $P(x_k)=0$, $k=2,\ldots,n$ (i.e. $P \in \mathscr{I}$) but $P(x_1-f(x_{n+1})) \neq 0$, which is a contradiction. □

Lemma 2. *Let V be a topologically simple normed triple system for which $Z(V)$ is a field and let G be a non-empty open subset of V. If $P \in M(V)$ and* $\dim_{Z(V)} P(V)=1$ *then, there exists $Q \in M(V)$ such that $(QP)^2(V) \cap G \neq \varnothing$.*

Proof. Let $y \in V$ such that $P(V)=Z(V)y$ and let $x \in V$ such that $P(x)=y$. Since $P \neq 0$, $G \not\subset \ker(P)$ and so $(V \backslash \ker(P)) \cap G$ is a nonempty open subset of V. From the topological simplicity of V it follows that there is $Q \in M(V)$ such that $Q(y) \in (V \backslash \ker(P)) \cap G$. Since $Q(y) \notin \ker(P)$ we have $Q(y)=f(x)+z$, for suitable $f \in Z(V)$ and $z \in \ker(P)$. Therefore

$$(QP)^2(f^{-1}x)=f^{-1}(QPQ)y=f^{-1}(QP)(fx+z)=f^{-1}Q(fy)=Q(y) \in G. \quad \square$$

Now we follow the traditional sliding hump procedure. To do this we construct as in [24] appropriate sequences having amazing properties, which allow us to put powerful automatic continuity principles into action.

Theorem 4. *Let V be a topologically simple normed triple system for which $Z(V)$ is a field and* $\dim_{Z(V)} V=\infty$ *and let G be a non-empty open subset of V which does not contain the zero. If V satisfies* C1, *then one of the following assertions holds*:

1. *There exist sequences $\{x_n\}$ in V and $\{P_n\}$ in $M(V)$ such that*:

$$P_{n+1} \cdots P_1 x_n = 0 \quad \textit{and} \quad P_n \cdots P_1 x_n \in G \; \forall n \in \mathbb{N}.$$

2. *There exists a sequence $\{Q_n\}$ in $M(V)$ such that*:

$$\dim_{Z(V)} Q_n(V)=1 \qquad \forall n \in \mathbb{N},$$

$$Q_n^2(V) \cap G \neq \varnothing \quad \forall n \in \mathbb{N},$$

$$Q_n Q_m = 0 \quad \text{if } m<n.$$

Proof. In the first step we prove that 1 is obtained if $\dim_{Z(V)} P(V) \geqslant 2$ for all $P \in M(V) \backslash \{0\}$. From the topological simplicity of V, we obtain $x_1 \in V$ and $P_1 \in M(V)$ such that $P_1 x_1 \in G$ and suppose that $x_1, \ldots, x_k$ and $P_1, \ldots, P_k$ have been choosen inductively such that $P_j \ldots P_1 x_j \in G$ and $P_j \ldots P_1 x_{j-1}=0$ for $j=2,\ldots,k$. Since $\dim_{Z(V)}(P_k \ldots P_1)(V) \geqslant 2$, there exists $x_{k+1} \in V$ such that $P_k \ldots P_1 x_k$ and $P_k \ldots P_1 x_{k+1}$ are $Z(V)$-linearly independent, and so by Lemma 1, there exists $P \in M(V)$ such that $PP_k \ldots P_1 x_k=0$ and $PP_k \ldots P_1 x_{k+1} \neq 0$. From the topological simplicity of V there is $Q \in M(V)$ such that $QPP_k \ldots P_1 x_{k+1} \in G$ and we define $P_{k+1}=QP$. The sequences $\{x_n\}$ and $\{P_n\}$ constructed in this way satisfy the requirements of the first assertion.

We assume that assertion 2 does not hold and we deduce the first one. Certainly we can assume that there is $P\in M(V)$ such that $\dim_{Z(V)}P(V)=1$ and we apply Lemma 2 to obtain $R\in M(V)$ with $\dim_{Z(V)}R(V)=1$ and $R^2(V)\cap G\neq\emptyset$. Let $\mathcal{Q}$ be the set of all finite sequences $\{Q_n\}_{n=1}^N$ in $M(V)$ satisfying:

$$\dim_{Z(V)}Q_n(V)=1 \qquad n=1,2,\ldots,N,$$

$$Q_n^2(V)\cap G\neq\emptyset \qquad n=1,2,\ldots,N,$$

$$Q_nQ_m=0 \text{ if } 1\leqslant m<n\leqslant N.$$

Since the singleton $\{R\}$ lies in $\mathcal{Q}$, this set is non-empty. Moreover, we consider in $\mathcal{Q}$ the partial order defined by:

$$\{Q_n\}_{n=1}^N\leqslant\{Q'_m\}_{m=1}^M \quad\text{iff}\quad N\leqslant M \quad\text{and}\quad Q_n=Q'_n \quad\text{for}\quad n=1,\ldots,N.$$

There is a maximal element $\{Q_n\}_{n=1}^N$ in $\mathcal{Q}$, because we assume that assertion 2 does not hold. For every $n=1,\ldots,N$, let $y_n\in V$ such that $Q_n(V)=Z(V)y_n$ and $Q_n(y_n)\neq 0$. From the equality $Q_nQ_m=0$ if $1\leqslant m<n\leqslant N$, we deduce that $\{y_1,\ldots,y_N\}$ is a family of $Z(V)$-linearly independent elements in V. Now let $x_1\in V$ such that $\{y_1,\ldots,y_N,x_1\}$ is a family of $Z(V)$-linearly independent elements and, by Lemma 1, let $P_1\in M(V)$ such that $P_1(y_1)=\cdots=P_1(y_N)=0$ and $P_1(x_1)\neq 0$. If $\dim_{Z(V)}P_1(V)=1$ then we apply Lemma 2 to obtain $Q\in M(V)$ such that the element $Q_{N+1}=QP_1$ has the property $Q_{N+1}^2(V)\cap G\neq\emptyset$, trivially $\dim_{Z(V)}Q_{N+1}(V)=1$, and also $Q_{N+1}Q_n=0$ for all $n=1,\ldots,N$. Thus, the sequence $\{Q_n\}_{n=1}^{N+1}$ lies in $\mathcal{Q}$ which contradicts the maximality of $\{Q_n\}_{n=1}^N$. Therefore $\dim_{Z(V)}P_1(V)\geqslant 2$, and thus there exists $x_2\in V$ such that $P_1(x_1)$ and $P_1(x_2)$ are $Z(V)$-linearly independent. Now let $P_2\in M(V)$ with $P_2P_1(x_1)=0$ and $P_2P_1(x_2)\in G$. Reasoning as above, we deduce that $\dim_{Z(V)}(P_2P_1(V)\geqslant 2$. We suppose that $x_1,\ldots,x_k$ and $P_1,\ldots,P_k$ have been choosen inductively such that $P_j,\ldots P_1x_{j-1}=0$ and $P_j\ldots P_1x_j\in G$ with $\dim_{Z(V)}(P_j\ldots P_1)(V)\geqslant 2$ for $j=2,\ldots,k$. We get $x_{k+1}\in V$ such that $P_k\ldots P_1x_k$ and $P_k\ldots P_1x_{k+1}$ are $Z(V)$-linearly independent. Therefore, by Lemma 1 and topological simplicity there exists an element $P_{k+1}\in M(V)$ such that $P_{k+1}\ldots P_1x_k=0$ and $P_{k+1}\ldots P_1x_{k+1}\in G$. If $\dim_{Z(V)}(P_{k+1}\ldots P_1)(V)=1$, we apply Lemma 2, to obtain an element $Q_{N+1}=QP_{k+1}\ldots P_1$ in $M(V)$ satisfying $Q_{N+1}^2(V)\cap G\neq\emptyset$, clearly $\dim_{Z(V)}Q_{N+1}(V)=1$, and $Q_{N+1}Q_n=0$ for $n=1,\ldots,N$. Consequently $\dim_{Z(V)}(P_{k+1}\ldots P_1)(V)\geqslant 2$. Finally the sequences $\{x_n\}$ in V and $\{P_n\}$ in $M(V)$ constructed in this way satisfy assertion 1. □

According to Corollary 1, we may apply the preceding theorem to get the following.

COROLLARY 2

Let U be an infinite-dimensional dense subtriple of a topqlogically simple H^*-triple system V with $Z(V)=\mathbb{K}$. Then one of the following conditions holds:

S1: *There exist sequences $\{x_n\}$ in U and $\{P_n\}$ in $M(U)$ such that:*

$$P_{n+1}\ldots P_1x_n=0 \quad \textit{and} \quad P_n\ldots P_1x_n\neq 0 \quad \forall n\in\mathbb{N}.$$

S2: *There exists a sequence $\{Q_n\}$ in $M(U)$ such that:*

$$\dim_{\mathbb{K}}Q_n(U)=1 \qquad \forall n\in\mathbb{N},$$

$$Q_n^2\neq 0 \qquad \forall n\in\mathbb{N},$$

$$Q_nQ_m=0 \quad \textit{if} \quad m<n.$$

3. Continuity of homomorphisms and homomorphism pairs

Let U and V be triple systems. A linear map $\Phi: U \to V$ is called an *homomorphism* if

$$\Phi([xyz]) = [\Phi(x)\Phi(y)\Phi(z)]$$

for all $x, y, z \in U$.

For a given homomorphism $\Phi: U \to V$, it is easy to check that the image $\Phi(U)$ is a subtriple of V and the kernel $\ker\Phi$ is an ideal in U.

Given $x, y, z \in U$, it follows easily that $\Phi L(x, y) = L(\Phi(x), \Phi(y))\Phi$, $\Phi M(x, z) = M(\Phi(x), \Phi(z))\Phi$, and $\Phi R(y, z) = R(\Phi(y), \Phi(z))\Phi$. So the subalgebra $\mathcal{A}$ of those $P \in M(U)$ for which there is $Q \in M_{\Phi(U)}(V)$ such that $\Phi P = Q\Phi$, contains all left, middle, and right multiplication operators on U and therefore equals $M(U)$. If U and V are normed triple systems and Φ is a densely valued homomorphism from U into V we obtain for every P in $M(U)$ a unique element $\phi(P)$ in $M_{\Phi(U)}(V)$ such that $\Phi P = \phi(P)\Phi$. So we can define an algebra homomorphism ϕ from $M(U)$ onto $M_{\Phi(U)}(V)$.

Recall that we can measure the continuity of an operator F acting between Banach spaces X and Y by considering its so-called *separating subspace*

$$\mathscr{S}(F) = \{y \in Y: \lim F(x_n) = y \text{ and } \lim x_n = 0\}.$$

By the closed graph theorem it follows that F is continuous if, and only if, $\mathscr{S}(F) = 0$.

It is easy to check that the separating subspace $\mathscr{S}(\Phi)$ for a densely valued homomorphism Φ between Banach triple systems U and V is a closed ideal in V.

For studying the continuity properties of densely valued homomorphisms into H^*-triple systems we first require two fundamental principles from automatic continuity. The first one was stated by M P Thomas in [22; Proposition 1.3], but it should be pointed out that the underlying principle was used earlier by many authors [15, 17, 23, 24].

PROPOSITION 3

Let X be a Banach space, $\{S_n\}$ a sequence of continuous linear operator from X into itself and $\{R_n\}$ be a sequence of continuous linear operators whose domain is X but which may map into others Banach spaces Y_n. If F is a possibly discontinuous linear operator from X into itself, such that $R_n F S_1 \ldots S_m$ is continuous for $m > n$, then $R_n F S_1 \ldots S_n$ is continuous for sufficiently large n.

The second principle seems to be unstated, but as before the underlying principle was used by many authors [2, 24].

PROPOSITION 4

Let X and Y be Banach spaces and F a possibly discontinuous linear operator from X into Y. If $\{S_n\}$ and $\{R_n\}$ are sequences of continuous linear operators on X and Y respectively such that $FS_n - R_n F$ is continuous for all $n \in \mathbb{N}$ and $S_n S_m = 0$ if $n < m$ then FS_n^2, and so $R_n^2 F$, is continuous for sufficiently large n.

Proof. Suppose that the result fails. It is then possible to choose a strictly increasing sequence $\{n_k\}$ of natural numbers and a sequence $\{x_k\}$ in X satisfying:

(i) $\|x_k\| < \dfrac{1}{2^k \|S_{n_k}\|} \forall k \in \mathbb{N}$,

(note that $S_{n_k} \neq 0$).

(ii) $\|FS_{n_k}^2(x_k)\| > k\|R_{n_k}\| + \left\|F\left(\sum_{j=1}^{k-1} S_{n_k}S_{n_j}(x_j)\right)\right\| + \|R_{n_k}F - S_{n_k}F\| \; \forall k\in\mathbb{N}$.

Defining x in X as $x = \sum_{j=1}^{\infty} S_{n_j}(x_j)$ we have, for all $k\in\mathbb{N}$:

$$FS_{n_k}(x) = F\left(\sum_{j=1}^{\infty} S_{n_k}S_{n_j}(x_j)\right) = FS_{n_k}^2(x) + F\left(\sum_{j=1}^{k-1} S_{n_k}S_{n_j}(x_j)\right),$$

and so

$$\begin{aligned}\|R_{n_k}F(x)\| &\geqslant \|FS_{n_k}x\| - \|(R_{n_k}F - FS_{n_k})x\| \\ &\geqslant \|FS_{n_k}^2 x\| - \left\|F\left(\sum_{j=1}^{k-1} S_{n_k}S_{n_j}(x_j)\right)\right\| - \|R_{n_k}F - FS_{n_k}\| \\ &> k\|R_{n_k}\|.\end{aligned}$$

Hence $\|F(x)\| > k$ for every $k\in\mathbb{N}$ which is impossible. □

Lemma 3. *Let Φ be a densely valued homomorphism from a Banach triple system V into an infinite-dimensional topologically simple H^*-triple system W. If $(\Phi(V))^*$ satisfies* S1, *then Φ is continuous.*

Proof. There exist sequences $\{x_n\}$ in $(\Phi(V))^*$ and $\{P_n\}$ in $M_{(\Phi(V))^*}(W) = (M_{\Phi(V)}(W))^*$ such that: $P_{n+1}\ldots P_1 x_n = 0$ and $P_n\ldots P_1 x_n \neq 0$ $\forall n\in\mathbb{N}$. For every $n\in\mathbb{N}$, we choose $S_n\in M(V)$ such that $\phi(S_n) = P_n^*$. Let R_n be the continuous linear operator defined on W by $R_n(x) = (x|x_n)$. For all $x\in V$ and $m,n\in\mathbb{N}$, we have:

$$\begin{aligned}R_n\Phi S_1\ldots S_m x &= (\Phi S_1\ldots S_m x|x_n) = (\phi(S_1\ldots S_m)\Phi x|x_n) \\ &= (\phi(S_1)\ldots\phi(S_m)\Phi x|x_n) = (P_1^*\cdots P_m^*\Phi x|x_n) \\ &= (\Phi x|P_m\ldots P_1 x_n)\end{aligned}$$

which equals zero if $m > n$. From Proposition 3 it follows the continuity of the operator $R_n\Phi S_1\ldots S_n$, and so the continuity of the functional $x\mapsto(\Phi x|P_n\ldots P_1 x_n)$ on V, for sufficiently large n. The linear subspace of W defined by $I = \{y\in W : x\mapsto(\Phi(x)|y)$ is continuous on $V\}$ is closed by the Banach-Steinhauss theorem and it is easy to check that

$$[I(\Phi(V))^*(\Phi(V))^*] + [(\Phi(V))^* I(\Phi(V))^*] + [(\Phi(V))^*(\Phi(V))^* I] \subset I.$$

Since $(\Phi(V))^*$ is dense in W, I is a closed ideal of W containing the non-zero elements $P_n\ldots P_1 x_n$, for sufficiently large n. Hence I equals W by topological simplicity. The continuity of every functional $x\mapsto(\Phi(x)|y)$ $(y\in W)$ shows that $\mathscr{S}(\Phi) = 0$ and consequently the continuity of Φ. □

Lemma 4. *Let Φ be a densely valued homomorphism from a Banach triple system V into a topologically simple H^*-triple system W with $Z(V) = \mathbb{K}$. Then* $\ker\Phi$ *is closed in V.*

Proof. It is obvious that $\ker\Phi$ is an ideal in V and $\overline{\Phi(\overline{\ker\Phi})}$ is a closed ideal in W. If $\overline{\Phi(\overline{\ker\Phi})} = 0$, then $\overline{\ker\Phi} = \ker\Phi$ and the claim follows. Otherwise, by topological simplicity, $\overline{\Phi(\overline{\ker\Phi})} = W$ and the restriction $\Psi = \Phi_{|\overline{\ker\Phi}}$ is a densely valued homomorphism

from $\overline{\ker\Phi}$ into W. Let us denote by ψ the algebra homomorphism from $M(\overline{\ker\Phi})$ onto $M_{\Psi(\overline{\ker\Phi})}(W)$ associated to Ψ.

Assume that W has infinite dimension. $(\Psi(\overline{\ker\Phi}))^*$ is infinite-dimensional and, by Corollary 2, satisfies either S1 or S2.

If $(\Psi(\overline{\ker\Phi}))^*$ satisfies S1, then the preceding lemma shows that Ψ is continuous and so $\ker\Psi$ is closed in $\overline{\ker\Phi}$. Since $\ker\Psi = \ker\Phi$, this is a contradiction.

On the other hand, if $(\Psi(\overline{\ker\Phi}))^*$ satisfies S2, then there is Q in $M_{(\Psi(\overline{\ker\Phi}))^*}(W)$ with $\dim_{\mathbb{K}} Q(W)=1$ and $Q^2 \neq 0$. Since $\dim_{\mathbb{K}} Q^\bullet(W)=1$ there exist $x_1, x_2 \in W$ such that $Q^\bullet(x)=(x|x_1)x_2 \forall x \in W$. Furthermore $(x_2|x_1)\neq 0$, since $(Q^\bullet)^2 \neq 0$. Set $S=(x_2|x_1)^{-1} Q^\bullet \in M_{\Psi(\overline{\ker\Phi})}(W)$ and note that $S\neq 0$ and $S^2=S$. Let $\overline{\ker\psi}$ denote the closure in $M(\overline{\ker\Phi})$ of $\ker\psi$, which is obviously a subalgebra of $M(\overline{\ker\Phi})$ containing all multiplication operators on $\overline{\ker\Phi}$. Therefore $\overline{\ker\psi} = M(\overline{\ker\Phi})$. Since $S \in M_{\Psi(\overline{\ker\Phi})}(W)$ there is P in $\overline{\ker\psi}$ such that $\psi(P)=S$ and we choose $R \in \ker\psi$ such that $\|R-P\|<1$. The completeness of the domain, $\overline{\ker\Phi}$, implies that $(I_{\overline{\ker\Phi}} - P + R)F = I_{\overline{\ker\Phi}}$ for a suitable invertible continuous operator F from $\overline{\ker\Phi}$ onto itself. Hence

$$
\begin{aligned}
\Psi &= \Psi(I_{\overline{\ker\Phi}} - P + R)F \\
&= (\Psi - \Psi P + \Psi R)F = (\Psi - \psi(P)\Psi + \psi(R)\Psi)F \\
&= (\Psi - S\Psi)F = (I_W - S)\Psi F
\end{aligned}
$$

and so $I_W - S$ is a densely valued continuous linear operator from W into itself. Since $S(I_W - S)=0$, we conclude that $S=0$, a contradiction.

Assume that W has finite dimension. Let $\{y_1,\ldots,y_m\}$ a basis in W. By Lemma 1, there is $Q \in M(W)$ such that $Q(y_1)\neq 0$ and $Q(y_k)=0$ $k=2,\ldots,m$. It is clear that $\dim_{\mathbb{K}} Q(W)=1$ and we apply Lemma 2 to get $R\in M(W)$ with $\dim_{\mathbb{K}} R(W)=1$ and $R^2\neq 0$. Since $\Psi(\overline{\ker\Phi}) = W$, $R^\bullet \in M_{\Psi(\overline{\ker\Phi})}(W)$ and we take $S=\psi^{-1}(R^\bullet)\in M(\overline{\ker\Phi})$. Now we argue as in the preceding step to get a contradiction. □

PROPOSITION 5

Let Φ be a densely valued homomorphism from a Banach triple system V into a topologically simple H^*-triple system W with $Z(W)=\mathbb{K}$. Then Φ is continuous.

Proof. By the preceding result we can drop the homomorphism Φ into a densely valued homomorphism Ψ from the Banach triple system $U = V/\ker\Phi$ into W. Let us denote by ψ the associated algebra isomorphism from $M(U)$ onto $M_{\Psi(U)}(W)$. It suffices to prove that Ψ is continuous. To do this we need only consider an infinite-dimensional W. In such a case $(\Psi(U))^*$ satisfies either S1 or S2.

If $(\Psi(U))^*$ satisfies S1, then according to Lemma 3, we have Ψ continuous.

Otherwise, $(\Psi(U))^*$ satisfies S2. For each $n\in\mathbb{N}$, we define $S_n=\psi^{-1}(Q_n^\bullet)$. Also we define $R_n = Q_n^\bullet$. The operators $\Psi S_n - R_n\Psi$ equals zero, and so they are continuous. Moreover, it is easy to check that $S_nS_m=0$ if $n<m$. Proposition 4 shows that the operator $R_n^2\Psi$ is continuous, and thus $\mathscr{S}(R_n^2\Psi)=0$, for sufficiently large n. By [21; Lemma 1.3] $\mathscr{S}(\Psi)\subset\ker(R_n^2)$ for sufficiently large n. Since $\mathscr{S}(\Psi)$ is a closed ideal in W and $R_n^2\neq 0$, we have $\mathscr{S}(\Psi)=0$. Therefore Ψ is continuous as required. □

Theorem 5. *Let Φ be a densely valued homomorphism from a Banach triple system V into an H^*-triple system W. Then the separating subspace $\mathcal{S}(\Phi)$ is contained in the annihilator of W. Accordingly, Φ is continuous if W has zero annihilator.*

Proof. We have divided the proof into many steps.

I. Assume that the ground field is $\mathbb{C}$. If W is an H^*-triple system with zero annihilator, then we apply Theorem 2. Let I be a minimal closed ideal of W and π_I the orthogonal projection from W onto I. We consider the densely valued homomorphism $\pi_I\Phi$ from the Banach triple system V into the topologically simple H^*-triple system I for which $Z(I)=\mathbb{C}$ ([10; Theorem 3]). It follows from Proposition 5 that $\pi_I\Phi$ is continuous. By [21; Lemma 1.3], $\mathcal{S}(\Phi)\subset\ker(\pi_I)$ and so $(\mathcal{S}(\Phi)|I)=0$. The equality $(\mathcal{S}(\Phi)|I)=0$ holds for every minimal closed ideal I of W and applying Theorem 2 we get $(\mathcal{S}(\Phi)|W)=0$ and hence $\mathcal{S}(\Phi)=0$.

The general case will be reduced to that proved above by means of Theorem 1. W is an orthogonal sum $W=\mathrm{Ann}(W)\oplus W_0$ where W_0 is a complex H^*-triple system with zero annihilator. Let π denote the orthogonal projection from W onto W_0. It can be easily verified that $\pi\Phi$ is a densely valued homomorphism from V into W_0. $\pi\Phi$ is continuous and so [20; Lemma 1.3] $\mathcal{S}(\Phi)\subset\ker\pi=\mathrm{Ann}(W)$.

II. Assume that the ground field is $\mathbb{R}$. We consider the algebraic complexified $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ of V and W, respectively, and the only complex homomorphism $\Phi_{\mathbb{C}}$ from $V_{\mathbb{C}}$ into $W_{\mathbb{C}}$ extending Φ. $W_{\mathbb{C}}$ is endowed with a natural structure of complex H^*-triple system [10; Proposition 5] and $V_{\mathbb{C}}$ may be endowed with a norm for which $V_{\mathbb{C}}$ becomes a complex Banach triple system (this can be made as in the algebra case [3; Proposition 13.3]). From what has already been proved, it may be concluded that $\mathcal{S}(\Phi_{\mathbb{C}})\subset\mathrm{Ann}(W_{\mathbb{C}})$. Since $\mathcal{S}(\Phi_{\mathbb{C}})=\mathcal{S}(\Phi)\oplus i\mathcal{S}(\Phi)$ and $\mathrm{Ann}(W_{\mathbb{C}})=\mathrm{Ann}(W)\oplus i\mathrm{Ann}(W)$, our claim follows. $\square$

A couple (V_+,V_-) of linear spaces over a field $\mathbb{K}$ together with trilinear triple products $[\cdots]_+: V_+\times V_-\times V_+\to V_+$ and $[\cdots]_-: V_-\times V_+\times V_-\to V_-$ is said to be a *pair*. *Normed pairs* are defined as those real or complex pairs whose underlying vector spaces are normed spaces and the triple products are jointly continuous. We define *Banach pairs* as those normed pairs (V_+,V_-) whose underlying normed spaces V_+ and V_- are Banach spaces.

Examples. 1. Let X and Y be normed spaces and let us denote by $BL(X,Y)$, the normed space of all continuous linear operators from X into Y. Then $(BL(X,Y), BL(Y,X))$ is the classical example of normed pair where the triple products are both defined by $[abc]=abc$.

2. If V is a triple system, then (V,V) becomes a pair for the products $[xyz]_+=[xyz]_-=[xyz]$.

Given two pairs (U_+,U_-) and (V_+,V_-), a couple (Φ_+,Φ_-) of linear mappings Φ_σ: $U_\sigma\to V_\sigma$ is said to be an *homomorphism pair* if

$$\Phi_\sigma([xyz]_\sigma)=[\Phi_\sigma(x)\Phi_{-\sigma}(y)\Phi_\sigma(z)]_\sigma$$

for all $x,z\in V_\sigma$, $y\in V_{-\sigma}$, and $\sigma\in\{+,-\}$.

Theorem 6. *Let (Φ_+,Φ_-) be a homomorphism pair from a Banach pair (V_+,V_-) into an H^*-triple system W. If the operators Φ_+ and Φ_- are densely valued into W, then the separating subspaces $\mathcal{S}(\Phi_+)$ and $\mathcal{S}(\Phi_-)$ are contained in the annihilator of W. Accordingly, Φ_+ and Φ_- are continuous if W has zero annihilator.*

Proof. We consider the Banach space $V = V_+ \oplus V_-$ and define a jointly continuous triple product on V by

$$[(x_+, x_-)(y_+, y_-)(z_+, z_-)] = ([x_+ y_- z_+]_+, [x_- y_+ z_-]_-).$$

The same construction applied to W gives the so-called H^*-triple system polarized of W (see [8]), and we denote it as W_p. It is a simple matter to see that the linear map Φ from V into W_p given by $\Phi(x_+, x_-) = (\Phi_+ x_+, \Phi_- x_-)$ is a densely valued homomorphism. From Theorem 5 it follows that $\mathscr{S}(\Phi) \subset \mathrm{Ann}(W_p) = \mathrm{Ann}(W) \oplus \mathrm{Ann}(W)$ and this completes the proof, since $\mathscr{S}(\Phi) = \mathscr{S}(\Phi_+) \oplus \mathscr{S}(\Phi_-)$. □

From Theorem 5 we deduce the Rodríguez's theorem on the continuity of densely valued homomorphisms into H^*-algebras. Recall that an *H^*-algebra* is a real or complex (possibly non-associative) algebra B with an algebra involution $*$, which is linear in the real case and conjugate-linear in the complex one, and whose underlying linear space is a Hilbert space in which the equalities $(ab|c) = (a|cb^*) = (b|a^*c)$ hold for all $a, b, c \in B$.

COROLLARY 3

Let Φ be a densely valued homomorphism from a complete normed algebra A into an H^*-algebra B. Then the separating subspace $\mathscr{S}(\Phi)$ for Φ is contained in the annihilator of B. Accordingly, Φ is continuous if B has zero annihilator.

Proof. We endow the Hilbert space $W = B \oplus B$ with the jointly continuous triple product given by $[(a_1, a_2)(b_1, b_2)(c_1, c_2)] = ((a_1 b_2)c_1, c_2(b_1 a_2))$ and the involution given by $(a_1, a_2)^* = (a_2^*, a_1^*)$. W becomes an H^*-triple system. In the same way the Banach space $V = A \oplus A$ becomes a Banach triple system. The map Ψ from V into W given by $\Psi(a_1, a_2) = (\Phi a_1, \Phi a_2)$ is a densely valued homomorphism and therefore $\mathscr{S}(\Phi) \oplus \mathscr{S}(\Phi) = \mathscr{S}(\Psi) \subset \mathrm{Ann}(W) = \mathrm{Ann}(B) \oplus \mathrm{Ann}(B)$, which ends the proof. □

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Four coplanar Griffith cracks moving in an infinitely long elastic strip under antiplane shear stress

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Abstract. This paper concerns with the problem of determining the anti-plane dynamic stress distributions around four coplanar finite length Griffith cracks moving steadily with constant velocity in an infinitely long finite width strip. The two-dimensional Fourier transforms have been used to reduce the mixed boundary value problem to the solution of five integral equations. These integral equations have been solved using the finite Hilbert transform technique to obtain the analytic form of crack opening displacement and stress intensity factors. Numerical results have also been depicted graphically.

Keywords. Moving Griffith cracks; elastic strip; Hilbert transform; stress intensity factor.

1. Introduction

In recent years, scattering of elastic waves by cracks of finite dimension in a strip of elastic material has been investigated by several investigators. The theory of cracks in 2-dimensional medium was first developed by Griffith [3]. Sih and Chen [7] solved the problem of a uniformly propagating finite crack in a strip of isotropic material under plane extension. Singh *et al* [8] also studied the problem of propagation for a finite length crack moving in a strip under anti-plane shear stress and gave the closed form solution. In the above analysis, the usual method of solving mixed boundary value problems by integral transforms is used to reduce the problem to a Fredholm integral equation of second kind and then proceed to its numerical solution.

As regards the crack problem research has been restricted mainly to the case of one or two cracks because of the severe mathematical complexity encountered in solving the problems of three or more cracks. Jain and Kanwal [6] solved the low frequency solution of diffraction of normally incident longitudinal waves by two co-planar Griffith cracks in an infinite isotropic elastic medium. Using a completely different technique Itou [5] solved the diffraction problem of elastic waves by two co-planar Griffith cracks in an infinite elastic medium. Problems on three coplanar Griffith cracks moving steadily in an elastic strip has been solved by Das and Sarkar [2].

To the best knowledge of the authors, the problem of stress distribution around four co-planar Griffith cracks in a strip has not been investigated so far. In this paper we have considered the problem of propagation of four co-planar Griffith cracks moving steadily in an infinitely long finite width strip under antiplane shear stress. Cracks are assumed to be moving steadily along a fixed direction with a constant speed V less than the shear wave velocity in the medium. The application of two-dimensional Fourier transforms reduced this problem to that of solving a set of five integral equations with cosine kernel and weight function. Employing finite Hilbert transform technique [9], the closed form solutions are obtained when the lateral boundaries are subjected to

shearing stresses. The dynamic stress intensity factors and the crack opening displacement have been evaluated numerically for various values of crack velocity and distance between the cracks and the results have been presented by means of graphs.

2. Formulation of the problem

We first consider a strip of elastic material occupying the region $-h' \leqslant Y' \leqslant h'$ referred to a fixed co-ordinate system (X', Y', Z'). The strip extends from $-\infty$ to ∞ in X'-direction and contains four coplanar Griffith cracks such that these cracks are located in the region $-d' \leqslant X' \leqslant c'$, $-b' \leqslant X' \leqslant -a'$, $a' \leqslant X' \leqslant b'$, $c' \leqslant X' \leqslant d'$, $|Z'| < \infty$, $Y' = 0$ moving at a constant speed v in the X'-direction.

In dynamic problem of antiplane shear, there exists a single non-vanishing component of displacement $W = W(X', Y', t)$ in the Z'-direction. The corresponding stress components are

$$\sigma_{X'Z'} = \mu \frac{\partial W}{\partial X'}, \quad \sigma_{Y'Z'} = \mu \frac{\partial W}{\partial Y'} \tag{2.1}$$

where μ is the shear modulus of elastic material.

The two dimensional wave equation for $W(X', Y', t)$ is given by

$$\frac{\partial^2 W}{\partial X'^2} + \frac{\partial^2 W}{\partial Y'^2} = \frac{1}{c_2^2} \frac{\partial^2 W}{\partial t^2} \tag{2.2}$$

where $c_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity and ρ is the density of the material.

Using Galilean transformation, $x' = X' - Vt$, $y' = Y'$, $z' = Z'$, $t' = t$ where (x', y', z') represents the translating co-ordinate system and also normalizing all the lengths with respect to d' so that $x' = d'x$, $y' = d'y$, $a' = ad'$, $b' = bd'$, $c' = cd'$, $h' = d'h$, $W = d'w$, (figure 1) equation (2.2) reduces to

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{2.3}$$

with

$$s^2 = 1 - v^2/c_2^2. \tag{2.4}$$

Since the geometry of the problem is symmetric about the y-axis, introducing Fourier

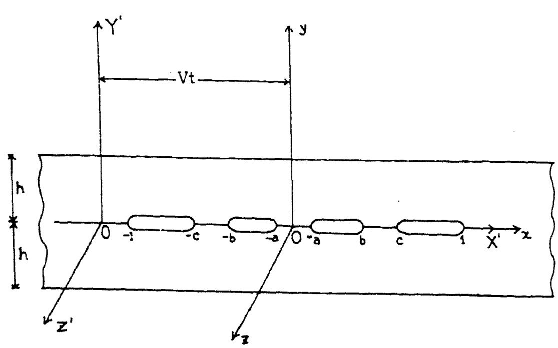

Figure 1. Geometry of the cracks.

cosine transform

$$A_1(\xi)=\int_0^\infty A(x)\cos(\xi x)\,\mathrm{d}x$$

and

$$A(x)=\frac{2}{\pi}\int_0^\infty A_1(\xi)\cos(\xi x)\mathrm{d}\xi$$

we obtain the solution of equation (2.3) as

$$w(x,y)=\pm\frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-\xi|y|s)+A_2(\xi)\exp(\xi|y|s)]\cos(\xi x)\mathrm{d}\xi \quad (2.5)$$

with ($y\gtrless 0$)

$$\sigma_{yz}(x,y)=-\frac{2\mu s}{\pi}\int_0^\infty [A_1(\xi)\exp(-\xi|y|s)-A_2(\xi)\exp(\xi|y|s)]\xi\cos(\xi x)\mathrm{d}\xi \quad (2.6)$$

where s is the positive root of equation (2.4) and $A_1(\xi)$, $A_2(\xi)$ are the unknown functions to be determined.

In our case uniform shearing stress p is applied to the upper and lower boundaries $y=\pm h$ of the strip. The equivalent problem in our case involves the application of the shear stress $-p$ to the crack faces at $y=\phi$. Accordingly, the boundary conditions are

$$\sigma_{yz}(x,\pm h)=0,\quad 0<x<\infty \quad (2.7)$$

$$w(x,0)=0,\quad x\in I_1,I_3,I_5 \quad (2.8a\text{–}c)$$

$$\sigma_{yz}(x,0)=-p,\quad x\in I_2,I_4 \quad (2.9a\text{–}b)$$

where $I_1=(0,a)$, $I_2=(a,b)$, $I_3=(b,c)$, $I_4=(c,1)$, $I_5=(1,\infty)$.

3. Solution of the problem

Using the expression for $w(x,y)$ from (2.5) in (2.7) it has been found that

$$A_1(\xi)=\frac{A(\xi)}{1+\exp(-2\xi hs)}$$

and

$$A_2(\xi)=\frac{A(\xi)\exp(-2\xi hs)}{1+\exp(-2\xi hs)}$$

where $A(\xi)$ is to be determined from the boundary conditions. With the help of boundary conditions (2.8) and (2.9), $A(\xi)$ is found to satisfy the following set of five integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)\mathrm{d}\xi=0,\quad x\in I_1,I_3,I_5 \quad (3.1a\text{–}c)$$

and

$$\int_0^\infty \xi H_1(\xi hs)A(\xi)\cos(\xi x)\mathrm{d}\xi=\frac{\pi p}{2\mu s},\quad x\in I_2,I_4 \quad (3.2a\text{–}b)$$

where

$$H_1(\xi hs) = \frac{1-\exp(-2\xi hs)}{1+\exp(-2\xi hs)} = \tanh(\xi hs). \tag{3.3}$$

In order to solve the set of five integral equations given by equations (3.1) and (3.2), let us take

$$A(\xi) = \frac{1}{\xi}\int_a^b g(u^2)\cosh(eu)\sin(\xi u)\mathrm{d}u + \frac{1}{\xi}\int_c^1 h(v^2)\cosh(ev)\sin(\xi v)\mathrm{d}v. \tag{3.4}$$

In (3.4), $g(u^2)$ and $h(v^2)$ are unknown functions to be determined from the boundary conditions and $e = \pi/2hs$.

Using the following result [4]

$$\int_0^\infty \frac{\sin(\xi u)\cos(\xi x)}{\xi}\mathrm{d}\xi = \begin{cases} \frac{\pi}{2}, & u > x > 0 \\ 0, & x > u > 0 \end{cases}$$

it is found that the choice of $A(\xi)$ satisfies equations (3.1a, c) if $g(u^2)$ and $h(v^2)$ satisfy

$$\int_a^b g(u^2)\cosh(eu)\mathrm{d}u = 0 \tag{3.5a}$$

and

$$\int_c^1 h(v^2)\cosh(ev)\mathrm{d}v = 0. \tag{3.5b}$$

Now (3.2a–b) may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \tanh(\xi hs)A(\xi)\sin(\xi x)\mathrm{d}\xi = \frac{\pi p}{2\mu s}, \quad x\in I_2, I_4. \tag{3.6a–b}$$

Substitution of (3.4) in (3.6a) and use of the following result [1]

$$\int_0^\infty \xi^{-1}\tanh(\xi hs)\sin(\xi u)\sin(\xi x)\mathrm{d}\xi = \frac{1}{2}\log\left|\frac{\sinh(ex)+\sinh(eu)}{\sinh(ex)-\sinh(eu)}\right|$$

yields

$$\int_A^B \frac{UG(U^2)}{U^2-X^2}\mathrm{d}U = \frac{\pi}{2}F(x), \quad (A < X < B) \tag{3.7}$$

where $\cosh(eu) = U$, $\cosh(ev) = S$, $X = \cosh(ex)$, $A = \cosh(ea)$, $B = \cosh(eb)$, $C = \cosh(ec)$, $D = \cosh(e)$, $g(u^2) = G(U^2)$, $h(v^2) = H(S^2)$ and

$$F(X) = \frac{p}{\mu s X} - \frac{2}{\pi}\int_C^D \frac{SH(S^2)}{S^2-X^2}\mathrm{d}S. \tag{3.8}$$

Using the finite Hilbert transform technique [9] the solution of equation (3.7) is

$$G(U^2) = \frac{2p}{\pi\mu s}\sqrt{\frac{U^2-A^2}{B^2-U^2}}\int_A^B \sqrt{\frac{B^2-X^2}{X^2-A^2}}\frac{\mathrm{d}X}{U^2-X^2} - \frac{2}{\pi}\sqrt{\frac{U^2-A^2}{B^2-U^2}}$$
$$\times \int_C^D \sqrt{\frac{S^2-B^2}{S^2-A^2}}\frac{SH(S^2)}{S^2-U^2}\mathrm{d}S + \frac{B_1}{\sqrt{(U^2-A^2)(B^2-U^2)}},$$
$$(A < U < B) \tag{3.9}$$

where B_1 is a constant to be determined from (3.5a). Substitution of expression for $A(\xi)$ from (3.4) in (3.6a) yields with aid of (3.9) the following singular integral equation involving $H(S^2)$

$$\int_C^D \sqrt{\frac{S^2-B^2}{S^2-A^2}}\frac{SH(S^2)}{S^2-U^2}\,\mathrm{d}S = \frac{\pi}{2}\left[\frac{p}{\mu sX}\sqrt{\frac{X^2-B^2}{X^2-A^2}} - \frac{2pA^2}{\pi\mu sBX^2}\right.$$
$$\left.\times\left\{\left(\frac{X^2-B^2}{X^2-A^2}\right)\Pi\left(\frac{\pi}{2},\frac{X^2(B^2-A^2)}{B^2(X^2-A^2)},q\right) - \frac{B^2}{A^2}F\left(\frac{\pi}{2},q\right)\right\} + \frac{B_1}{X^2-A^2}\right] \tag{3.10}$$

where $q=(B^2-A^2)^{1/2}/B$ and $F(\phi,k)$, $\Pi(\phi,n,k)$ are elliptic integrals of first and third kind respectively.

Again, using finite Hilbert transform technique [9] it is found that

$$H(S^2) = -\frac{2}{\pi}\sqrt{\frac{(S^2-A^2)(S^2-C^2)}{(S^2-B^2)(D^2-S^2)}}\left[\frac{p}{\mu s}\left\{\int_C^D\sqrt{\frac{(D^2-X^2)(X^2-B^2)}{(X^2-C^2)(X^2-A^2)}}\right.\right.$$
$$\left.\times\frac{\mathrm{d}X}{(X^2-S^2)} - \int_C^D\sqrt{\frac{(D^2-Y^2)(B^2-Y^2)}{(Y^2-A^2)(C^2-Y^2)}}\frac{\mathrm{d}Y}{(S^2-Y^2)}\right\}$$
$$\left.-\frac{\pi}{2}\sqrt{\frac{D^2-A^2}{C^2-A^2}}\frac{B_1}{(S^2-A^2)}\right] + \frac{B_2\sqrt{S^2-A^2}}{\sqrt{(S^2-B^2)(S^2-C^2)(D^2-S^2)}}$$
$$(C<S<D) \tag{3.11}$$

the constant B_2 occurring in (3.11) is to be determined using the condition given by equation (3.5b).

Next, substituting the value of $H(S^2)$ from equation (3.11) in equation (3.9) $G(U^2)$ may be written in the following form

$$G(U^2) = \frac{2}{\pi}\sqrt{\frac{U^2-A^2}{B^2-U^2}}\frac{p}{\mu s}\left[\frac{(B^2-U^2)}{BU^2(U^2-A^2)}\left\{A^2\Pi\left(\frac{\pi}{2},\frac{X^2(B^2-A^2)}{B^2(X^2-A^2)},q\right)\right.\right.$$
$$\left.+(U^2-A^2)F\left(\frac{\pi}{2},q\right)\right\} + \frac{1}{B}F\left(\frac{\pi}{2},q\right)$$
$$-\sqrt{\frac{C^2-U^2}{D^2-U^2}}\left\{\int_C^D\sqrt{\frac{(D^2-X^2)(X^2-B^2)}{(X^2-C^2)(X^2-A^2)}}\frac{\mathrm{d}X}{(X^2-U^2)}\right.$$
$$\left.+\int_A^B\sqrt{\frac{(D^2-Y^2)(B^2-Y^2)}{(Y^2-A^2)(C^2-Y^2)}}\frac{\mathrm{d}Y}{(Y^2-U^2)}\right\}$$
$$\left.+\int_A^B\sqrt{\frac{(B^2-Y^2)}{(Y^2-A^2)}}\frac{\mathrm{d}Y}{(Y^2-U^2)}\right] + \left(\frac{(D^2-A^2)(C^2-U^2)}{(C^2-A^2)(D^2-U^2)}\right)^{1/2}$$
$$\times\frac{B_1}{[(U^2-A^2)(B^2-U^2)]^{1/2}} - \frac{B_2(U^2-A^2)^{1/2}}{[(B^2-U^2)(C^2-U^2)(D^2-U^2)]^{1/2}}$$
$$(A<U<B) \tag{3.12}$$

To determine the values of the unknown constants B_1 and B_2, we substitute $H(S^2)$ and

$G(U^2)$ given by (3.11) and (3.12) in (3.5a, b) and obtain

$$B_1 = \frac{p}{\mu s}\left\{\frac{K_3(K_{1,2}-K_{1,1})-K_6(K_{1,3}+K_{2,3})}{RK_4K_6+K_3K_5}\right\} \tag{3.13a}$$

$$B_2 = \frac{p}{\mu s}\left\{\frac{RK_4(K_{1,1}-K_{1,2})-K_5(K_{1,3}+K_{2,3})}{RK_4K_6+K_3K_5}\right\} \tag{3.13b}$$

where

$$K_{1,1} = \int_C^D M_1(X)\mathrm{d}X \int_C^D \frac{M_2(S)}{X^2-S^2}\mathrm{d}S \tag{3.14}$$

$$K_{1,2} = \int_A^B M_1(Y)\mathrm{d}Y \int_C^D \frac{M_2(S)}{S^2-Y^2}\mathrm{d}S \tag{3.15}$$

$$K_{1,3} = \int_C^D M_1(X)\mathrm{d}X \int_A^B \frac{M_2(U)}{X^2-U^2}\mathrm{d}U \tag{3.16}$$

$$K_{2,3} = \int_A^B M_1(Y)\mathrm{d}Y \int_A^B \frac{M_2(U)}{Y^2-U^2}\mathrm{d}U \tag{3.17}$$

$$K_3 = \frac{\pi}{2}\int_A^B \frac{M_2(U)}{C^2-U^2}\mathrm{d}U, \quad K_4 = \int_A^B \frac{M_2(U)}{U^2-A^2}\mathrm{d}U \tag{3.18}$$

$$K_5 = R\int_C^D \frac{M_2(S)}{S^2-A^2}\mathrm{d}S, \quad K_6 = \frac{\pi}{2}\int_C^D \frac{M_2(S)}{S^2-C^2}\mathrm{d}S \tag{3.19}$$

$$M_1(T) = \left[\frac{(D^2-T^2)(T^2-B^2)}{(T^2-C^2)(T^2-A^2)}\right]^{1/2},$$

$$M_2(T) = \left[\frac{(T^2-A^2)(T^2-C^2)}{(T^2-B^2)(D^2-T^2)}\right]^{1/2}\frac{T}{\sqrt{T^2-1}} \tag{3.20}$$

and

$$R = -\frac{\pi}{2}\sqrt{\frac{D^2-A^2}{C^2-A^2}}. \tag{3.21}$$

4. Stress intensity factors

The corresponding displacement and stress components in the plane of the cracks may be written as

$$\begin{aligned} w(x,0) &= \frac{1}{e}\int_X^B \frac{UG(U^2)}{\sqrt{U^2-1}}\mathrm{d}U, \quad x\in I_2 \\ &= \frac{1}{e}\int_X^D \frac{SH(S^2)}{\sqrt{S^2-1}}\mathrm{d}S, \quad x\in I_4 \end{aligned} \tag{4.1a, b}$$

and

$$[\sigma_{yz}(x,0)]_{0<x<a} = -\frac{2\mu sX}{\pi}\left[\int_A^B \frac{UG(U^2)}{U^2-X^2}\mathrm{d}U + \int_C^D \frac{SH(S^2)}{S^2-X^2}\mathrm{d}S\right] \tag{4.2a}$$

$$[\sigma_{yz}(x,0)]_{b<x<c} = \frac{2\mu sX}{\pi}\left[\int_A^B \frac{UG(U^2)}{X^2-U^2}\mathrm{d}U + \int_C^D \frac{SH(S^2)}{S^2-X^2}\mathrm{d}S\right] \tag{4.2b}$$

$$[\sigma_{yz}(x,0)]_{x>1}=\frac{2\mu s X}{\pi}\left[\int_A^B \frac{UG(U^2)}{X^2-U^2}\,\mathrm{d}U+\int_C^D \frac{SH(S^2)}{X^2-S^2}\,\mathrm{d}S\right]. \tag{4.2c}$$

Using the results given by (3.11) and (3.12) the expressions (4.2a–c) yield after some algebraic manipulation, the results

$$[\sigma_{yz}(x,0)]_{0<x<a}=\frac{2\mu s}{\pi}[F_1(X)-F_2(X)-F_3(X)+F_4(X)$$
$$-F_5(X)-F_6(X)-F_7(X)-F_8(X)] \tag{4.3a}$$

$$[\sigma_{yz}(x,0)]_{b<x<c}=\frac{2\mu s}{\pi}[F_1(X)-F_2(X)+F_3(X)+F_4(X)$$
$$-F_5(X)-F_6(X)-F_7(X)-F_8(X)] \tag{4.3b}$$

$$[\sigma_{yz}(x,0)]_{x>1}=\frac{2\mu s}{\pi}[F_1(X)-F_2(X)+F_3(X)+F_4(X)$$
$$-F_5(X)-F_6(X)-F_7(X)+F_8(X)] \tag{4.3c}$$

where

$$F_1(X)=\frac{2pX}{\pi\mu S}\int_C^D\left[\frac{(D^2-Y^2)(Y^2-B^2)}{(Y^2-C^2)(Y^2-A^2)}\right]^{1/2}\left[\frac{\pi}{2(Y^2-X^2)}\left\{\left[\frac{Y^2-A^2}{Y^2-B^2}\right]^{1/2}\right.\right.$$
$$\left.\left.-\left[\frac{A^2-X^2}{B^2-X^2}\right]^{1/2}\right\}\left[\frac{C^2-B^2}{D^2-B^2}\right]^{1/2}+I^{B,D}_{A,C}(X,Y)\right]\mathrm{d}Y \tag{4.4a}$$

$$F_2(X)=\frac{2pX}{\pi\mu s}\int_A^B\left[\frac{(D^2-Y^2)(B^2-Y^2)}{(C^2-Y^2)(Y^2-A^2)}\right]^{1/2}\left[\frac{\pi}{2(Y^2-X^2)}\right.$$
$$\left.\times\left[\frac{(C^2-B^2)(A^2-X^2)}{(D^2-B^2)(B^2-X^2)}\right]^{1/2}+L^{B,D}_{A,C}(X,Y)\right]\mathrm{d}Y \tag{4.4b}$$

$$F_3(X)=\frac{B_1X}{X_1}\left[\frac{\pi}{2}\left[\frac{(C^2-B^2)}{(D^2-B^2)}\right]^{1/2}+J^{B,D}_{A,C}(X)\right]\left[\frac{(D^2-A^2)}{(C^2-A^2)}\right]^{1/2} \tag{4.4c}$$

$$F_4(X)=B_2X\left[\frac{\pi}{2[(C^2-B^2)(D^2-B^2)]^{1/2}}\left\{1-\left[\frac{(A^2-X^2)}{(B^2-X^2)}\right]^{1/2}\right\}\right.$$
$$\left.+K^{B,D}_{A,C}(X)\right] \tag{4.4d}$$

$$F_5(X)=\frac{2pX}{\pi\mu s}\int_C^D\left[\frac{(D^2-Y^2)(Y^2-B^2)}{(Y^2-C^2)(Y^2-A^2)}\right]^{1/2}$$
$$\times\left[\frac{\pi}{2(Y^2-X^2)}\left[\frac{(D^2-A^2)(C^2-X^2)}{(D^2-B^2)(D^2-X^2)}\right]^{1/2}-L^{D,B}_{C,A}(X,Y)\right]\mathrm{d}Y \tag{4.4e}$$

$$F_6(X)=\frac{2pX}{\pi\mu s}\int_A^B\left[\frac{(D^2-Y^2)(B^2-Y^2)}{(C^2-Y^2)(Y^2-A^2)}\right]^{1/2}\left[\frac{\pi}{2(Y^2-X^2)}\left\{\left[\frac{C^2-X^2}{D^2-X^2}\right]^{1/2}\right.\right.$$
$$\left.\left.-\left[\frac{C^2-Y^2}{D^2-Y^2}\right]^{1/2}\right\}\left[\frac{D^2-A^2}{D^2-B^2}\right]^{1/2}+I^{D,B}_{C,A}(X,Y)\right]\mathrm{d}Y \tag{4.4f}$$

$$F_7(X) = \frac{B_1 X}{(A^2 - X^2)}\left[\frac{\pi}{2}\left[\frac{(D^2 - A^2)}{(D^2 - B^2)}\right]^{1/2}\left\{\left[\frac{C^2 - X^2}{D^2 - X^2}\right]^{1/2} - \left[\frac{C^2 - A^2}{D^2 - A^2}\right]^{1/2}\right\} + I^{D,B}_{C,A}(X, A)\right]\left[\frac{(D^2 - A^2)}{(C^2 - A^2)}\right]^{1/2} \tag{4.4g}$$

$$F_8(X) = \frac{B_1 X}{X_2}\left[\frac{\pi}{2}\left[\frac{(D^2 - A^2)}{(D^2 - B^2)}\right]^{1/2} - J^{B,D}_{C,A}(X)\right] \tag{4.4h}$$

$$I^{Q,S}_{P,R}(X, Y) = \int_P^Q \left(\frac{S^2 - R^2}{Y^2 - X^2}\right)\left\{\left[\frac{(Y^2 - P^2)}{(Y^2 - Q^2)}\right]^{1/2} \tan^{-1}\left[\frac{(U^2 - P^2)(Y^2 - Q^2)}{(Q^2 - U^2)(Y^2 - P^2)}\right]^{1/2} - \left[\frac{(P^2 - X^2)}{(Q^2 - X^2)}\right]^{1/2} \tan^{-1}\left[\frac{(U^2 - P^2)(Q^2 - X^2)}{(Q^2 - U^2)(P^2 - X^2)}\right]^{1/2}\right\} \times \frac{U\,dU}{[(R^2 - U^2)(S^2 - U^2)^3]^{1/2}} \tag{4.4i}$$

$$L^{Q,S}_{P,R}(X, Y) = \int_P^Q \left(\frac{S^2 - R^2}{Y^2 - X^2}\right)\left\{\left[\frac{(P^2 - X^2)}{(Q^2 - X^2)}\right]^{1/2} \tan^{-1}\left[\frac{(U^2 - P^2)(Q^2 - X^2)}{(Q^2 - U^2)(P^2 - X^2)}\right]^{1/2} + \frac{1}{2}\left[\frac{(Y^2 - P^2)}{(Q^2 - Y^2)}\right]^{1/2} \times \log\left|\frac{[(U^2 - P^2)(Q^2 - Y^2)]^{1/2} - [(Q^2 - U^2)(Y^2 - P^2)]^{1/2}}{[(U^2 - P^2)(Q^2 - Y^2)]^{1/2} - [(Q^2 - U^2)(Y^2 - P^2)]^{1/2}}\right|\right\} \times \frac{U\,dU}{[(R^2 - U^2)(S^2 - U^2)^3]^{1/2}} \tag{4.4j}$$

$$J^{Q,S}_{P,R}(X) = \int_P^Q \left\{\tan^{-1}\left[\frac{(U^2 - P^2)(Q^2 - X^2)}{(Q^2 - U^2)(P^2 - X^2)}\right]^{1/2}\right\} \frac{U(S^2 - R^2)\,dU}{[(R^2 - U^2)(S^2 - U^2)^3]^{1/2}} \tag{4.4k}$$

$$K^{Q,S}_{P,R}(X) = \int_P^Q \left\{\tan^{-1}\left[\frac{(U^2 - P^2)}{(Q^2 - U^2)}\right]^{1/2} - \left[\frac{(P^2 - X^2)}{(Q^2 - X^2)}\right]^{1/2} \times \tan^{-1}\left[\frac{(U^2 - P^2)(Q^2 - X^2)}{(Q^2 - U^2)(P^2 - X^2)}\right]^{1/2}\right\} \frac{U(2U^2 - R^2 - S^2)\,dU}{[(R^2 - U^2)^3(S^2 - U^2)^3]^{1/2}} \tag{4.4l}$$

$$X_1 = [(A^2 - X^2)(B^2 - X^2)]^{1/2}, \quad X_2 = [(C^2 - X^2)(D^2 - X^2)]^{1/2}. \tag{4.4m}$$

The dynamic stress intensity factors are given by

$$N_a = \operatorname*{Lt}_{x \to a-} [2(a - x)]^{1/2} \left|\frac{\sigma_{yz}(x, 0)}{P}\right|_{0<x<a} \tag{4.5a}$$

$$N_b = \operatorname*{Lt}_{x \to b+} [2(x - b)]^{1/2} \left|\frac{\sigma_{yz}(x, 0)}{P}\right|_{b<x<c} \tag{4.5b}$$

$$N_c = \operatorname*{Lt}_{x \to c-} [2(c - x)]^{1/2} \left|\frac{\sigma_{yz}(x, 0)}{P}\right|_{b<x<c} \tag{4.5c}$$

$$N_1 = \operatorname*{Lt}_{x \to 1+} [2(x-1)]^{1/2} \left| \frac{\sigma_{yz}(x,0)}{P} \right|_{x>1}. \tag{4.5d}$$

With the aid of the results given by (4.3) in (4.5) it follows that

$$N_a = -\frac{\mu s\sqrt{A}}{[e(A^2-1)^{1/2}(B^2-A^2)]^{1/2}} B_1 \tag{4.6a}$$

$$\begin{aligned} N_b = {} & -\frac{\mu s\sqrt{B}}{\sqrt{e(B^2-1)^{1/2}}} \left[-\frac{2p}{\pi \mu s} \left[\frac{(B^2-A^2)(C^2-B^2)}{(D^2-B^2)} \right]^{1/2} \left\{ \int_A^B G_1(Y)\mathrm{d}Y \right.\right. \\ & \left. + \int_C^D G_1(Y)\mathrm{d}Y \right\} + \left[\frac{(C^2-B^2)(D^2-A^2)}{(B^2-A^2)(C^2-A^2)(D^2-B^2)} \right]^{1/2} B_1 \\ & \left. - \left[\frac{(B^2-A^2)}{(C^2-B^2)(D^2-B^2)} \right]^{1/2} B_2 \right] \end{aligned} \tag{4.6b}$$

$$N_c = -\frac{\mu s[C(C^2-A^2)]^{1/2}}{[e(C^2-1)^{1/2}(C^2-B^2)(D^2-C^2)]^{1/2}} B_2 \tag{4.6c}$$

$$\begin{aligned} N_1 = {} & -\frac{\mu s\sqrt{D}}{\sqrt{e(D^2-1)^{1/2}}} \left[-\frac{2p}{\pi \mu s} \left[\frac{(D^2-A^2)(D^2-C^2)}{(D^2-B^2)} \right. \right. \\ & \times \left\{ \int_A^B G_2(Y)\mathrm{d}Y + \int_C^D G_2(Y)\mathrm{d}Y \right\} + \left[\frac{(D^2-C^2)}{(D^2-B^2)(C^2-A^2)} \right]^{1/2} B_1 \\ & \left. + \left[\frac{(D^2-A^2)}{(D^2-C^2)(D^2-B^2)} \right]^{1/2} B_2 \right] \end{aligned} \tag{4.6d}$$

where

$$G_1(Y) = \frac{[(D^2-Y^2)]^{1/2}}{[(Y^2-A^2)(Y^2-B^2)(Y^2-C^2)]^{1/2}} \tag{4.7a}$$

$$G_2(Y) = \frac{[(B^2-Y^2)]^{1/2}}{[(Y^2-A^2)(C^2-Y^2)(D^2-Y^2)]^{1/2}}. \tag{4.7b}$$

The crack opening displacements are obtained by using the expressions for $G(U^2)$ and $H(S^2)$ from (3.12) and (3.11) in (4.1a, b).

Again letting $a \to 0$ and simplifying, it may be noted that the results (4.6b), (4.6c) and (4.6d) become those given by equations (3.14) of Das [1].

5. Numerical results

The numerical values of stress intensity factors (SIF) N_a, N_b, N_c and N_1 given by (4.6a–d) at the tips of the crack have been plotted against crack speed (V/c_2) for different values of crack lengths, separating distances of the cracks and strip width (h). Keeping the length of the outer cracks and distance between inner and outer cracks fixed ($b = 0{\cdot}6$, $c = 0{\cdot}8$) SIFs at the tips of the cracks have been plotted against crack speed ($0{\cdot}1 \leqslant V/c_2 < 1$) for different lengths of the inner cracks ($a = 0{\cdot}2, 0{\cdot}4$) and strip width ($h = 1, 3, 5$). It is found from the graphs (figures 2–5) that SIFs increase

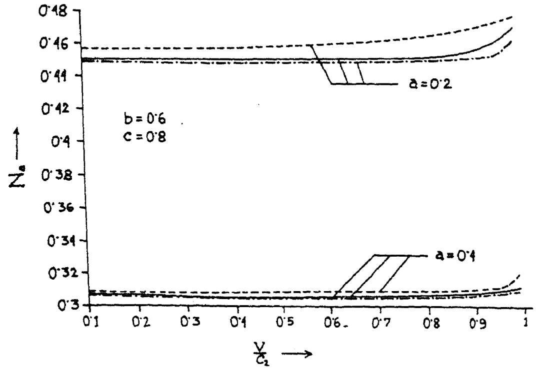

Figure 2. Stress intensity factor N_a vs. V/c_2.
(--- $h=1$, —— $h=2$, -·-·- $h=5$)

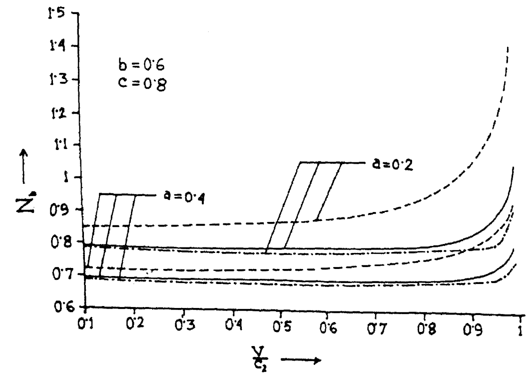

Figure 3. Stress intensity factor N_b vs. V/c_2.
(--- $h=1$, —— $h=2$, -·-·- $h=5$)

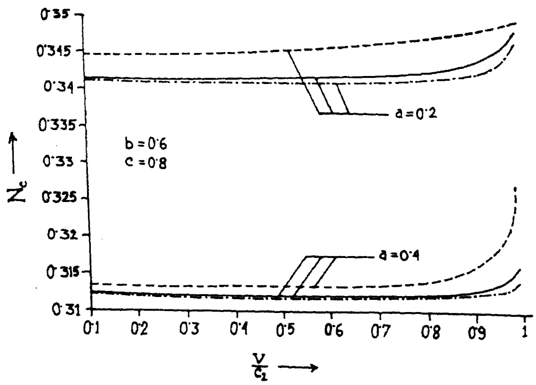

Figure 4. Stress intensity factor N_c vs. V/c_2.
(--- $h=1$, —— $h=2$, -·-·- $h=5$)

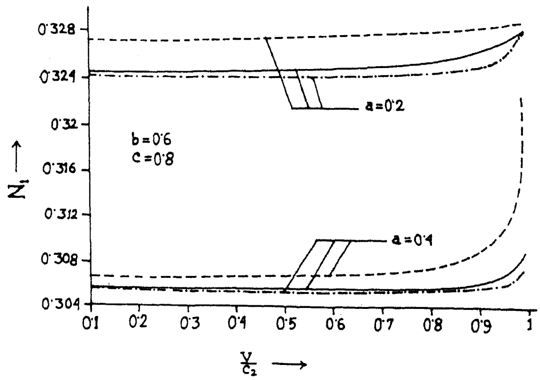

Figure 5. Stress intensity factor N_1 vs. V/c_2.
(--- $h = 1$, —— $h = 2$, -·-·- $h = 5$)

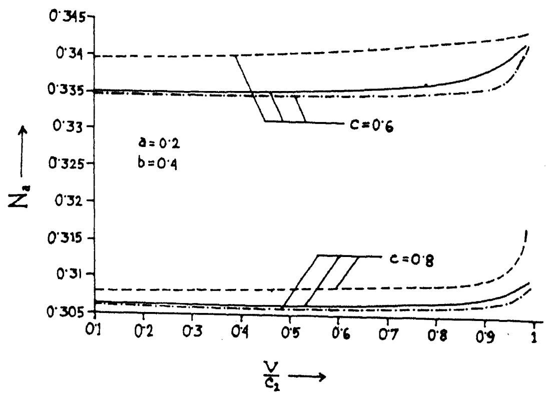

Figure 6. Stress intensity factor N_a vs. V/c_2.
(--- $h = 1$, —— $h = 2$, -·-·- $h = 5$)

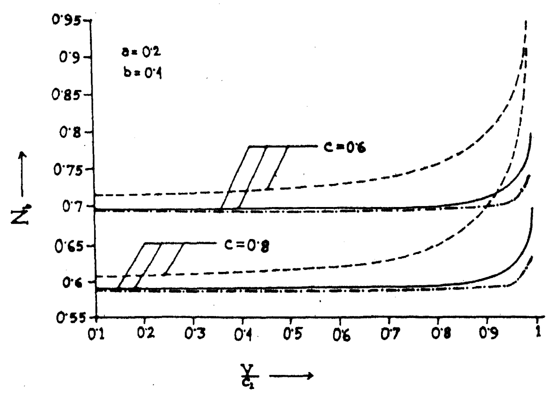

Figure 7. Stress intensity factor N_b vs. V/c_2.
(--- $h = 1$, —— $h = 2$, -·-·- $h = 5$)

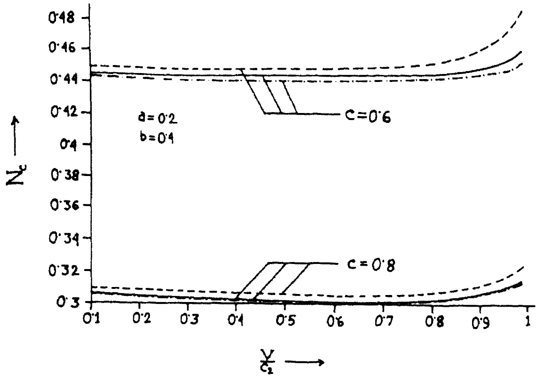

Figure 8. Stress intensity factor N_c vs. V/c_2. (--- $h=1$, —— $h=2$, -·-·- $h=5$)

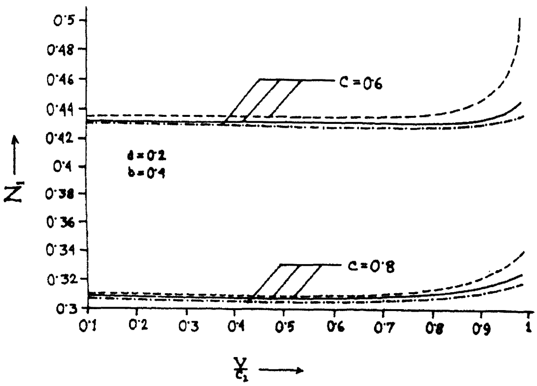

Figure 9. Stress intensity factor N_1 vs. V/c_2. (--- $h=1$, —— $h=2$, -·-·- $h=5$)

rapidly as $V/c_2 \to 1$ and with the decrease in the value of inner crack length i.e. with the increase in the value of the distance between inner cracks the value of SIF decreases.

Similar effect on SIFs can be found with the increase in the value of b when lengths of the outer cracks and the distance between inner cracks are kept fixed.

Next, keeping the lengths of the inner cracks fixed ($a = 0{\cdot}2$, $b = 0{\cdot}4$) it is seen from the graphs (figures 6–9) that the value of SIF N_b is higher for higher values of $c(0{\cdot}6, 0{\cdot}8)$. But the nature is opposite in case of N_a, N_b and N_1.

In all the cases mentioned above the SIFs increase with the increase in the value of V/c_2 gradually at a slow rate in the beginning but increase rapidly as $V/c_2 \to 1$. Also the value of SIFs are higher for lower values of h in these cases.

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Limit distributions of expanding translates of certain orbits on homogeneous spaces

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Abstract. Let L be a Lie group and Λ a lattice in L. Suppose G is a non-compact simple Lie group realized as a Lie subgroup of L and $\overline{G\Lambda} = L$. Let $a \in G$ be such that $\mathrm{Ad}\,a$ is semisimple and not contained in a compact subgroup of $\mathrm{Aut}(\mathrm{Lie}(G))$. Consider the expanding horospherical subgroup of G associated to a defined as $U^+ = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to \infty\}$. Let Ω be a non-empty open subset of U^+ and $n_i \to \infty$ be any sequence. It is showed that $\overline{\cup_{i=1}^{\infty} a^{n_i} \Omega \Lambda} = L$. A stronger measure theoretic formulation of this result is also obtained. Among other applications of the above result, we describe G-equivariant topological factors of $L/\Lambda \times G/P$, where the real rank of G is greater than 1, P is a parabolic subgroup of G and G acts diagonally. We also describe equivariant topological factors of unipotent flows on finite volume homogeneous spaces of Lie groups.

Keywords. Limit distributions; unipotent flow; horospherical patches; symmetric subgroups; continuous equivariant factors.

1. Introduction

Let G be a connected semisimple Lie group with no compact factors and of $\mathbb{R}$-rank $\geqslant 2$, P a parabolic subgroup of G, and Γ an irreducible lattice in G. It was proved by Margulis [M1] that if $\phi: G/P \to Y$ is a measure class preserving Γ-equivariant factor of G/P then there exist a parabolic subgroup Q containing P and a measurable isomorphism $\psi: Y \to G/Q$ such that $\psi \circ \phi$ is the canonical quotient map. The topological analogue of this result was obtained by Dani [D3], who proved that, in the above notation, if ϕ is continuous then ψ can be chosen to be a homeomorphism. On the other hand the result of Margulis was generalized by Zimmer [Z1] in the measure theoretical category. This result was later used in [SZ] for describing faithful and properly ergodic finite measure preserving G-actions. It was suggested by Stuck [St] that the following question, which is a topological analogue of Zimmer's result, is of importance for studying locally free minimal G-actions.

Question 1.1. Let G be a simple Lie group of $\mathbb{R}$-rank $\geqslant 2$. Suppose that G acts minimally and locally freely on a compact Hausdorff space X. Suppose there are G-equivariant continuous surjective maps $X \times G/P \xrightarrow{\phi} Y \xrightarrow{\psi} X$ such that $\psi \circ \phi$ is the projection onto X, where G acts diagonally on $X \times G/P$. Does there exist a parabolic subgroup Q containing P and a G-equivariant homeomorphism $\rho: Y \to X \times G/Q$ such that $\rho \circ \phi$ is the canonical quotient map?

The above mentioned result of Dani says that this question has the affirmative answer if $X = G/\Gamma$, Γ being a lattice in G. In this paper we consider the case when G is a Lie subgroup of a Lie group L acting on $X = L/\Lambda$ by translations, Λ being a lattice in L. To analyze this case we follow the method of the proof of Dani [D3]. To adapt

Dani's proof for the general case one needs the following theorem 1.1, which is a non-trivial generalization of its particular case of $L=G$ (cf.[D3, Lemma 1.1]). Its proof involves, in an essential way, Ratner's theorem [Ra1] on classification of finite ergodic invariant measures of unipotent flows on homogeneous spaces.

For the results stated in the introduction, let L denote a connected Lie group, Λ a lattice in L, $\pi: L \to L/\Lambda$ the natural quotient map, and μ_L the (unique) L-invariant probability measure on L/Λ.

Theorem 1.1. *Let G be a connected semisimple Lie group. Let $a\in G$ be a semi-simple element; that is,* Ad(a) *is a semi-simple endomorphism of the Lie algebra of G. Consider the expanding horospherical subgroup U^+* of *G associated to a which is defined as*

$$U^+ = \{u\in G : a^{-n}ua^n \to e \ \text{ as } \ n\to\infty\}.$$

Assume that U^+ is not contained in any proper closed normal subgroup of G. Suppose that G is realized as a Lie subgroup of L and that $\pi(G)=\overline{L/\Lambda}$. Then

$$\overline{\pi(\cup_{n=1}^{\infty} a^n U^+)} = L/\Lambda.$$

In particular, if P is any parabolic subgroup of G and $\overline{\pi(G)} = L/\Lambda$, then $\overline{\pi(P)} = L/\Lambda$.

In the case of $L=G$ this result is well-known (see [DR, Prop. 1.5]). Actually theorem 1.1 is a straightforward consequence of a technically much stronger result stated later in the introduction as theorem. 1.4.

Using the techniques of [D3] along with theorem 1.1 and the result of Ratner [Ra2] on closures of orbits of unipotent flows on finite volume homogeneous spaces, in the next result we provide an affirmative answer to Question 1.1 in case when $X = L/\Lambda$. In this case we are able to relax certain other conditions in the question as well.

Theorem 1.2. *Let G be a semisimple Lie group of $\mathbb{R}$-rank $\geqslant 2$ and with finite center. Suppose that G is realized as a Lie subgroup of L such that the G-action is ergodic with respect to μ_L, and that $\overline{G_1 x}=\overline{Gx}$ for any $x\in L/\Lambda$ and any closed normal connected subgroup G_1 of G such that $\mathbb{R}$-rank$(G/G_1)\leqslant 1$. Let P be a parabolic subgroup of G and consider the diagonal action of G on $L/\Lambda \times G/P$. Let Y be a Hausdorff space with a continuous G-action and $\phi: L/\Lambda \times G/P \to Y$ a continuous G-equivariant map. Then there exist a parabolic subgroup $Q\supset P$, a locally compact Hausdorff space X with a continuous G-action, a continuous surjective G-equivariant map $\phi_1: L/\Lambda \to X$, and a continuous G-equivariant map $\psi: X\times G/Q\to Y$ such that the following holds:*

1. *If we define $\rho: L/\Lambda \times G/P \to X\times G/Q$ as $\rho(x,gP)=(\phi_1(x), gQ)$ for all $x\in L/\Lambda$ and $g\in G$, then*

$$\phi = \psi\circ\rho.$$

2. *There exists an open dense G-invariant set $X_0\subset L/\Lambda$ such that if we put $Z_0=\phi_1(X_0)\times G/Q$* and *$Y_0=\psi(Z_0)$, then $Z_0=\psi^{-1}(Y_0)$ and $\psi|_{Z_0}$ is injective.*

Furthermore if Y is a locally compact second countable space and ϕ is surjective, then Y_0 is open and dense in Y and $\psi|_{Z_0}$ is a homeomorphism onto Y_0.

In the next result we classify the G-equivariant factors of L/Λ, in particular we describe the factor $\phi_1: L/\Lambda \to X$ appearing in the statement of theorem 1.2. The proof of

this result uses the theorem of Ratner on orbit closures of unipotent flows and the main result of [MS].

DEFINITION 1.1

Let Λ_1 be a closed subgroup of L. A homeomorphism τ on L/Λ_1 is called an affine automorphism of L/Λ_1 if there exists $\sigma \in \mathrm{Aut}(L)$ such that $\tau(gx) = \sigma(g)\tau(x)$ for all $x \in L/\Lambda_1$. The group of all affine automorphisms of L/Λ_1 is denoted by $\mathrm{Aff}(L/\Lambda_1)$. It is endowed with the compact-open topology; i.e. its open sub-base consists of sets of the form $\{\tau \in \mathrm{Aff}(L/\Lambda_1) : \tau(C) \subset U\}$, where C is a compact subset of L/Λ_1 and U is an open subset of L/Λ_1.

Remark 1.1. (1) $\mathrm{Aff}(L/\Lambda_1)$ is a locally compact topological group acting continuously on L/Λ_1. (2) If $\sigma \in \mathrm{Aut}(L)$ is such that $\sigma(\Lambda_1) = \Lambda_1$ and if $g \in L$, then the map τ on L/Λ_1 defined by $\tau(h\Lambda_1) = g\sigma(h)\Lambda_1$ for all $h \in L$ is an affine automorphism. (3) Let Λ_1' be the maximal closed normal subgroup of L contained in Λ_1. Define $\bar{L} = L/\Lambda_1'$ and $\bar{\Lambda}_1 = \Lambda_1/\Lambda_1'$. Then we have natural isomorphisms $L/\Lambda \cong \bar{L}/\bar{\Lambda}_1$ and $\mathrm{Aff}(L/\Lambda_1) = \mathrm{Aff}(\bar{L}/\bar{\Lambda}_1)$.

Theorem 1.3. *Let G be a subgroup of L which is generated by one-parameter unipotent subgroups of L contained in G. Suppose that G acts ergodically on L/Λ. Let X be a Hausdorff locally compact space with a continuous G-action and $\phi: L/\Lambda \to X$ a continuous surjective G-equivariant map. Then there exists a closed subgroup Λ_1* containing *Λ, a compact group K contained in the centralizer of the subgroup of translations by elements of G in $\mathrm{Aff}(L/\Lambda_1)$, and a G-equivariant continuous surjective map $\psi: K\backslash L/\Lambda_1 \to X$ such that the following holds:*

1. *If $\rho: L/\Lambda \to K\backslash L/\Lambda_1$ is defined by $\rho(g\Lambda) = K(g\Lambda_1), \forall g \in L$, then ρ is G-equivariant and*
$$\phi = \psi \circ \rho.$$
2. *Given a neighbourhood Ω of e in $Z_L(G)$, there exists an open dense G-invariant subset X_0 of L/Λ_1 such that for any $x \in X_0$ and $y \in L/\Lambda_1$ if $\psi(K(x)) = \psi(K(y))$ then $y \in K(\Omega x)$. In this situation, further if $Gx = L/\Lambda_1$, then $K(y) = K(x)$.*

The above description of topological factors of unipotent flows is also of independent interest. The measurable factors of unipotent flows were described by Witte [W].

The next result is an immediate consequence of theorems 1.2 and 1.3.

COROLLARY 1.1

Let L be a Lie group, Λ a lattice in L, and G a connected semisimple Lie group with finite center, realized as a closed subgroup of L. Suppose that the action of G_1 on L/Λ is minimal for any closed normal subgroup G_1 of G such that $\mathbb{R}$-rank$(G/G_1) \leqslant 1$. Let Y be a locally compact Hausdorff space with a continuous G-action, P a parabolic subgroup of G, and $\phi: L/\Lambda \times G/P \to Y$ a continuous surjective G-equivariant map, where G acts diagonally on $L/\Lambda \times G/P$. Then there exist a parabolic subgroup Q of G containing P, a closed subgroup Λ_1 of L containing Λ, and a compact group K contained in the centralizer of the image of G in $\mathrm{Aff}(L/\Lambda_1)$, such that Y is G-equivariantly homeomorphic to $(K\backslash L/\Lambda_1) \times (G/Q)$ and ϕ is the natural quotient map.

In particular if, as in question 1.1, there exists a map $\psi: Y \to L/\Lambda$ such that $\psi \circ \phi$ is the projection on the first factor, then $\Lambda_1 = \Lambda$ and K is trivial. Hence Y is G-equivariantly homeomorphic to $L/\Lambda \times G/Q$ and ϕ is the natural quotient map.

For the purpose of other applications, we obtain a stronger measure of theoretic version of theorem 1.1. Before the statement, we recall some definitions.

For any Borel map $T:X \to Y$ of Borel spaces and a Borel measure λ on X, the Borel measure $T_*\lambda$ defined by $T_*\lambda(E) = \lambda(T^{-1}(E))$, for all Borel sets $E \subset Y$, is called the image of λ under T.

For any Borel measure μ on L/Λ and any $g \in L$, the translated measure $g \cdot \mu$ on L/Λ is the image of μ under the map $x \mapsto gx$ on L/Λ.

On a locally compact space X, for a sequence $\{\mu_i\}$ of finite Borel measures and μ a finite Borel measure, we say that $\mu_i \to \mu$ as $i \to \infty$, if and only if for all bounded continuous function f on X, $\int_X f\,\mathrm{d}\mu_i \to \int_X f\,\mathrm{d}\mu$ as $i \to \infty$.

Notation 1.1. Let G be a connected semisimple real algebraic group. Let A be an $\mathbb{R}$-split torus in G such that the set of real roots on A for the adjoint action on the Lie algebra of G forms a root system. Fix an order on this set of roots and let Δ be the corresponding system of simple roots. Let $\bar{A}^+$ be the closure of the positive Weyl chamber in A. Let $\{a_i\}_{i \in \mathbb{N}}$ be a sequence in $\bar{A}^+$ such that for any $\alpha \in \Delta$, either $\sup_{i \in \mathbb{N}} \alpha(a_i) < \infty$ or $\alpha(a_i) \to \infty$ as $i \to \infty$. Put

$$U^+ = \{g \in G : a_i^{-1} g a_i \to e \text{ as } i \to \infty\}.$$

Theorem 1.4. *Consider the notation* 1.1. *Assume that U^+ is not contained in any proper closed normal subgroup of G. Suppose that G is realized as a Lie subgroup of L and that $\pi(G)$ is dense in L/Λ. Then for any probability measure λ on U^+ which is absolutely continuous with respect to a Haar measure on U^+,*

$$a_i \cdot \pi_*(\lambda) \to \mu_L, \quad \textit{as} \quad i \to \infty.$$

In other words, for any bounded continuous function f on L/Λ,

$$\lim_{i \to \infty} \int_{U^+} f(a_i \pi(\omega))\,\mathrm{d}\lambda(\omega) = \int_{L/\Lambda} f\,\mathrm{d}\mu_L.$$

In particular, for any Borel set Ω of U^+ having strictly positive Haar measure,

$$\overline{\bigcup_{i \in \mathbb{N}} a_i \cdot \pi(\Omega)} = L/\Lambda.$$

Using this theorem we obtain the following generalization of a result due to Duke, Rudnick and Sarnak [DRS]; their result corresponds to the case of $L = G$. First we need a definition.

Let G be a semisimple Lie group. A subgroup S of G is said to be *symmetric* if there exists an involution σ of G (i.e. σ is a continuous automorphism and $\sigma^2 = 1$) such that $S = \{g \in G : \sigma(g) = g\}$. For example, any maximal compact subgroup of G is a symmetric subgroup, for it is the fixed point set of a Cartan involution of G.

COROLLARY 1.2

Let G be a connected real algebraic semisimple Lie group realized as a Lie subgroup of L, S the connected component of the identity of a symmetric subgroup of G, and $\{g_i\}_{i \in \mathbb{N}}$ a sequence contained in G. Suppose that $\pi(S)$ is closed and admits an S-invariant probability measure, say μ_S. Also suppose that $\pi(G_1)$ is dense in L/Λ. for any closed normal subgroup G_1 of G such that the image of $\{g_i\}$ in $G/(SG_1)$ admits a convergent subsequence. Then the sequence of measures $g_i \cdot \mu_S$ converges to μ_L; that is, for every

bounded continuous function f on L/Λ,

$$\lim_{i\to\infty}\int_{\pi(S)} f(g_i x)\,d\mu_S(x) = \int_{L/\Lambda} f\,d\mu_L.$$

In the case of $L = G$, Eskin and McMullen [EM] gave a proof of this result using the mixing property of geodesic flows. The main technical observation in their proof is what they call 'a wave front lemma'. In the general case of $L \supset G$, our analogue of the wave front lemma is theorem 1.4.

Using the arguments of the proof of corollary 1.2, one can also deduce the following result from theorem 1.4.

COROLLARY 1.3

Let G be a connected real algebraic semisimple group realized as a Lie subgroup of L. Let $\{g_i\}$ be a sequence in G. Suppose that $\pi(G_1)$ is dense in L/Λ for any closed normal subgroup G_1 of G such that the image of $\{g_i\}$ in G/G_1 admits a convergent subsequence. Then for any Borel probability measure λ on G which is absolutely continuous with respect to a Haar measure on G,

$$g_i\pi_*(\lambda)\to\mu_L \quad as \quad i\to\infty.$$

In particular, for any Borel set Ω of G having strictly positive Haar measure,

$$\overline{\bigcup_{i\in\mathbb{N}} g_i\cdot\pi(\Omega)} = L/\Lambda.$$

The main result of this paper is theorem 1.4 and other results (except theorem 1.3) are derived from it. The main steps of its proof are as follows. First suppose that the set of probability measures $\{a_i\pi_*(\lambda): i\in\mathbb{N}\}$ is not relatively compact in the space of all probability measures on L/Λ. Using an extension of a result of Dani and Margulis [DM2], in § 2 we see that there exist a nonempty open set $\Omega\subset U^+$, a finite dimensional representation V of L, a discrete set $\{\mathbf{v}_i: i\in\mathbb{N}\}\subset V$, and a compact set $K\subset V$ such that $a_i\Omega\cdot\mathbf{v}_i\subset K$ for infinitely many $i\in\mathbb{N}$. Via some observations about representations of semisimple Lie groups, in § 5 we show that the conditions mentioned above lead to a contradiction when we restrict the representation to G. Now let a probability measure μ be a limit distribution of the sequence $\{a_i\cdot\pi_*(\lambda)\}$. We observe that μ is U^+-invariant. Using Ratner's [Ra1] description of finite measures on L/Λ which are ergodic and invariant under the action of a unipotent subgroup, in § 3 we conclude that either $\mu=\mu_L$, or μ is non-zero when restricted to the image under π of some strictly lower dimensional 'algebraic subvariety' of L. Using techniques developed in [DM1, Sh1, DM3, MS], in § 4 we see that in the later case the above type of condition on a finite dimensional representation of L must hold, and this again leads to a contradiction. Thus $\mu=\mu_L$ and hence μ_L is the only limit distribution of $\{a_i\cdot\mu_\Omega\}$.

2. A condition for returning to compact sets

In [DM2] Dani and Margulis proved that large compact sets in finite volume homogeneous spaces have relative measures close to 1 on the trajectories of unipotent flows starting from a fixed compact set. This result was generalized in [EMS1] to a larger class of higher dimensional trajectories. In these results one considered only the case of arithmetic lattices in algebraic semisimple Lie groups defined over $\mathbb{Q}$. Here we modify them to include the case of any lattice in any Lie group.

Notation 2.1. Let G be a Lie group and $\mathfrak{g}$ the Lie algebra associated to G. For $d, m \in \mathbb{N}$, let $\mathscr{P}_{d,m}(G)$ denote the set of continuous maps $\Theta:\mathbb{R}^m \to G$ such that for all $\mathbf{c}, \mathbf{a} \in \mathbb{R}^m$ and $X \in \mathfrak{g}$, the map

$$t \in \mathbb{R} \mapsto \mathrm{Ad} \circ \Theta(t\mathbf{c} + \mathbf{a})(X) \in \mathfrak{g}$$

is a polynomial of degree at most d in each co-ordinate of $\mathfrak{g}$ (with respect to any basis).

We shall write $\mathscr{P}_d(G)$ for the set $\mathscr{P}_{d,1}(G)$.

Theorem 2.1 (Dani, Margulis). *Let G be a Lie group, Γ a lattice in G, and $\pi: G \to G/\Gamma$ the natural quotient map. Then given a compact set $C \subset G/\Gamma$, an $\varepsilon > 0$, and a $d \in \mathbb{N}$, there exists a compact subset $K \subset G/\Gamma$ with the following property: For any $\Theta \in \mathscr{P}_{d,m}(G)$ and any bounded open convex set $B \subset \mathbb{R}^m$, one of the following conditions hold:*

1. $(1/\nu(B))\nu(\{\mathbf{t} \in B: \pi(\Theta(\mathbf{t})) \in K\}) \geqslant 1 - \varepsilon$, *where ν denotes the Lebesgue measure on $\mathbb{R}^m$.*
2. $\pi(\Theta(B)) \cap C = \emptyset$.

Proof. See [Sh2, Theorem 3.1]. □

The usefulness of the above result is enhanced by the following theorem which provides an algebraic condition in place of the geometric condition $\pi(\Theta(B)) \cap C = \emptyset$.

Notation 2.2. Let G be a connected Lie group and $\mathfrak{g}$ denote the Lie algebra associated to G. Let $V_G = \oplus_{k=1}^{\dim \mathfrak{g}} \wedge^k \mathfrak{g}$, the direct sum of exterior powers of $\mathfrak{g}$, and consider the linear G-action on V_G via the representation $\oplus_{l=1}^{\dim \mathfrak{g}} \wedge^l \mathrm{Ad}$, the direct sum of exterior powers of the adjoint representation of G on $\mathfrak{g}$.

Fix any Euclidean norm on $\mathfrak{g}$ and let $\mathscr{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_{\dim \mathfrak{g}}\}$ denote an orthonormal basis of $\mathfrak{g}$. There is a unique Euclidean norm $\|\cdot\|$ on V_G such that the associated basis of V_G given by

$$\{\mathbf{e}_{l_1} \wedge \cdots \wedge \mathbf{e}_{l_r}: 1 \leqslant l_1 < \cdots < l_r \leqslant \dim \mathfrak{g}, \quad r = 1, \ldots, \dim \mathfrak{g}\}$$

is orthonormal. This norm is independent of the choice of $\mathscr{B}$.

To any Lie subgroup W of G and the associated Lie subalgebra $\mathfrak{w}$ of $\mathfrak{g}$ we associate a unit-norm vector $\mathbf{p}_W \in \wedge^{\dim \mathfrak{w}} \mathfrak{w} \in V_G$.

Theorem 2.2 (Cf. [DM2]). *Let G be a connected Lie group, Γ a lattice in G, and $\pi: G \to G/\Gamma$ the natural quotient map. Let M be the smallest closed normal subgroup of G such that $\bar{G} = G/M$ is a semisimple group with trivial center and without nontrivial compact normal subgroups. Let $q: G \to \bar{G}$ be the quotient homomorphism. Then there exist finitely many closed subgroups $W_1, \ldots, W_r$ of G such that each W_i is of the form $q^{-1}(U_i)$ with U_i the unipotent radical of a maximal parabolic subgroup of $\bar{G}$, $\pi(W_i)$ is compact and the following holds: Given $d, m \in \mathbb{N}$ and reals $\alpha, \varepsilon > 0$, there exists a compact set $C \subset G/\Gamma$ such that for any $\Theta \in \mathscr{P}_{d,m}(G)$, and a bounded open convex set $B \subset \mathbb{R}^m$, one of the following conditions is satisfied:*

1. *There exist $\gamma \in \Gamma$ and $i \in \{1, \ldots, r\}$ such that*

$$\sup_{\mathbf{t} \in B} \| \Theta(\mathbf{t})\gamma \cdot \mathbf{p}_{W_i} \| < \alpha.$$

2. $\pi(\Theta(B)) \cap C \neq \emptyset$, *and hence condition* (1) *of theorem* 2.1 *holds.*

Proof. Let R be the radical of G, C the maximal connected compact normal subgroup of G/R, $S = (G/R)/C$ and Z the center of S. Note that S is a semisimple Lie group

without nontrivial compact connected normal subgroups. Clearly $S/Z \cong G/M$. Therefore M is the inverse image of Z in G.

Let $H = \overline{R\Gamma^0}$. Then $H\Gamma$ is closed and $H \cap \Gamma$ is a lattice in H (see [R, Lemma 1.7]). By Auslander's theorem [R, Theorem 8.24] H is solvable, and so is its image in S. By Borel's density theorem [R, Lemma 5.4, Corollary 5.16] the image is a normal subgroup of S and therefore it has to be trivial. Hence $H \subset M^0$, and since $R \subset H$, M^0/H is compact. Since H is solvable, by Mostow's theorem [R, Theorem 3.1] $H/(H \cap \Gamma)$ is compact. Therefore $M^0/H \cap \Gamma$ is compact. So $M^0\Gamma/\Gamma$ is compact and $M^0\Gamma$ is closed.

Therefore the image Δ of Γ in S is discrete, and hence a lattice in S. Therefore by Borel's density theorem [R, Corollary 5.18] $Z\Delta$ is discrete. Hence Δ is of finite index in $Z\Delta$ and hence $M^0\Gamma$ is of finite index in $M\Gamma$. Hence $M\Gamma/\Gamma$ is compact, i.e. $\pi(M)$ is compact.

Thus $\bar{\Gamma} = q(\Gamma)$ is a lattice in $\bar{G}$ and the fibers of the map $\bar{q}: G/\Gamma \to \bar{G}/\bar{\Gamma}$ are compact M-orbits. Therefore without loss of generality, we may assume that $\bar{G} = G$.

Then there are finitely many normal connected subgroups $G_1, \ldots, G_r$ of G such that $G = G_1 \times \cdots \times G_r$ and each $\Gamma_i = G_i \cap \Gamma$ is an irreducible lattice in G_i (see [R, Sect. 5.22]). In proving the theorem without loss of generality we may replace Γ by its finite-index subgroup $\Gamma_1 \times \cdots \times \Gamma_r$. In order to prove the theorem for G, it is enought to prove it for each G_i separately. Thus without loss of generality we may assume that Γ is an irreducible lattice.

The result in the case of $\mathbb{R}$-rank$(G) = 1$ can be deduced from the arguments in [D2, (2.4)].

Next suppose that $\mathbb{R}$-rank$(G) \geqslant 2$. Then by the arithmeticity theorem of Margulis [M2], Γ is an arithmetic lattice. Therefore there exist a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ and a surjective homomorphism $\rho: \mathbf{G}(\mathbb{R})^0 \to G$ with compact kernel such that, for $\Lambda = \mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0$, the subgroup $\Gamma \cap \rho(\Lambda)$ is a subgroup of finite index in both Γ and $\rho(\Lambda)$. Again without loss of generality we may replace G by $\mathbf{G}(\mathbb{R})^0$ and Γ by Λ. In this case the result follows from [EMS1, Thm. 3.6].

3. Description of measures invariant under a unipotent flow

In this and the next section, let G denote a Lie group, Γ a lattice in G, and $\pi: G \to G/\Gamma$ the natural quotient map.

A subgroup U of G is called unipotent if $\mathrm{Ad}u$ is a unipotent endomorphism of the Lie algebra of G for every $u \in U$.

Let $\mathscr{H}_\Gamma$ denote the collection of all closed connected subgroup H of G such that (1) $H \supset \Gamma$, (2) $H/H \cap \Gamma$ admits a finite H-invariant measure, and (3) the subgroup generated by all one-parameter unipotent subgroups of H acts ergodically on $H/H \cap \Gamma$ with respect to the H-invariant probability measure. In particular, the Zariski closure of $\mathrm{Ad}(H \cap \Gamma)$ contains $\mathrm{Ad}(H)$ (see [Sh1, Theorem 2.3]).

Theorem 3.1 ([Ra1, Theorem 1.1]). *The collection $\mathscr{H}_\Gamma$ is countable.*

Let W be a subgroup of G which is generated by one-parameter unipotent subgroups of G contained in W. For any $H \in \mathscr{H}_\Gamma$ define

$$N_G(H, W) = \{g \in G : W \subset gHg^{-1}\},$$

$$S_G(H, W) = \bigcup_{\substack{H' \in \mathscr{H}_\Gamma, H' \subset H \\ \dim H' < \dim H}} N_G(H', W).$$

Note that (see [MS, Lemma 2.4]),

$$\pi(N_G(H, W)\backslash S_G(H, W)) = \pi(N_G(H, W))\backslash\pi(S_G(H, W)). \tag{1}$$

We reformulate Ratner's classification [Ra1] of finite measures which are invariant and ergodic under unipotent flows on homogeneous spaces of Lie groups, using the above definitions (see [MS, Theorem 2.2]).

Theorem 3.2. *Let W be a subgroup as above and μ a W-invariant probability measure on G/Γ. For every $H \in \mathscr{H}_\Gamma$, let μ_H denote the restriction of μ on $\pi(N_G(H, W)\backslash S_G(H, W))$. Then the following holds.*

1. *The measure μ_H is W-invariant, and any W-ergodic component of μ_H is of the form $g \cdot \lambda$, where $g \in N_G(H, W)\backslash S_G(H, W)$ and λ is a H-invariant measure on $H\Gamma/\Gamma$.*
2. *For any Borel measurable set $A \subset G/\Gamma$,*

$$\mu(A) = \sum_{H \in \mathscr{H}^*_\Gamma} \mu_H(A),$$

where $\mathscr{H}^*_\Gamma \subset \mathscr{H}_\Gamma$ is a countable set consisting of one representative from each Γ-conjugacy class of elements in $\mathscr{H}_\Gamma$.

In particular, if $\mu(\pi(S(G, W)) = 0$ then μ is the unique G-invariant probability measure on G/Γ.

4. Linear presentation of G-actions near singular sets

Let $C \subset \pi(N_G(H, W)\backslash S_G(H, W))$ be any compact set. It turns out that on certain neighborhoods of C in G/Γ, the G-action is equivariant with the linear G-action on certain neighbourhoods of a compact subset of a linear subspace in a finite dimensional linear G-space. We study unipotent trajectories in those thin neighbourhoods of C via this linearization. This type of technique is developed in ([DM1, Sh1, DM3, Sh2, MS, EMS2]).

Let V_G be the representation of G as described in notation 2.2. For $H \in \mathscr{H}_\Gamma$, let $\eta_H : G \to V_G$ be the map defined by $\eta_H(g) = g\mathbf{p}_H = (\wedge^d \mathrm{Ad}g)\mathbf{p}_H$ for all $g \in G$. Let $N_G(H)$ denote the normalizer of H in G. Define

$$N^1_G(H) = \eta_H^{-1}(\mathbf{p}_H) = \{g \in N_G(H) : \det(\mathrm{Ad}g|_{\mathfrak{h}}) = 1\}.$$

PROPOSITION 4.1 ([DM3, Theorem 3.4])

The orbit $\Gamma . \mathbf{p}_H$ is closed, and hence discrete. In particular, the orbit $N^1_G(H)\Gamma/\Gamma$ is closed in G/Γ.

Let W be a subgroup which is generated by one-parameter unipotent subgroups of G contained in W.

PROPOSITION 4.2 ([DM3, Prop. 3.2])

Let $V_G(H, W)$ denote the linear span of $\eta(N_G(H, W))$ in V_G. Then

$$\eta_H^{-1}(V_G(H, W)) = N_G(H, W).$$

Theorem 4.1. *Let $\varepsilon > 0$, d, $m \in \mathbb{N}$, and a compact set $C \subset \pi(N_G(H, W)\backslash S_G(H, W))$ be given. Then there exists a compact set $D \subset V_G(H, W)$ such that given any neighbourhood*

Φ *of* D in V_G, *there exists a neighbourhood* Ψ *of* C *in* G/Γ *such that for any* $\Theta \in \mathscr{P}_{d,m}(G)$, *and a bounded open convex set* $B \subset \mathbb{R}^m$, *one of the following conditions is satisfied:*

1. $\Theta(B)\gamma \cdot \mathbf{p}_H \subset \Phi$ *for some* $\gamma \in \Gamma$.
2. $$\frac{1}{\nu(B)}\nu(\{\mathbf{t}\in B:\Theta(\mathbf{t})\Gamma/\Gamma\in\Psi\})<\varepsilon.$$

Proof. The result is easily deduced from [Sh2, Prop. 5.4]. See also the proof of [Sh2, Thm. 5.2]. □

Some related results on unipotent flows

We recall a theorem of Ratner [Ra2] on closures of individual orbits of unipotent flows.

Theorem 4.2 (Ratner). *Let* G, Γ and W *be as above. Then for any* $x \in G/\Gamma$, *there exists a closed subgroup* F *of* G *containing* W *such that* $\overline{Wx} = Fx$ *and the orbit* Fx *admits a unique* F*-invariant probability measure, say* μ_F. *Also* μ_F *is* W*-ergodic.*

Next we recall a version of the main result of [MS].

Theorem 4.3 ([MS]). *Let* $x \in G/\Gamma$, *and sequences* $\{F_i\}$ *of closed subgroups of* G *and* $g_i \to e$ *in* G *be such that each of the orbits* $F_i(g_i x)$ *is closed, and admits an* F_i*-invariant probability measure, say* μ_i. *Suppose that the subgroup generated by all unipotent one-parameter subgroups of* G *contained in* F_i *acts ergodically with respect to* μ_i, $\forall i \in \mathbb{N}$. *Then there exists a closed subgroup* F *of* G *such that the orbit* Fx *is closed, and admits a* F*-invariant probability measure, say* μ, *and a subsequence of* $\{\mu_i\}$ *converges to* μ. *Moreover if* $\mu_i \to \mu$ *as* $i \to \infty$, *then* $g_i^{-1} F_i g_i \subset F$ *for all large* $i \in \mathbb{N}$.

5. Some results on linear representations

In view of proposition 4.1, in order to obtain further consequences when either condition 1 of theorem 2.2 or condition 1 of theorem 4.1 holds for a sequence $\{\Theta_i\} \subset \mathscr{P}_{d,m}(G)$, the following observation is very useful.

Linear actions of unipotent subgroups

Lemma 5.1. *Let* V *be a finite dimensional real vector space equipped with a Euclidean norm. Let* $\mathfrak{n}$ *be a nilpotent Lie subalgebra of* End (V). *Let* N *be the associated unipotent subgroup of* Aut(V). *Let* $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ *be a basis of* $\mathfrak{n}$. *Consider the map* $\Theta : \mathbb{R}^m \to N$ *defined as*

$$\Theta(t_1,\ldots,t_m)=\exp(t_m\mathbf{b}_m)\cdots\exp(t_1\mathbf{b}_1),\quad \forall (t_1,\ldots,t_m)\in\mathbb{R}^m.$$

For $\rho > 0$, *define*

$$B_\rho=\{\Theta(t_1,\ldots,t_m)\in N:0\leqslant t_k<\rho, k=1,\ldots,m\}.$$

Put

$$W=\{\mathbf{v}\in V:n\cdot\mathbf{v}=\mathbf{v},\forall n\in N\}.$$

Let pr_W *denote the orthogonal projection on* W. *Then for any* $\rho > 0$, *there exists* $c > 0$

such that for every $\mathbf{v}\in V$,

$$\|\mathbf{v}\| \leqslant c\cdot \sup_{\mathbf{t}\in B_\rho} \|\mathrm{pr}_W(\Theta(\mathbf{t})\cdot\mathbf{v})\|.$$

Proof. For $k=1,\ldots,m$, let $n_k\in\mathbb{N}$ be such that $\mathbf{b}_k^{n_k}=0$. Let

$$\mathscr{I}=\{I=(i_1,\ldots,i_m):0\leqslant i_k\leqslant n_k-1,\ k=1,\ldots,m\}.$$

For $\mathbf{t}=(t_1,\ldots,t_m)\in\mathbb{R}^m$ and $I=(i_1,\ldots,i_m)\in\mathscr{I}$, define

$$\mathbf{t}^I=t_m^{i_m}\ldots t_1^{i_1} \quad\text{and}\quad \mathbf{b}^I=\frac{\mathbf{b}_m^{i_m}\ldots\mathbf{b}_1^{i_1}}{i_m!\ldots i_1!}.$$

Then for all $\mathbf{v}\in V$ and $\mathbf{t}\in\mathbb{R}^m$, we have

$$\Theta(\mathbf{t})\cdot\mathbf{v}=\sum_{I\in\mathscr{I}}\mathbf{t}^I\cdot(\mathbf{b}^I\mathbf{v}). \tag{2}$$

We define a transformation $T:V\to\bigoplus_{I\in\mathscr{I}}W$ by

$$T(\mathbf{v})=(\mathrm{pr}_W(\mathbf{b}^I\cdot\mathbf{v}))_{I\in\mathscr{I}},\quad \forall\mathbf{v}\in V. \tag{3}$$

We claim that T is injective. To see this, suppose there exists $\mathbf{v}\in V\setminus\{0\}$ such that $T(\mathbf{v})=0$. Then $N\cdot\mathbf{v}\subset W^\perp$, the orthogonal complement of W. Hence $W^\perp$ contains a non-trivial N-invariant subspace. Then by Lie–Kolchin theorem, $W^\perp$ contains a non-zero vector fixed by N. Then $W\cap W^\perp\neq\{0\}$, which is a contradiction.

We consider $\bigoplus_{I\in\mathscr{I}}V$ equipped with a box norm; that is

$$\|(v_I)_{I\in\mathscr{I}}\|=\sup_{I\in\mathscr{I}}\|v_I\|,\quad\text{where}\quad v_I\in V,\ \forall\, I\in\mathscr{I}.$$

Since T is injective, there exists a constant $c_1>0$ such that

$$\|\mathbf{v}\|\leqslant c_1\cdot\|T(\mathbf{v})\|,\quad\forall\ \mathbf{v}\in V.$$

For all $k=1,\ldots,m$, and $j_k=1,\ldots,n_k$, fix $0<t_{k,1}<\cdots<t_{k,n_k}<\rho$ and put $M_k=(t_{k,j_k}^{i_k})_{0\leqslant i_k\leqslant n_k-1,1\leqslant j_k\leqslant n_k}$ for $k=1,\ldots,m$. Then det M_k is a Vandermonde determinant and hence M_k is invertible.

Let

$$\mathscr{J}=\{J=(j_1,\ldots,j_m):1\leqslant j_k\leqslant n_k,k=1,\ldots,m\}.$$

For $J=(j_1,\ldots,j_m)\in\mathscr{J}$, put

$$\mathbf{t}_J=(t_{1,j_1},\ldots,t_{m,j_m})\quad\text{and}\quad M=(\mathbf{t}_J^I)_{(I,J)\in\mathscr{I}\times\mathscr{J}}.$$

Take $\mathbf{v}\in V$. Put

$$X_{\mathscr{I}}=T(\mathbf{v})\quad\text{and}\quad Y_{\mathscr{J}}=(\mathrm{pr}_W(\Theta(\mathbf{t}_J)\mathbf{v}))_{J\in\mathscr{J}}.$$

Then by (2) and (3),

$$M\cdot X_{\mathscr{I}}=Y_{\mathscr{J}}.$$

Since $M=M_1\otimes\cdots\otimes M_m$ and each M_k is invertible, we have that M is invertible. Hence

$$X_{\mathscr{I}}=M^{-1}\cdot Y_{\mathscr{J}}.$$

Put $c_2 = \|M^{-1}\|$ and $c = c_1 c_2$. Then

$$\|\mathbf{v}\| \leqslant c_1 \|T(\mathbf{v})\| = c_1 \|X_{\mathscr{J}}\| \leqslant c_1 c_2 \|Y_{\mathscr{J}}\| = c \cdot \sup_{J \in \mathscr{J}} \|\mathrm{pr}_W(\Theta(\mathbf{t}_J)\mathbf{v}\|.$$

This completes the proof. □

Linear actions of semisimple groups

We fix the following setup for the rest of this section.

Notation 5.1. Consider the notation 1.1. Put

$$\Phi = \{\alpha \in \Delta : \alpha(a_i) \to \infty \quad \text{as} \quad i \to \infty\}.$$

Let P^+ be the standard parabolic subgroup associated to the set of roots $\Delta \backslash \Phi$. Then $U^+ = \{g \in G : a_i^{-1} g a_i \to e \text{ as } i \to \infty\}$ is the unipotent radical of P^+. Let P^- denote the standard opposite parabolic subgroup for P^+ and let U^- be the unipotent radical of P^-. Note that

$$P^- = \{g \in G : \overline{\{a_i g a_i^{-1} : i \in \mathbb{N}\}} \text{ is compact}\}. \tag{4}$$

Also put $Z = P^- \cap P^+$. Then $P^- = U^- Z$. Let $\mathfrak{g}$, $\mathfrak{u}^-$, $\mathfrak{z}$, and $\mathfrak{u}^+$ denote the Lie algebras associated to G, U^-, Z, and U^+, respectively. Then

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{z} \oplus \mathfrak{u}^+. \tag{5}$$

Lemma 5.2. *Consider a continuous nontrivial irreducible representation of G on a finite dimensional normed vector space V. Let $W = \{\mathbf{v} \in V : W \cdot \mathbf{v} = \mathbf{v}\}$. Let $\{\mathbf{v}_i\} \subset W$ be a sequence such that* $\inf_{i \in \mathbb{N}} \|\mathbf{v}_i\| > 0$. *Then*

$$\|a_i \cdot \mathbf{v}_i\| \to \infty \quad \textit{as } i \to \infty.$$

Proof. Since A is $\mathbb{R}$-split, there is a finite set Λ of real characters on A such that for each $\lambda \in \Lambda$, if we define

$$V_\lambda = \{\mathbf{v} \in V : a \cdot \mathbf{v} = \lambda(a)\mathbf{v}, \forall a \in A\},$$

then $V = \oplus_{\lambda \in \Lambda} V_\lambda$. After passing to an appropriate subsequence, if we define

$$\Lambda_+ = \{\lambda \in \Lambda : \lambda(a_i) \to \infty \text{ as } i \to \infty\}$$
$$\Lambda_- = \{\lambda \in \Lambda : \lambda(a_i) \to 0 \text{ as } i \to \infty\}, \qquad \text{and}$$
$$\Lambda_0 = \{\lambda \in \Lambda : \lambda(a_i) \to c \text{ for some } c > 0 \text{ as } i \to \infty\},$$

then $\Lambda = \Lambda_+ \cup \Lambda_0 \cup \Lambda_-$.

Since U^+ is normalized by A, we have that W is invariant under the action of A. Therefore

$$W = \oplus_{\lambda \in \Lambda}(W \cap V_\lambda).$$

Suppose that there exists $\mathbf{w} \in W \cap V_\lambda \backslash \{0\}$ for some $\lambda \in \Lambda_0 \cup \Lambda_-$. For any $g \in P^-$, we have $a_i g a_i^{-1} \to g_0$ for some $g_0 \in P^-$. Therefore as $i \to \infty$,

$$a_i(g\mathbf{w}) = a_i g a_i^{-1}(a_i \mathbf{w}) \to c(g_0 \mathbf{w}) \quad \text{for some } c \geqslant 0.$$

Hence $P^- \mathbf{w} \subset \oplus_{\lambda \in \Lambda_0 \cup \Lambda_-} V_\lambda$. Now $U^+ \mathbf{w} = \mathbf{w}$ and by notation 5.1 $P^- U^+$ is open in G. Therefore $G \cdot \mathbf{w} \subset \oplus_{\lambda \in \Lambda_0 \cup \Lambda_-} V_\lambda$. Since V is irreducible, $\Lambda = \Lambda_0 \cup \Lambda_-$. Now since G is semisimple, $\det g = 1$ for all $g \in G$ and hence $\Lambda_- = \emptyset$. Thus $\Lambda = \Lambda_0$.

Now for any relatively compact neighbourhood Ω of U^+ and any $\mathbf{v} \in V_\lambda$, there exists a compact ball $B \subset V$ such that for all $i \in \mathbb{N}$,

$$B \supset a_i \Omega \cdot \mathbf{v} = (a_i \Omega a_i^{-1}) a_i \cdot \mathbf{v} = \lambda(a_i)(a_i \Omega a_i^{-1}) \mathbf{v}.$$

Since $\lambda(a_i) \to c$ for some $c > 0$ and $\cup_{i \in \mathbb{N}} a_i \Omega a_i^{-1} = U^+$, we have $U^+ \cdot \mathbf{v} \subset c^{-1} B$. Since U^+ acts on V by unipotent linear transformations, we obtain that $U^+ \cdot \mathbf{v} = \mathbf{v}$. Thus U^+ acts trivially on V. Since the kernel of G action on V is a normal subgroup of G containing U^+, it is equal to G by our assumption. This contradicts our hypothesis in the lemma that the action of G is non-trivial. This proves that $W \subset \Sigma_{\lambda \in \Lambda_+} V_\lambda$, and the conclusion of the lemma follows. □

COROLLARY 5.1

Consider a continuous representation of G on a finite dimensional vector space V with a Euclidean norm. Let $L = \{\mathbf{v} \in V : G \cdot \mathbf{v} = \mathbf{v}\}$. Let $\{\mathbf{v}_i\}$ be a discrete subset of V contained in $V \backslash L$. Then for any non-empty open set $\Omega \subset U^+$,

$$\sup_{\omega \in \Omega} \| a_i \omega \cdot \mathbf{v}_i \| \to \infty \quad \text{as} \quad i \to \infty. \tag{6}$$

Proof. Let L' be the sum of all G-invariant irreducible subspaces of $\dim \geqslant 2$. After passing to a subsequence, one of the following holds:

$$\text{(A)} \quad \| \mathrm{pr}_L(\mathbf{v}_i) \| \to \infty, \quad \text{or} \quad \text{(B)} \quad \inf_{i \in \mathbb{N}} \| \mathrm{pr}_{L'}(\mathbf{v}_i) \| > 0.$$

If (A) holds then (6) is obvious. If (B) holds, then there exists an irreducible G-subspace $V_1 \subset L'$ such that $\inf_{i \in \mathbb{N}} \| \mathrm{pr}_{V_1}(\mathbf{v}_i) \| > 0$. Therefore, without loss of generality, by replacing $\{\mathbf{v}_i\}$ by $\{\mathrm{pr}_{V_1}(\mathbf{v}_i)\}$ and V by V_1 we may assume that G acts non-trivially and irreducibly on V and $\inf_{i \in \mathbb{N}} \| \mathbf{v}_i \| > 0$.

Let $\omega_0 \in \Omega$. Then $\inf_{i \in \mathbb{N}} \| \omega_0 \mathbf{v}_i \| > 0$. Therefore replacing $\{\mathbf{v}_i\}$ by $\{\omega_0 \mathbf{v}_i\}$ and Ω by $\Omega \omega_0^{-1}$, we may assume that $e \in \Omega$.

Let $W = \{\mathbf{v} \in V : U^+ \cdot \mathbf{v} = \mathbf{v}\}$. By lemma 5.1, there exists $c > 0$ such that for all $i \in \mathbb{N}$,

$$\sup_{\omega \in \Omega} \| \mathrm{pr}_W(\omega \cdot \mathbf{v}_i) \| \geqslant c \| \mathbf{v}_i \| \geqslant c \cdot \inf_{j \in \mathbb{N}} \| \mathbf{v}_j \|.$$

Since $\inf_{j \in N} \| \mathbf{v}_j \| > 0$, by lemma 5.2,

$$\sup_{\omega \in \Omega} \| a_i \cdot \omega \mathbf{v}_i \| \geqslant \sup_{\omega \in \Omega} \| a_i \cdot \mathrm{pr}_W(\omega \cdot \mathbf{v}_i) \| \to \infty \quad \text{as} \quad i \to \infty.$$ □

6. Proofs of the main results

Translates of horospherical patches

Proof of theorem 1.4. Since U^+ is σ-compact, without loss of generality we may assume that $\mathrm{supp}(\lambda)$ is compact. Let $\mathfrak{u}^+$ denote the Lie algebra of U^+. We identify $\mathfrak{u}^+$ with $\mathbb{R}^m$

($m = \dim \mathfrak{u}^+$). Let B be a ball in $\mathfrak{u}^+$ around the origin such that $\mathrm{supp}(\lambda) \subset \exp(B)$. Let ν be the restriction of the Lebesgue measure on B. By our hypothesis, λ is absolutely continuous with respect to $\exp_*(\nu)$, denoted by $\lambda \ll \exp_*(\nu)$.

For each $i \in \mathbb{N}$, define $\Theta_i : \mathbb{R}^m \to G \subset L$ as $\Theta_i(\mathbf{t}) = a_i \exp(\mathbf{t})$, $\forall \mathbf{t} \in \mathbb{R}^m \cong \mathfrak{u}^+$. Since $\mathfrak{u}^+$ is a nilpotent Lie algebra, there exists $d \in \mathbb{N}$ such that $\Theta_i \in \mathscr{P}_{d,m}(L)$, $\forall i \in \mathbb{N}$.

Claim 6.1. Given $\delta > 0$ there exists a compact set $K \subset L/\Lambda$ such that

$$(a_i \pi_*(\lambda))(K) > 1 - \delta, \quad \forall i \in \mathbb{N}.$$

Suppose that the claim fails to hold. Since $\lambda \ll \exp_*(\nu)$, there exists an $\varepsilon > 0$ such that for any compact set $K \subset L/\Lambda$,

$$\frac{1}{\nu(B)}(\Theta_i)_*(\nu)(K) < 1 - \varepsilon, \quad \text{for } i \text{ in a subsequence.}$$

We apply theorems 2.1 and 2.2 for the Lie group L, the lattice Λ, and the polynomial maps $\Theta_i \in \mathscr{P}_{d,m}(L)$, $\forall i \in \mathbb{N}$. Then by passing to a subsequence, there exists a continuous representation of L on a finite dimensional vector space V with a Euclidean norm and a non-zero vector $\mathbf{p} \in V$ such that the following holds: (1) the orbit $\Gamma \cdot \mathbf{p}$ is discrete (see proposition 4.1), and (2) for each $i \in \mathbb{N}$ there exists $\mathbf{v}_i \in \Gamma \cdot \mathbf{p}$ such that

$$\sup_{\omega \in \exp(B)} \|a_i \omega \cdot \mathbf{v}_i\| \to 0 \quad \text{as} \quad i \to \infty. \tag{7}$$

After passing to a subsequence, we may assume that $G \cdot \mathbf{v}_i \neq \mathbf{v}_i$, $\forall i \in \mathbb{N}$. Then corollary 5.1 contradicts (7). This proves the claim.

By claim 6.1, after passing to a subsequence, we may assume that the sequence $a_i \cdot \pi_*(\lambda) \to \mu$ as $i \to \infty$, where μ is a probability measure on L/Λ.

Claim 6.2. The measure μ is U^+-invariant.

To prove the claim, let $u \in U^+$. Then for all $i \in \mathbb{N}$,

$$u(a_i \pi_*(\lambda)) = a_i(u_i \pi_*(\lambda)) = a_i \pi_*(u_i \lambda), \tag{8}$$

where $u_i = a_i^{-1} u a_i \in U^+$. Note that $u_i \to e$ as $i \to \infty$.

Let η be a Haar measure on U^+. Since $\lambda \ll \eta$, there exists $h \in L^1(U, \eta)$ such that $\mathrm{d}\lambda = h\mathrm{d}\eta$. Now for any bounded continuous function f on L/Λ,

$$\begin{aligned}
&|\textstyle\int f \mathrm{d}[a_i \pi_*(u_i \lambda))] - \int f \mathrm{d}[a_i \pi_*(\lambda)]| \\
&\quad = |\textstyle\int_{U^+} f(a_i \pi(u_i \omega)) \mathrm{d}\lambda(\omega) - \int_{U^+} f(a_i \pi(\omega)) \mathrm{d}\lambda(\omega)| \\
&\quad = |\textstyle\int_{U^+} f(a_i \pi(u_i \omega) h(\omega) \mathrm{d}\eta(\omega) - \int_{U^+} f(a_i \pi(\omega)) h(\omega) \mathrm{d}\eta(\omega)| \\
&\quad = |\textstyle\int_{U^+} f(a_i \pi(\omega)) h(u_i^{-1} \omega) \mathrm{d}\eta(\omega) - \int_{U^+} f(a_i \pi(\omega)) h(\omega) \mathrm{d}\eta(\omega)| \\
&\quad \leqslant \sup|f| \cdot \textstyle\int_{U^+} |h(u_i^{-1} \omega) - h(\omega)| \mathrm{d}\eta(\omega) \to 0 \quad \text{as} \quad i \to \infty,
\end{aligned} \tag{9}$$

because the left regular representation of U^+ on $L^1(U^+, \eta)$ is continuous.

Since $a_i \pi_*(\lambda) \to \mu$ as $i \to \infty$, by (9), we get $a_i \pi_*(u_i \lambda) \to \mu$ as $i \to \infty$. Therefore by (8), $u\mu = \mu$. This completes the proof of the claim.

In view of claim 6.2, we apply theorem 3.2 to $W = U^+$. Then there exists a closed subgroup H of L in the collection $\mathscr{H}_\Lambda$, such that

$$\mu(\pi(S_L(H, U^+)) = 0 \quad \text{and} \quad \mu(\pi(N_L(H, U^+))) > 0.$$

Let a compact set $C \subset \pi(N_L(H, U^+)) \backslash \pi(S_L(H, U^+))$ be such that $\mu(C) > 0$. Since $\lambda \ll \exp_*(\nu)$, there exists $\varepsilon > 0$ such that for any Borel measurable set $E \subset U^+$,

$$\frac{1}{\nu(B)} \exp_*(\nu)(E) < \varepsilon \Rightarrow \lambda(E) < \mu(C)/2. \tag{10}$$

Let the finite dimensional vector space V_L and the unit vector $\mathbf{p}_H \in V_L$ be as described in notation 2.2, for L in place of G there. We apply theorem 4.1 for $\varepsilon > 0$, $d \in \mathbb{N}$ and $m \in \mathbb{N}$ chosen as above, and the compact set $C \subset \pi(N_L(H, U^+)) \backslash \pi(S_L(H, U^+))$ as above. Then there exists a relatively compact set $\Phi \subset V_L$ and an open neighbourhood Ψ of C in L/Λ such that for each $i \in \mathbb{N}$, applying the theorem to Θ_i in place of Θ, one of the following conditions holds:

1. There exists $\mathbf{v}_i \in \Lambda \cdot \mathbf{p}_H$ such that

$$a_i \exp(B) \cdot \mathbf{v}_i \subset \Phi.$$

2.

$$\frac{1}{\nu(B)} \nu(\{\mathbf{t} \in B : \pi(a_i \exp(\mathbf{t})) \in \Psi\}) < \varepsilon.$$

Since $a_i \pi_*(\lambda) \to \mu$ as $i \to \infty$ and Ψ is a neighbourhood of C, there exists $i_0 \in \mathbb{N}$ such that $\lambda(\pi^{-1}(a_i^{-1}\Psi) \cap U^+) > \mu(C)/2$ for all $i \geqslant i_0$. Therefore by (10) condition 1 must hold for all $i \geqslant i_0$. Now by passing to a subsequence, there exists $\mathbf{v}_i \in \Lambda \cdot \mathbf{p}_H$ for each $i \in \mathbb{N}$ such that

$$a_i \exp(B) \cdot \mathbf{v}_i \subset \Phi. \tag{11}$$

By proposition 4.1, the sequence $\{\mathbf{v}_i\}$ is discrete. By corollary 5.1 and (11), there exists $i_0 \in \mathbb{N}$ such that $G \cdot \mathbf{v}_{i_0} = \mathbf{v}_{i_0}$. Let $\gamma \in \Lambda$ such that $\mathbf{v}_{i_0} = \gamma \mathbf{p}_H$. Then

$$G \cdot \gamma \cdot \mathbf{p}_H = \gamma \cdot \mathbf{p}_H.$$

Thus $G \subset \gamma N_L^1(H) \gamma^{-1}$. But $\pi(N_L^1(H))$ is closed in L/Λ by proposition 4.1, and $\pi(G)$ is dense in L/Λ. Therefore we conclude that H is a normal subgroup of L. Since $N_L(H, U^+) \supset C \neq \emptyset$, this implies in particular that U^+ is contained in H. Thus $U^+ \subset G \cap H$ and $G \cap H$ is normal in G. Therefore by our hypothesis $G \cap H = G$, or in other words $G \subset H$. Again since $\pi(G)$ is dense in L/Λ, we have $H = L$. Therefore $\mu(\pi(S(L, U^+))) = 0$. Hence by theorem 3.2, we have that μ is L-invariant. This completes the proof of the theorem. □

Translates of orbits of symmetric subgroups

First we make some observations. For the results stated below, let (U, ν_1) and (V, ν_2) be locally compact second countable spaces with Borel measures.

PROPOSITION 6.1

Let λ be a Borel probability measure on $U \times V$ which is absolutely continuous with respect to $\nu_1 \times \nu_2$, denoted by $\lambda \ll \nu_1 \times \nu_2$. Then there exists a probability measure $\lambda_1 \ll \nu_1$ on U, and for almost all $u \in (U, \lambda_1)$, there exists a probability measure $\lambda_u \ll \delta_u \times \nu_2$ on $\{u\} \times V$, where δ_u is the point mass at $\{u\}$, such that the following holds: For any bounded continuous function f on $U \times V$, the map $u \mapsto \int_{\{u\} \times V} f \, d\lambda_u$ is λ_1-measurable, and

$$\int_{U \times V} f \, d\lambda = \int_U \left(\int_{\{u\} \times V} f \, d\lambda_u \right) d\lambda_1(u).$$

Proof. Let $h = \mathrm{d}\lambda/\mathrm{d}(\nu_1 \times \nu_2) \geqslant 0$ be the Radon-Nikodym derivative. For any $u \in U$, put $\alpha(u) = \int_V h(u,v)\mathrm{d}\nu_2(v)$. Let $C = \{u \in U : \alpha(u) > 0\}$. Let λ_1 be the restriction of ν_1 to C. For almost any $u \in (U, \lambda_1)$, let λ_u be the Borel measure on $\{u\} \times V$ such that $\mathrm{d}\lambda_u/\mathrm{d}[\delta_u \times \nu_2] = h(u,\cdot)/\alpha(u)$. Now the conclusion of the proposition follows from Fubini's theorem. □

For the propositions stated below, let G be a locally compact topological group acting continuously on a locally compact space X. Let $\{a_i\}$ be a sequence in G and μ a Borel probability measure on X.

PROPOSITION 6.2

Let λ be a probability measure on X such that $a_i\lambda \to \mu$ as $i \to \infty$. Let $b \in G$ such that $\{a_i b a_i^{-1} : i \in \mathbb{N}\}$ is compact. If μ is G-invariant, then $a_i(b\lambda) \to \mu$ as $i \to \infty$.

Proof. First observe that there is no loss of generality in passing to a subsequence. Therefore we may assume that $a_i b a_i^{-1} \to g$ for some $g \in G$. Now

$$a_i(b\lambda) = (a_i b a_i^{-1})(a_i\lambda) \to g\mu \quad \text{as} \quad i \to \infty.$$

Since $g\mu = \mu$, the proof is complete. □

For the next two propositions, assume that G contains the spaces U and V. Fix $x_0 \in X$, and let $\rho: U \times V \to X$ be the map given by $\rho(u,v) = uvx_0$, $\forall (u,v) \in U \times V$.

PROPOSITION 6.3

Let the notation be as in proposition 6.1. *Suppose that for almost all $u \in (U, \lambda_1)$, we have $a_i\rho_*(\lambda_u) \to \mu$ as $i \to \infty$. Then $a_i\rho_*(\lambda) \to \mu$ as $i \to \infty$.*

Proof. Let f be a bounded continuous function on X. Then

$$\begin{aligned}
\int_X f\,\mathrm{d}[a_i\rho_*(\lambda)] &= \int_{U\times V} f(a_i\rho(\omega))\mathrm{d}\lambda(\omega)\\
&= \int_V \mathrm{d}\lambda_1(u)\cdot\int_{\{u\}\times V} f(a_i\rho(\omega))\mathrm{d}\lambda_u(\omega)\\
&= \int_V \mathrm{d}\lambda_1(u)\cdot\int_X f\,\mathrm{d}[a_i\rho_*(\lambda_u)]\\
&\to \int_V \mathrm{d}\lambda_1(u)\cdot\int_X f\,\mathrm{d}\mu \quad \text{as } i\to\infty\\
&= \int_X f\,\mathrm{d}\mu.
\end{aligned}$$

□

By similar arguments we obtain the following result.

PROPOSITION 6.4

Suppose that $a_i u a_i^{-1} \to e$ as $i \to \infty$ for all $u \in U$. Then as $i \to \infty$,

$$a_i\rho_*(\nu_2) \to \mu \Leftrightarrow a_i\rho_*(\nu_1 \times \nu_2) \to \mu.$$

Proof of corollary 1.2. Using the results in [S, § 7.1] there exist an $\mathbb{R}$-split torus $A \subset G$ and a maximal compact subgroup K of G such that the following holds: (1) $\sigma(a) = a^{-1}$, $\forall a \in A$, (2) the set of real roots of A for the adjoint action on the Lie algebra of G forms a root system, and (3) G admits a decomposition $G = K\bar{A}^+S$, were $\bar{A}^+$ denotes the closure of the positive Weyl chamber with respect to a system Δ of simple roots on A.

Using this decomposition and by passing to a subsequence, without loss of generality we may assume the following: (1) $g_i = a_i \in \bar{A}^+$ for all $i \in \mathbb{N}$; (2) $\{a_i\}_{i\in\mathbb{N}}$ has no convergent subsequence, (because otherwise $G_1 = \{e\}$ and $\pi(e)$ cannot be dense in L/Λ); and (3) for any $\alpha \in \Delta$, either $\sup_{i\in\mathbb{N}} \alpha(a_i) < \infty$ or $\alpha(a_i) \to \infty$ as $i \to \infty$.

For the rest of the proof, consider the notation 5.1.

Let G_1 be the smallest closed normal subgroup of G containing U^+. Then it is straightforward to verify that the projection of $\{a_i\}$ on G/G_1 is relatively compact. Therefore by our hypothesis, $\overline{\pi(G_1)} = L/\Lambda$.

Take any $g_0 \in S$ and define $\rho(h) = \pi(hg_0)$ for all $h \in L$. Since any closed connected normal subgroup of G_1 is also normal is G, we can apply theorem 1.4 to G_1 in place of G and ρ in place of π. Then for any probability measure ν on U^+ which is absolutely continuous with respect to a Haar measure on U^+, we have

$$a_i \rho_*(\nu) \to \mu_L, \quad \text{as } i \to \infty. \tag{12}$$

Since $\sigma(a) = a^{-1} (\forall a \in A)$, for any $X \in \mathfrak{u}^+$, we have $\sigma(X) \in \mathfrak{u}^-$ and $X + \sigma(X) \in \mathfrak{s}$. Also $\sigma(\mathfrak{z}) = \mathfrak{z}$. Now by (5),

$$\mathfrak{u}^- \oplus \mathfrak{s} = \mathfrak{u}^- \oplus (\mathfrak{s} \cap \mathfrak{z}) \oplus \mathfrak{u}^+. \tag{13}$$

Then by implicit function theorem, there exist relatively compact neighbourhoods Ω^-, Ω^0, Ω^+ and Φ of e in $U^-, (Z \cap S)U^-, U^+$ and S, respectively, such that for any open set Ψ of Φ, we have that $\Omega^- \Psi$ is an open subset of $\Omega^0 \Omega^+$. Also we may assume that under the multiplication map $\Omega^- \times \Phi \cong \Omega^- \Phi$ and $\Omega^0 \times \Omega^+ \cong \Omega^0 \Omega^+$.

Let ν_- and ν' be probability measures obtained by restricting Haar measures of U^- and S to Ω^- and Ψ, respectively. Then $\lambda = \nu_- \times \nu'$ is a smooth measure on $\Omega^- \times \Psi$. By choosing Ψ small enough, we can ensure that $\rho_*(\nu')$ is a multiple of μ_S restricted to $\rho(\Psi)$. Since $g_0 \in S$ chosen in the definition of ρ is arbitrary and since there is enough flexibility in the choices of Φ and Ψ, to prove that $a_i \mu_S \to \mu_L$, it is enough to show that $a_i \rho_*(\nu') \to \mu_L$ as $i \to \infty$.

By proposition 6.4, as $i \to \infty$, $a_i \rho_*(\nu') \to \mu_L$ if and only if $a_i \rho_*(\lambda) \to \mu_L$. Therefore to complete the proof of the corollary, it is enough to show the following.

Claim 6.3. As $i \to \infty$, $a_i \rho_*(\lambda) \to \mu_L$.

Since $\Omega^- \Psi \subset \Omega^0 \Omega^+$, λ can be treated as a measure on $\Omega^0 \times \Omega^+$. Let ν_1 and ν_2 be the probability measures obtained by restricting the Haar measures on $(Z \cap S)U^-$ and U^+ to Ω^0 and Ω^+, respectively. Since λ is a smooth measure, $\lambda \ll \nu_1 \times \nu_2$ (see (13)). Decompose λ as in proposition 6.1. Then for almost all $\omega \in (\Omega^0, \lambda_1)$, we have $\lambda_\omega \ll \omega \nu_2$. Put $\nu_\omega = \omega^{-1} \lambda_\omega$. Then $\nu_\omega \ll \nu_2$. Hence by (4), (12) and proposition 6.2,

$$a_i \rho_*(\lambda_\omega) = a_i(\omega \rho_*(\nu_\omega)) \to \mu_L \quad \text{as } i \to \infty.$$

Now by proposition 6.3, $a_i \pi_*(\lambda) \to \mu_L$ as $i \to \infty$. This completes the proof of the claim, and also the proof of the corollary. □

Continuous G-equivariant factors of $G/P \times L/\Lambda$

First we recall the following result from [D3, § 2].

PROPOSITION 6.5 (Dani)

Let G be a semisimple group with finite center and $\mathbb{R}$-rank$(G) \geq 2$. Let P be a parabolic subgroup of G. Then given $g \in G \backslash P$, there exist $k \in \mathbb{N}$ ($k < \mathbb{R}$-rank(G)), elements

$g_1,\dots,g_{k+1}$ in G, and one-parameter unipotent subgroups $\{u_1(t)\},\dots,\{u_k(t)\}$ of G contained in P such that the following holds:

1. *$g_1 = g$, $g_k \notin P$, and $g_{k+1} = e$.*
2. *For each $i = 1,\dots,k$,*

$$u_i(t)g_iP \to g_{i+1}P \text{ in } G/P \text{ as } t \to \infty.$$

3. *There exists a semisimple element a of G in $g_kPg_k^{-1} \cap P$ such that if U^+ is the associated horospherical subgroup then $U^+ \subset g_kPg_k^{-1} \cap P$, and if G_1 denotes the smallest normal subgroup of G containing U^+, then $\mathbb{R}$-rank$(G/G_1) \leqslant 1$.*

Proof. Apply [D3, Corollary 2.3] iteratively. Also use the proofs of [D3, Corollary 2.6 and Lemma 2.7]. □

Now we obtain the analogue of [D3, Lemma 1.4] by using theorem 1.1 in place of [D3, Lemma 1.1]. Also we use the recurrence conclusion of theorem 4.2 of Ratner in place of [D3, Lemma 1.6].

PROPOSITION 6.6

Let the notation and assumptions be as in theorem 1.2. Let $x, y \in L/\Lambda$ and $g \in G \backslash P$. If $\phi(x, gP) = \phi(y, P)$, there exists a parabolic subgroup Q containing $\{g\} \cup P$ such that $\phi(z, P) = \phi(z, qP)$ for all $z \in \overline{Gx}$ and $q \in Q$. Moreover, $\phi(y, qP) = \phi(y, P)$ for all $q \in Q$.

Proof. Let $k \in \mathbb{N}$, elements $g_1,\dots,g_{k+1}$ in G, the one-parameter unipotent subgroups $\{u_i(t)\}$ contained in P, and a semisimple element a and G and the associated expanding horospherical subgroup U^+ be as in proposition 6.5. For each $i = 1,\dots,k$, by Ratner's theorem 4.2 applied to the diagonal action of $\{u_i(t)\}$ on $L/\Lambda \times L/\Lambda$, there exists a sequence $t_n \to \infty$ such that $(u_i(t_n)x, u_i(t_n)y) \to (x, y)$ as $n \to \infty$. Now for any $i \in \{1,\dots,k\}$,

$$\phi(x, g_iP) = \phi(y, P) \Rightarrow \phi(u_i(t_n)x, u_i(t_n)g_iP) = \phi(u_i(t_n)y, P), \quad \forall n \in \mathbb{N}.$$

In the limit as $n \to \infty$, we get $\phi(x, g_{i+1}P) = \phi(y, P)$. Since $g_1 = g$, by induction on i, we get that $\phi(x, g_iP) = \phi(y, P)$ for all $1 \leqslant i \leqslant k+1$.

In particular, since $g_{k+1} = e$,

$$\phi(x, g_kP) = \phi(y, P) = \phi(x, P).$$

Since $F = \{a^n : n \in \mathbb{N}\} \cdot U^+ \subset g_kPg_k^{-1} \cap P$, we have that

$$\phi(fx, g_kP) = \phi(fx, P), \quad \forall f \in F.$$

Let G_1 be the smallest closed normal subgroup of G containing U^+. Then by the choice of a as in Propositon 6.5, $\mathbb{R}$-rank$(G/G_1) \leqslant 1$. Therefore by the hypothesis in theorem 1.2, $\overline{G_1x} = \overline{Gx}$. By theorem 4.2, $\overline{Gx}$ is an orbit of a closed subgroup, say L', of L containing G, and the stabilizer of x in L', say Λ', is a lattice in L'. Applying theorem 1.1 to L' and Λ' in place of L and Λ, respectively, we conclude that $\overline{Fx} = \overline{Gx}$. Thus

$$\phi(z, g_kP) = \phi(z, P), \quad \forall z \in \overline{G_1x} = \overline{Gx}.$$

Put

$$Q = \{h \in G : \phi(z, fhP) = \phi(z, fP), \quad \forall z \in \overline{Gx} \text{ and } \forall f \in G\}. \tag{14}$$

Then Q is a closed subgroup of G containing $P \cup \{g_k\}$. Since $g_k \notin P$,

$$Q \neq P. \tag{15}$$

Now if $g \notin Q$, then replacing P by Q and L/Λ by $\overline{Gx}$, we repeat the above argument. Note that by definition the new set given by (14) still turns out to be same as Q. This fact contradicts the new equation (15). This completes the proof. □

Proof of theorem 1.2. Define the equivalence relation

$$R = \{(x, y) \in L/\Lambda \times L/\Lambda : \phi(x, gP) = \phi(y, gP) \text{ for some } g \in G\}$$

on L/Λ. Clearly R is a closed subset of $L/\Lambda \times L/\Lambda$ invariant under the diagonal action of G. Let X be the space of equivalence classes of R and let $\phi_1 : L/\Lambda \to X$ be the map taking any element of L/Λ to its equivalence class. Equip X with the quotient topology. Then X is a locally compact Hausdorff space.

For any $x \in L/\Lambda$, put

$$\mathcal{Q}(x) = \{h \in G : \phi(x, gP) = \phi(x, ghP), \forall g \in G\}.$$

Observe that $\mathcal{Q}(x)$ is a closed subgroup of G containing P and for any $y \in \overline{Gx}$, we have $\mathcal{Q}(y) \supset \mathcal{Q}(x)$. Let $x_0 \in L/\Lambda$ such that $\overline{Gx_0} = L/\Lambda$ and put $Q = \mathcal{Q}(x_0)$. Then $\mathcal{Q}(y) \supset Q$ for all $y \in L/\Lambda$. Since Q is a parabolic subgroup of G, there are only finitely many closed subgroups of G containing Q. Therefore the set $X_Q := \{x \in L/\Lambda : \mathcal{Q}(x) = Q\}$ is open in L/Λ. Also X_Q is non-empty and G-invariant. Now since G acts ergodically on L/Λ, the set $L/\Lambda \backslash X_Q$ is closed and nowhere dense.

Note that for any $x, y \in L/\Lambda$, if $\phi_1(x) = \phi_1(y)$ then by proposition 6.6, we have that $\mathcal{Q}(x) = \mathcal{Q}(y)$. Let $\rho : L/\Lambda \times G/P \to X \times G/Q$ be the (G-equivariant) map defined by $\rho(x, gP) = (\phi_1(x), gQ)$ for all $x \in L/\Lambda$ and $g \in G$. Then there exists a uniquely defined map $\psi : X \times G/Q \to Y$ such that $\phi = \psi \circ \rho$. It is straightforward to verify that ψ is continuous and G-equivariant.

Take any $x \in X_Q$, $y \in L/\Lambda$, and $g, h \in G$ such that $\phi(x, ghP) = \phi(y, gP)$. Then $\phi_1(y) = \phi_1(x)$, and hence $h \in \mathcal{Q}(y) = \mathcal{Q}(x) = Q$. This proves that ψ restricted to $\phi_1(X_Q) \times G/Q$ is injective and $y \in X_Q$.

Now if Y is a locally compact second countable space and ϕ is surjective, then using Baire's category theorem for Hausdorff locally compact second countable spaces, one can show that ϕ is an open map. This completes the proof of the theorem. □

Continuous G-equivariant factors of L/Λ

Proof of theorem 1.3. Define $\Lambda_1 = \{h \in L : \phi(gh\Lambda) = \phi(g\Lambda), \forall g \in L\}$. Then Λ_1 is a closed subgroup of L containing Λ. Since G-acts ergodically on L/Λ, $\mathrm{Ad}(\Lambda)$ is Zariski dense in $\mathrm{Ad}(L)$ (see [Sh1, Theorem 2.3]). Therefore Λ_1^0 is a normal subgroup of L. Let Λ_1' be the largest subgroup of Λ_1 which is normal in L. In view of 1.1 (3), replacing L by L/Λ_1', Λ by Λ_1/Λ_1', and G by its image in L/Λ_1', without loss of generality we may assume that $\Lambda_1' = \{e\}$ and $\Lambda_1 = \Lambda$.

Define the equivalence relation

$$R = \{(x, y) \in L/\Lambda \times L/\Lambda : \phi(x) = \phi(y)\}$$

on L/Λ. Then R is closed and $\Delta(G)$-invariant, where $\Delta : L \to L \times L$ denotes the diagonal embedding of L in $L \times L$.

Let

$$K=\{\tau\in \mathrm{Aff}(L/\Lambda):(z,\tau(z))\in R \text{ and } \tau(gz)=g\tau(z), \forall z\in L/\Lambda, \forall g\in G\}$$

and

$$X_1=\{x\in L/\Lambda:\overline{Gx}=L/\Lambda\}.$$

Note that $X_1\neq\emptyset$, since G acts ergodically on L/Λ.

Claim 6.4. Let $(x,y)\in R$. If $x\in X_1$, then $y\in X_1$ and there exists $\tau\in K$ such that $y=\tau(x)$.

The claim is proved as follows. Since $\Delta(G)$ is generated by one-parameter unipotent-subgroups of $L\times L$, by Ratner's theorem 4.2 there exists a closed subgroup F of $L\times L$ containing $\Delta(G)$ such that

$$\overline{\Delta(G)\cdot(x,y)}=F\cdot(x,y)$$

and $F\cdot(x,y)$ admits an F-invariant probability measure, say λ.

Let $p_i:L\times L\to L$ denote the projection on the i-th coordinate, where $i=1,2$. Then $(\pi\circ p_1)_*(\lambda)$ is a $p_1(F)$-invariant probability measure on $p_1(F)x$. Hence the orbit $p_1(F)x$ is closed (see [R, theorem 1.13]). Since $G\subset p_1(F)$ and $\overline{Gx}=L/\Lambda$, we have that $p_1(F)=L$. Let $N_1=p_1(F\cap\ker(p_2))$. Then N_1 is a normal subgroup of $p_1(F)=L$ and $(N_1z,w)\subset R$ for all $(z,w)\in F\cdot(x,y)$. Therefore $N_1\subset\Lambda_1'=\{e\}$. Thus $F\cap\ker(p_2)=N_1\times\{e\}=\{e\}$. Now since $p_1(F)=L$ and $p_2|_F$ is injective, $\dim(p_2(F))=\dim(L)$. Since L is connected, $p_2(F)=L$. Thus $p_2|_F$ is an isomorphism.

Now $\overline{Gy}\supset p_2(F)y=L/\Lambda$. Hence $y\in X_1$. Now interchanging the roles of x and y in the above argument, we conclude that $p_1|_F$ is an isomorphism. Let $\sigma=p_2\circ(p_1|_F)^{-1}$. Then $\sigma\in\mathrm{Aut}(L)$ and

$$F=\{(g,\sigma(g))\in L\times L:g\in L\}.$$

Thus $(gx,\sigma(g)y)\in R$ for all $g\in L$. Now for any $\delta\in L$, if $\delta x=x$, then $(gx,\sigma(g)\sigma(\delta)y)\in R$ for all $g\in L$. Let $h\in L$ such that $y=h\Lambda$. Then $\phi(\sigma(g)h\Lambda)=\phi(\sigma(g)\sigma(\delta)h\Lambda)$ for all $g\in L$. Since $\sigma(L)h=L$, we conclude that $h^{-1}\sigma(\delta)h\in\Lambda_1$. Now since $\Lambda_1=\Lambda$, we have that $\sigma(\delta)y=y$. Therefore the map $\tau:L/\Lambda\to L/\Lambda$, given by $\tau(gx)=\sigma(g)y$ for all $g\in L$, is well defined and $\tau\in\mathrm{Aff}(L/\Lambda)$.

Therefore

$$F\cdot(x,y)=\{(z,\tau(z)):z\in L/\Lambda\}.$$

Since $\Delta(G)\subset F$, we have that $\sigma(g)=g$ and hence $\tau(gz)=gz$, for all $g\in G$. Thus $\tau\in K$, and the proof of the claim is complete.

Claim 6.5. The group K is compact.

We prove the claim as follows. Clearly, K is a closed subset of $\mathrm{Aff}(L/\Lambda)$, and hence it is locally compact. Let μ_L denote the L-invariant probability measure on L/Λ. Then $\mu_L(X_1)=1$. For any $x\in X_1$, if $y\in\overline{K\cdot x}$ then $(x,y)\in R$, and by claim 6.4 there exists $\tau\in K$ such that $y=\tau(x)$. Thus $K\cdot x$ is closed for all $x\in X_1$. Therefore by Hedlund's Lemma and the ergodic decomposition of μ_L with respect to the action of K on L/Λ, we have that almost all K-ergodic components are supported on closed K-orbits. Thus for almost all $x\in L/\Lambda$, the orbit $K\cdot x$ supports a K-invariant probability measure.

For any $x\in L/\Lambda$, put $K_x=\{\tau\in K:\tau(x)=x\}$. Let $\xi:K/K_x\to L/\Lambda$ be the map defined by $\xi(\tau K_x)=\tau(x)$ for all $\tau\in K$. Since $\mathrm{Aff}(L/\Lambda)$ acts continuously on L/Λ, we have that ξ is

a continuous injective K-equivariant map. Let $x \in X_1$ be such that $K \cdot x$ supports a K-invariant probability measure. Since ξ is injective, the measure can be lifted to a K-invariant probability measure on K/K_x. Let $\tau \in K_x$. Then for any $g \in G$, we have $\tau(gx) = g\tau(x) = gx$. Now since $\overline{Gx} = L/\Lambda$, we have that $\tau(y) = y$ for all $y \in L/\Lambda$. Hence K_x is the trivial subgroup of $\mathrm{Aff}(L/\Lambda)$. Thus K admits a finite Haar measure. Hence K is a compact group, and the claim is proved.

Let Ω be any neighbourhood of e in $Z_L(G)$. Put

$$R' = \{(x, y) \in R : y \notin K \cdot \Omega x\}.$$

Let X_c be the closure of the projection of R' on the first factor of $L/\Lambda \times L/\Lambda$. Put $X_0 = (L/\Lambda) \backslash X_c$.

Claim 6.6. $X_1 \subset X_0$.

Suppose the claim does not hold. Then there exists a sequence $\{(x_i, y_i)\} \subset R'$ converging to $(x, y) \in R$ with $x \in X_1$. By claim 6.4, there exists $\tau \in K$ such that $y = \tau(x)$. Therefore, after passing to a subsequence, there exists a sequence $g_i \to e$ in L such that $y_i = \tau(g_i x_i)$ for all $i \in \mathbb{N}$. By the definition of $R', g_i \notin \Omega \subset Z_L(G)$ for all $i \in \mathbb{N}$. Also $(x_i, g_i x_i) \in R$ for all $i \in \mathbb{N}$. By Ratner's theorem 4.2, there exists a $\Delta(G)$-invariant $\Delta(G)$-ergodic probability measure μ_i on $(L/\Lambda) \times (L/\Lambda)$ such that $\overline{\Delta(G)(x_i, g_i x_i)} = \mathrm{supp}(\mu_i)$. Let $h_i \to e$ be a sequence in L such that $x_i = h_i x$ for all $i \in \mathbb{N}$. By theorem 4.3, after passing to a subsequence, there exists a probability measure μ on $L/\Lambda \times L/\Lambda$ such that $\mu_i \to \mu$ as $i \to \infty$, and the following holds: $\mathrm{supp}(\mu) = F \cdot (x, x)$, where F is a closed subgroup of $L \times L$, and

$$(h_i^{-1}, h_i^{-1} g_i^{-1}) \Delta(G)(h_i, g_i h_i) \subset F, \quad \forall i \in \mathbb{N}. \tag{16}$$

In particular, $F \cdot (x, x) \subset R$ and $\Delta(G) \subset F$. Since $x \in X_1$, we have that $F \supset \Delta(L)$. By an argument as in the proof of claim 6.4, we conclude that $F \cap \ker(p_i) = \{e\}$ for $i = 1, 2$. Therefore $F = \Delta(L)$. Hence from (16) we conclude that $g_i \in Z_L(G)$, which is a contradiction. This completes the proof of the claim, and the proof of the theorem. □

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Note added in proof

Equidistribution of translates of measures is also considered in a recent preprint "Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture" of A Eskin, G A Margulis and S Mozes. In the context of the preprint it seems worthwhile to remark that the method in the present paper can be used to obtain 'uniform versions' of theorem 1.4 and corollary 1.2, as done in the above mentioned paper, for certain results.

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On the zeros of polynomials

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Abstract. In this paper we extend a classical result due to Cauchy and its improvement due to Datt and Govil to a class of lacunary type polynomials.

Keywords. Moduli of zeros; ring shaped region; lacunary type polynomials.

1. Introduction and statement of results

A classical result due to Cauchy [1] concerning the bounds for the moduli of the zeros of a polynomial $P(z)$ can be stated as

Theorem A. *If*

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n and

$$M = \max|a_j|, \quad j = 0, 1, 2, \ldots, n-1,$$

then all the zeros of $P(z)$ *lie in a circle*

$$|z| \leqslant 1 + M. \tag{1}$$

In the literature [3–6], there exists some improvements and generalizations of Cauchy's theorem. Recently Datt and Govil [2] have obtained the following improvement to Theorem A.

Theorem B. *If*

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n and

$$A = \max|a_j|, \quad j = 0, 1, 2, \ldots, n-1,$$

then all the zeros of $P(z)$ *lie in a ring shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}(1+An)} \leqslant |z| \leqslant 1 + \lambda_0 A \tag{2}$$

where λ_0 *is the unique root of the equation*

$$x = 1 - \frac{1}{(1+Ax)^n}$$

in the interval $(0, 1)$. *The upper bound in* (2) *is best possible and is attained for the*

polynomial

$$P(z)=z^n-A(z^{n-1}+z^{n-2}+\cdots+z+1).$$

The purpose of this paper is to extend the above results for a class of lacunary type polynomials. We start by proving the generalization of Theorem A.

Theorem 1. *If*

$$P(z)=a_nz^n+a_pz^p+\cdots+a_1z+a_0,\quad 0\leqslant p\leqslant n-1$$

is a polynomial of degree n and

$$M=\max\left|\frac{a_j}{a_n}\right|,\quad j=0,1,\ldots,p,$$

then all the zeros of $P(z)$ *lie in* $|z|<K$, *where K is a unique positive root of the trinomial equation*

$$x^{n-p}-x^{n-p-1}-M=0. \tag{3}$$

For $p=n-1$, *this reduces to Theorem A.*

The following corollary is obtained by taking $p=n-2$ in Theorem 1.

COROLLARY 1

If

$$P(z)=a_nz^n+a_{n-2}z^{n-2}+\cdots+a_1z+a_0,$$

is a polynomial of degree n and

$$M=\max\left|\frac{a_j}{a_n}\right|,\quad j=0,1,\ldots,n-1,$$

then all the zeros of $P(z)$ *lie in the circle*

$$|z|<\frac{1+\sqrt{(1+4M)}}{2}.$$

From Corollary 1, we can easily deduce Corollary 2.

COROLLARY 2

If

$$P(z)=a_nz^n+a_{n-2}z^{n-2}+\cdots+a_1z+a_0,$$

is a polynomial of degree n, such that

$$|a_j|\leqslant|a_n|,\quad j=0,1,2,\ldots,n-2,$$

then all the zeros of $P(z)$ *lie in*

$$|z|\leqslant\frac{1+\sqrt{5}}{2}.$$

Next we present the following generalization of Theorem B.

Theorem 2. *If*

$$P(z)=z^n+a_p z^p+\cdots+a_1 z+z_0, \quad 0\leqslant p\leqslant n-1$$

is a polynomial of degree n and

$$A=\max|a_j|, \quad j=0,1,\ldots,p$$

then $P(z)$ has all its zeros in the ring shaped region

$$\frac{|a_0|}{2(1+A)^{n-1}\{1+(p+1)A\}}\leqslant|z|\leqslant 1+\alpha_0 A \tag{4}$$

where α_0 is the unique root of the equation

$$x=1-\frac{1}{(1+Ax)^{p+1}}$$

in the interval $(0,1)$.

The upper bound $1+\alpha_0 A$ in (4) is best possible and is attained for the polynomial

$$P(z)=z^n-A(z^p+z^{p-1}+\cdots+z+1).$$

2. Lemma

For the proof of Theorem 2, we need the following lemma.

Lemma. Let

$$f(x)=x-\left[\frac{1}{(1+Ax)^{n-p-1}}-\frac{1}{(1+Ax)^n}\right]$$

where n is a positive integer and $A>0$. If $(p+1)A>1$, then $f(x)$ has a unique root in the interval $(0,1)$.

Proof of the Lemma. Consider

$$\begin{aligned}(1+Ax)^n f(x)&=(1+Ax)^n x-(1+Ax)^{p+1}+1, \quad \text{where } p\leqslant n-1\\
&=\binom{n}{0}x+\binom{n}{1}Ax^2+\binom{n}{2}A^2x^3+\cdots+\binom{n}{n}A^n x^{n+1}\\
&\quad-\left\{\binom{p+1}{1}Ax+\binom{p+1}{2}A^2x^2+\cdots\right.\\
&\quad\left.+\binom{p+1}{p}A^p x^p+(Ax)^{p+1}\right\}\\
&=(1-(p+1)A)x+\sum_{k=2}^{p+1}\frac{(p+1)!\,A^{k-1}x^k}{k!(p-k+2)!}\\
&\quad\times\left\{k\left(\frac{n(n-1)\cdots(n-k+2)}{(p+1)p\cdots(p-k+3)}+A\right)-(p+2)A\right\}\\
&\quad+\sum_{k=p+2}^{n+1}\binom{n}{k-1}A^{k-1}x^k\end{aligned}$$

$$= (1-(p+1)A)x + \sum_{k=2}^{p+1} \frac{(p+1)!A^{k-1}x^k}{k!(p-k+2)!}\{k(A+l_k)-(p+2)A\}$$

$$+ \sum_{k=p+2}^{n+1} \binom{n}{k-1} A^{k-1}x^k, \tag{5}$$

where $l_k = \dfrac{n(n-1)\cdots(n-k+2)}{(p+1)p\cdots(p-k+3)} \geqslant 1$ for all $k = 2, 3, \ldots, p+1$, as $p \leqslant n-1$. Since $1-(p+1)A < 0$, the coefficients of $x^{p+2}, \ldots, x^{n+1}$ are positive and $k(A+l_k)-(p+2)A$ are monotonically increasing for $k = 2, 3, \ldots, p+1$, it follows from Descartes rule of signs that $(1+Ax)^n f(x) = 0$ has exactly one positive root. Since

$$f(x) = x - \left[\frac{1}{(1+Ax)^{n-p-1}} - \frac{1}{(1+Ax)^n}\right]$$

then

$$f'(x) = 1 - \left[\frac{(1+p-n)A}{(1+Ax)^{n-p}} + \frac{nA}{(1+Ax)^{n+1}}\right]. \tag{6}$$

If $(p+1)A > 1$, then it is clear from (6) that $f'(0) < 0$. Thus there exists a $\delta > 0$ such that $f'(x) < 0$ in $(0, \delta)$. Also $f(1) > 0$, hence $f(x) = 0$ has one and only one positive root in $(0, 1)$ and the lemma follows.

3. Proof of the theorems

Proof of Theorem 1. We have

$$P(z) = a_n z^n + a_p z^p + \cdots + a_1 z + a_0, \quad 0 \leqslant p \leqslant n-1,$$

so that for $|z| > 1$,

$$|P(z)| \geqslant |a_n||z|^n \left\{1 - \left(\frac{|a_p|}{|a_n|}\frac{1}{|z|^{n-p}} + \cdots + \frac{|a_1|}{|a_n|}\frac{1}{|z|^{n-1}} + \frac{|a_j|}{|a_n|}\frac{1}{|z|^n}\right)\right\}.$$

Since $|(a_j/a_n)| \leqslant M\ \forall j = 0, 1, 2, .., n-1$, it follows that

$$|P(z)| \geqslant |a_n||z|^n \left\{1 - \frac{M}{|z|^{n-p}}\left(1 + \frac{1}{|z|} + \frac{1}{|z|^2} + \cdots + \frac{1}{|z|^p}\right)\right\}$$

$$> |a_n||z|^n \left\{1 - \frac{M}{|z|^{n-p}}\left(1 + \frac{1}{|z|} + \cdots\right)\right\}$$

$$= |a_n||z|^n \left\{1 - \frac{M}{|z|^{n-p-1}} \cdot \frac{1}{(|z|-1)}\right\}$$

$$\geqslant 0,$$

if

$$|z|^{n-p} - |z|^{n-p-1} - M \geqslant 0.$$

This implies

$$|P(z)| > 0 \quad \text{if } |z| \geqslant K,$$

where K is the (unique) positive root of the trinomial equation defined by (3) in $(1, \infty)$.

Hence all the zeros of $P(z)$ whose modulus is greater than 1 lie in $|z| < K$. Since all those zeros whose modulus is less or equal to 1 already lie in $|z| < K$, the desired result follows.

Proof of Theorem 2. We shall first prove that $P(z)$ has all its zeros in $|z| \leqslant 1 + \alpha_0 A$, and for this it is sufficient to consider the case when $(p+1)A > 1$ (for if $(p+1)A \leqslant 1$, then on $|z| = R > 1$, $|P(z)| \geqslant R^n - (p+1)A|z|^p \geqslant R^n - R^p > 0$).

If

$$A = \max|a_j|, \quad j = 0, 1, \ldots, p$$

and

$$P(z) = z^n + a_p z^p + a_{p-1} z^{p-1} + \cdots + a_0, \quad 0 \leqslant p \leqslant n-1,$$

then

$$\begin{aligned} |P(z)| &\geqslant |z|^n \left\{ 1 - \left(\frac{|a_p|}{|z|^{n-p}} + \frac{|a_{p-1}|}{|z|^{n-p+1}} + \cdots + \frac{|a_0|}{|z|^n} \right) \right\} \\ &\geqslant |z|^n \left\{ 1 - \frac{A}{|z|^{n-p}} \left(\frac{|z|^{p+1} - 1}{(|z|-1)|z|^p} \right) \right\} \\ &= |z|^n - A \left(\frac{|z|^{p+1} - 1}{|z| - 1} \right). \end{aligned}$$

Hence for every $\alpha > 0$, we have on $|z| = 1 + A\alpha$,

$$|P(z)| \geqslant (1 + A\alpha)^n - \frac{(1 + A\alpha)^{p+1} - 1}{\alpha} > 0$$

if

$$\alpha(1 + A\alpha)^n > (1 + A\alpha)^{p+1} - 1$$

which implies

$$\begin{aligned} \alpha &> \frac{(1 + A\alpha)^{p+1}}{(1 + A\alpha)^n} - \frac{1}{(1 + A\alpha)^n} \\ &= \frac{1}{(1 + A\alpha)^{n-p-1}} - \frac{1}{(1 + A\alpha)^n}. \end{aligned} \tag{7}$$

Thus if α_0 is the unique root of the equation

$$x = \frac{1}{(1 + Ax)^{n-p-1}} - \frac{1}{(1 + Ax)^n}, \quad \text{(by above lemma)},$$

in $(0, 1)$, then every $\alpha > \alpha_0$ satisfies (7) and hence $|P(z)| > 0$ on $|z| = 1 + A\alpha$ which implies that $P(z)$ has all its zeros in

$$|z| \leqslant 1 + A\alpha_0. \tag{8}$$

Next we prove that $P(z)$ has no zero in

$$|z| < \frac{|a_0|}{2(1 + A)^{n-1}\{1 + (p+1)A\}}.$$

Let us denote the polynomial $g(z)$ by $(1 - z)P(z)$, then

$$\begin{aligned} g(z) &= a_0 + \sum_{j=1}^{p} (a_j - a_{j-1}) z^j + z^n - a_p z^{p+1} - z^{n+1} \\ &= a_0 + h(z) \quad \text{(say)} \end{aligned}$$

if

$$R = 1 + A$$

then

$$\begin{aligned}\max_{|z|=R} |h(z)| &\leqslant R^{n+1} + R^n + |a_p| R^{p+1} + \sum_{j=1}^{p} |a_j - a_{j-1}| R^j \\ &\leqslant R^n[R + 1 + A + 2Ap] \\ &= 2R^n[R + Ap] \\ &= 2(1 + A)^n[1 + (p + 1)A]. \qquad (9)\end{aligned}$$

Hence on $|z| \leqslant R$,

$$\begin{aligned}|g(z)| &= |a_0 + h(z)| \geqslant |a_0| - |h(z)| \\ &\geqslant |a_0| - \frac{|z|}{(1 + A)} \max_{|z|=R>1} |h(z)| \quad \text{(by Schwarz lemma)} \\ &\geqslant |a_0| - \frac{|z|}{(1 + A)} 2(1 + A)^n\{1 + (p + 1)A\} \quad \text{(by (9))} \\ &> 0,\end{aligned}$$

if

$$|z| < \frac{|a_0|}{2(1 + A)^{n-1}[1 + (p + 1)A]}.$$

Hence all the zeros of $P(z)$ lie in

$$|z| \geqslant \frac{|a_0|}{2(1 + A)^{n-1}[1 + (p + 1)A]}. \qquad (10)$$

Combining (8) and (10), we get all the zeros of $P(z)$ to be in the ring shaped region

$$\frac{|a_0|}{2(1 + A)^{n-1}[1 + (p + 1)A]} \leqslant |z| \leqslant 1 + A\alpha_0.$$

This completes the proof of Theorem 2.

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Quasi-parabolic Siegel formula

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Abstract. The result of Siegel that the Tamagawa number of SL_r over a function field is 1 has an expression purely in terms of vector bundles on a curve, which is known as the Siegel formula. We prove an analogous formula for vector bundles with quasi-parabolic structures. This formula can be used to calculate the Betti numbers of the moduli of parabolic vector bundles using the Weil conjuctures.

Keywords. Siegel formula; quasi-parabolic divisors; Betti numbers; Weil conjucture.

1. Introduction

The Betti numbers of the moduli of stable vector bundles on a complex curve, in all the cases where the rank and degree are coprime, were first determined by Harder and Narasimhan [H-N] as an application of the Weil conjuctures. For this, they made use of the result of Siegel that the Tamagawa number of the special linear group over a function field is 1. In their refinement of the same Betti number calculation in [D-R], Desale and Ramanan expressed the result of Siegel in purely vector bundle terms. This result about the Tamagawa number, called the Siegel formula, was later given a simple proof in the language of vector bundles by Ghione and Letizia [G-L], by introducing a notion of effective divisors of higher rank on a curve, and counting the number of effective divisors which correspond to a given vector bundle. The purpose of this note is to introduce the notion of a quasi-parabolic divisor of higher rank on a curve (Definition 3.1 below), and to prove a quasi-parabolic analogue (Theorem 3.4 below) of the Siegel formula, which is done here by suitably generalizing the method of [G-L]. In a note to follow, this formula is used to calculate the zeta function and thereby the Betti numbers of the moduli of parabolic bundles in the case 'stable = semistable' (these Betti numbers have already been calculated by a guage theoretic method for genus $\geqslant 2$ in [N] and for genus 0 and 1 by Furuta and Steer in [F-S]).

2. Divisors supported on $X - S$

Let X be an absolutely irreducible, smooth projective curve over the finite field $k = \mathbf{F}_q$, and let S be any closed subset of X whose points are k-rational. Let K denote the function field of X, and let K_X denote the constant sheaf K on X. Let g denote the genus of X. Let r be a positive integer. Recall that (see [G-L]) a coherent subsheaf $D \subset K_X^r$ of generic rank r is called an r-divisor, and the r-divisor is called effective (or positive) if $\mathcal{O}_X^r \subset D$. The support of the divisor is by definition the support of the quotient $D/\mathcal{O}_X^r$, which is a torsion sheaf. The length n of $D/\mathcal{O}_X^r$ is called the degree of the divisor. Note that D is a locally free sheaf of rank r and degree n.

Remark 2.1. Let $Z_X(t)$ be the zeta function of X. Then as S consists of k-rational points, it can be seen that the zeta function Z_{X-S} of $X-S$ is given by the formula

$$Z_{X-S}(t) = (1-t)^s Z_X(t) \tag{1}$$

where s is the cardinality of S.

Note that an effective r-divisor on $X-S$ is the same as an effective r-divisor on X whose support is disjoint from S. Part (1) of the proposition 1 of [G-L] gives the following, with $X-S$ in place of X.

PROPOSITION 2.2

Let $b_n^{(r)}$ be the number of effective r-divisors of degree n on X whose support is disjoint from S. Let $Z_{X-S}^{(r)}(t) = \Sigma_{n\geq 0} b_n^{(r)} t^n$. Then we have

$$Z_{X-S}^{(r)}(t) = \prod_{1\leq j\leq r} Z_{X-S}(q^{j-1}t). \tag{2}$$

In order to have the analogue of part (2) of the proposition 1 of [G-L], we need the following lemmas.

Lemma 2.3. *Let V be a finite dimensional vector space over $k = \mathbf{F}_q$, and s a positive integer. For any $1\leq i\leq s$, let $\pi_i: k^s \to k$ be the linear projection. For any surjective linear map $\phi: V\to k^s$, let V_i be the kernel of $\pi_i\phi: V\to k$, which is a hyperplane in V as ϕ is surjective. Let $P = P(V)$, and $P_i = P(V_i)$ denote the corresponding projective spaces. Let $N(\phi)$ denote the number of k-rational points of $P - \cup_{1\leq i\leq s} P_i$. Then for any other surjective $\psi: V\to k^s$, we have $N(\phi) = N(\psi)$. In other words, given s, this number depends only on* $\dim(V)$.

Proof. Given any two surjective maps $\phi, \psi: V\to k^s$, there exists an $\eta\in GL(V)$ such that $\phi\eta = \psi$. From this, the results follows.

Lemma 2.4. *Let n be a positive integer, such that $n > 2g-2+s$ where g is the genus of X and s is the cardinality of s. Let b_n be the total number of effective* 1*-divisors of degree n supported on $X-S$. Then for any line bundle L on X of degree n, the number of effective* 1*-divisors supported on $X-S$ which define L is $b_n/P_X(1)$, where $P_X(1)$ is the number of isomorphism classes of line bundles of any fixed degree on X.*

(Here, $P_X(t)$ is the polynomial $(1-t)(1-qt)Z_X(t)$.)

Proof. Let L be any line bundle on X of degree n, where $n > 2g-2+s$. Then $H^1(X, L(-S)) = 0$, so the natural map $\phi: H^0(X,L)\to H^0(X, L|S)$ is surjective. Let $V = H^0(X,L)$. Then $\dim(V) = n+1-g$. Choose a basis for each fiber L_P for $P\in S$. This given an identification of $H^0(X, L|S)$ with k^s. Now it follows that the number $N(\phi)$ defined in the preceeding lemma depends only on n, and is independent of the choice of L as long as it has degree n. But $N(\phi)$ is precisely the number of effective 1-divisors supported on $X-S$, which define the line bundle L on X.

Using the above lemma, the following proposition follows, by an argument similar to the proof of part (2) of proposition 1 in [G-L]. The proof in [G-L] expresses the number of r-divisors in terms of the number of 1-divisors, and the above lemma tells us the number of 1-divisors with support in $X-S$ corresponding to a given line bundle on X.

PROPOSITION 2.5

For L a line bundle of degree n, let $b_n^{(r,L)}$ be the number of effective r-divisors on X supported on $X-S$, having determinant isomorphic to L. Then provided that $n > 2g-2+s$, we have

$$b_n^{(r,L)} = b_n^{(r)}/P_X(1). \tag{3}$$

PROPOSITION 2.6

$$\lim_{n\to\infty} \frac{b_n^{(r)}}{q^{rn}} = P_X(1)\frac{(q-1)^{s-1}}{q^{g-1+s}} Z_{X-S}(q^{-2})\dots Z_{X-S}(q^{-r}). \tag{4}$$

Proof. The above statement is the analogue of proposition 2 of [G-L], with the following changes. Instead of all r-divisors on X in [G-L], we consider only those which are supported over $X-S$, and instead of $Z_X(t)$, we use $Z_{X-S}(t)$. As $Z_{X-S}(t) = (1-t)^s Z_X(t)$, the property of $Z_X(t)$ that it has a simple pole at $t = q^{-1}$ and is regular at $1/q^j$ for $j \geqslant 2$ is shared by $Z_{X-S}(t)$. Hence the proof in [G-L] also works in our case, proving the proposition.

Remark 2.7. There is a minor misprint in the equation labeled (1) in [G-L] (p. 149); the factor q^{g-1} should be read as q^{1-g}.

Let L be any given line bundle on X. Choose any closed point $P \in X-S$, and let l denote its degree. For any $\mathcal{O}_X$ module E, set $E(m) = E \otimes \mathcal{O}_X(mP)$. If a vector bundle E of rank r, degree n has determinant L, then $E(m)$ has determinant $L(rm)$, degree $n+rml$ and Euler characteristic $\chi(m) = n + rml + r(1-g)$.

The equations (3) and (4) above imply the following

$$\lim_{m\to\infty} \frac{b_{n+rml}^{(r,L(rm))}}{q^{r\chi(m)}} = (q-1)^{s-1} q^{(r^2-1)(g-1)-s} Z_{X-S}(q^{-2})\dots Z_{X-S}(q^{-r}). \tag{5}$$

3. Quasi-parabolic divisors

For basic facts about parabolic bundles, see [S] and [M-S]. We now introduce the notion of a quasi-parabolic effective divisor of rank r. Let $S \subset X$ be a finite subset consisting of k-rational points. For each $P_i \in S$, let there be given positive integers p_i and $r_{i,1},\dots,r_{i,p_i}$, with $r_{i,1}+\dots+r_{i,p_i} = r$. This will be called, as usual, the quasi-parabolic data. Recall that a quasi-parabolic structure on a vector bundle E of rank r on X by definition consists of flags $E_{P_i} = F_{i,1} \supset \dots \supset F_{i,p_i} \supset F_{i,p_i+1} = 0$ of vector subspaces in the fibers over the points of S such that $\dim(F_{i,j}/F_{i,j+1}) = r_{i,j}$ for each j from 1 to p_i.

DEFINITION 3.1

Let X, S and the numerical data $(r_{i,j})$ be as above. A positive quasi-parabolic divisor (F,D) on X consists of (i) a quasi-parabolic structure F on the trivial bundle $\mathcal{O}_X^r$, consisting of flags F_i in k^r at points $P_i \in S$ of the given numerical type $(r_{i,j})$, together with (ii) an effective r-divisor D on X, supported on $X-S$.

Note that if (F,D) is a quasi-parabolic r-divisor, then the rank r vector bundle D has a quasi-parabolic structure given by F. We denote by $P_E^{(r)}$ the set of all effective quasi-parabolic r-divisors whose associated quasi-parabolic bundle is isomorphic to

a given quasi-parabolic bundle E. For any vector bundle E of rank r, let $\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)$ denote the set of all injective sheaf homomorphisms $\mathcal{O}^r_X \to E$ which are injective when restricted to S. For any quasi-parabolic bundle E, the group of all quasi-parabolic automorphisms of E will be denoted by $\mathrm{ParAut}(E)$. Then $\mathrm{ParAut}(E)$ acts on $\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)$ by composition. This action is free, and $P^{(r)}_E$ has a canonical bijection with the quotient set $\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)/\mathrm{ParAut}(E)$. Hence the cardinality of $P^{(r)}_E$ is given by

$$|P^{(r)}_E| = \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)|}{|\mathrm{ParAut}(E)|}. \tag{6}$$

For $1 \leqslant i \leqslant s$, let Flag_i be the variety of flags in k^r of the numerical type $(r_{i,1}, \ldots, r_{i,p_i})$. Let $\mathrm{Flag}_S = \Pi_{1 \leqslant i \leqslant s} \mathrm{Flag}_i$. Let $f(q, r_{i,j})$ denote the number of k-rational points of Flag_S. If $a^{(r,L)}_n$ denotes the number of quasi-parabolic divisors of flag data $(r_{i,j})$ with degree n, rank r and determinant L, then we have

$$a^{(r,L)}_n = f(q, r_{i,j}) b^{(r,L)}_n. \tag{7}$$

Now let $J(r, L)$ denote the set of all isomorphism classes of quasi-parabolic vector bundles of rank r, degree n, determinant L having the given quasi-parabolic data $(r_{i,j})$ over S. Hence (6) implies the following

$$a^{(r,L)}_n = \sum_{E \in J(r,L)} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)|}{|\mathrm{ParAut}(E)|}. \tag{8}$$

For any integer m, the map from $J(r, L) \to J(r, L(rm))$ which sends E to $E(m) = E \otimes \mathcal{O}_X(mP)$ is a bijection which preserves $|\mathrm{ParAut}|$. Hence for each m, we have

$$a^{(r,L(rm))}_{n+rml} = \sum_{E \in J(r, L)} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|}{|\mathrm{ParAut}(E)|}. \tag{9}$$

Lemma 3.2. *With the above notations,*

$$\lim_{m \to \infty} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|}{q^{r\chi(E(m))}} = 1. \tag{10}$$

Proof. Let m be large enough, so that $E(m)$ is generated by global sections. Then the subset $\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m)) \subset \mathrm{Hom}_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))$ is the intersection of the subset $\mathrm{Hom}_{\mathrm{inj}}(\mathcal{O}^r_X, E(m)) \subset \mathrm{Hom}(\mathcal{O}^r_X, E(m))$ with the complement of a union of sr number of r-codimensional linear subspaces of $\mathrm{Hom}(\mathcal{O}^r_X, E(m))$. (Here, the values sr and r are not important: all that matters is that the number sr of these linear subspaces is a constant independent of m, and each is a proper subspace.) Hence the above limit equals $\lim_{m \to \infty}(|\mathrm{Hom}_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|/q^{r\chi(E(m))})$, which has the value 1 by lemma 3 of [G-L].

Lemma 3.3. *The following sum and limit can be interchanged to give*

$$\sum_{E \in J(r, L)} \lim_{m \to \infty} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|}{q^{r\chi(E(m))}|\mathrm{ParAut}(E)|} = \lim_{m \to \infty} \sum_{E \in J(r, L)} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|}{q^{r\chi(E(m))}|\mathrm{ParAut}(E)|}.$$

This lemma has a proof entirely analogous to the corresponding statement in [G-L], so we omit the details.

By (10), the left hand side in the above lemma equals

$$\sum_{E\in J(r,L)} \frac{1}{|\mathrm{ParAut}(E)|}.$$

On the other hand, by (9), the right hand side is $\lim_{m\to\infty} a_{n+rml}^{(r,L(rm))}/q^{r\chi(m)}$. By (5) and (7), this limit has the following value

$$f(q, r_{i,j})(q-1)^{s-1} q^{((r^2-1)(g-1)-s} Z_{X-S}(q^{-2}) \dots Z_{X-S}(q^{-r}).$$

Hence we get theorem 3.4.

Theorem 3.4. (*Quasi-parabolic Siegel formula*)

$$\sum_{E\in J(r,L)} \frac{1}{|\mathrm{ParAut}(E)|} = f(q, r_{i,j})(q-1)^{s-1} q^{(r^2-1)(g-1)-s} Z_{X-S}(q^{-2}) \dots Z_{X-S}(q^{-r})$$

Remark 3.5. Using the expression $Z_{X-S}(t) = (1-t)^s Z_X(t)$, the above equation becomes

$$\sum_{E\in J(r,L)} \frac{1}{|\mathrm{ParAut}(E)|} = f(q, r_{i,j})(q-1)^{s-1} q^{(r^2-1)(g-1)-s} \prod_{2\leqslant j\leqslant r} (1-q^{-j}) Z_X(q^{-j}).$$

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Degree of approximation of functions by their Fourier series in the generalized Hölder metric

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Abstract. The paper studies the degree of approximation of functions by matrix means of their Fourier series in the generalized Hölder metric, generalizing many known results in the literature.

Keywords. Banach space; Hölder metric; generalized Hölder metric; infinite matrix; deferred Cesàro mean.

1. Definition

Let f be a periodic function of period 2π and let $f \in L_p[0, 2\pi]$ for $p \geqslant 1$. Let the Fourier series of f at $t = x$ be given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1.1}$$

In the case $0 < p < 1$, we can still regard (1.1) as the Fourier series of f by further assuming that $f(t)\cos nt$ and $f(t)\sin nt$ are integrable (see [21] p. 9).

The space $L_p[0, 2\pi]$ where $p = \infty$ includes the space $C_{2\pi}$ of all continuous functions defined over $[0, 2\pi]$. We write

$$\|f\|_c = \sup_{t \in [0, 2\pi]} |f(t)| \quad (p = \infty)$$

$$\|f\|_p = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} |f(t)|^p \mathrm{d}t \right)^{1/p} & (p \geqslant 1) \\ \displaystyle\int_0^{2\pi} |f(t)|^p \mathrm{d}t & (0 < p < 1). \end{cases}$$

We write

$$w(\delta) = w(\delta, f) = \sup_{0 \leqslant h \leqslant \delta} \|f(x+h) - f(x)\|_c \tag{1.2}$$

$$w_p(\delta) = w_p(\delta, f) = \sup_{0 \leqslant h \leqslant \delta} \|f(x+h) - f(x)\|_p \tag{1.3}$$

$$w_p^{(2)}(\delta) = w_p^{(2)}(\delta, f) = \sup_{0 \leqslant h \leqslant \delta} \|f(x+h) + f(x-h) - 2f(x)\|_p \tag{1.4}$$

which are respectively called modulus of continuity, integral modulus of continuity and integral modulus of smoothness (see [21], p. 42).

In the case $0<\beta\leqslant 1$ and $w(\delta,f)=0(\delta^{\beta})$ we write $f\in\mathrm{Lip}\,\beta$, and if $w_p(\delta,f)=0(\delta^{\beta})$ we write $f\in\mathrm{Lip}(\beta,p)$. The case $\beta>1$ is of no interest as in this case f turns out to be constant. The class $\mathrm{Lip}(\beta,p)$ with $p=\infty$ will be taken as $\mathrm{Lip}\,\beta$.

Let

$$H_\alpha=\{f\in C_{2\pi}:|f(x)-f(y)|\leqslant K|x-y|^{\alpha}\}$$

where K is a positive constant, not necessarily the same at each occurrence. It is known [15] that H_α is a Banach space with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha=\|f\|_c+\sup_{x\neq y}\Delta^{\alpha}f(x,y) \tag{1.5}$$

where

$$\Delta^{\alpha}f(x,y)=\frac{|f(x)-f(y)|}{|x-y|^{\alpha}}\quad (x\neq y)$$

and

$$\Delta^{0}f(x,y)=0.$$

The metric induced by the norm (1.5) on H_α is called the Hölder metric. Since

$$\|f\|_\beta\leqslant(2\pi)^{\alpha-\beta}\|f\|_\alpha,\quad 0\leqslant\beta<\alpha\leqslant 1$$

it follows that $H_\alpha\subseteq H_\beta\subseteq C_{2\pi}$; that is, $\{H_\alpha,\|\cdot\|_\alpha\}$ is a family of Banach space which decreases as α increases.

With a view to generalize Hölder metric, we proceed as follows. We define for $0<\alpha\leqslant 1$

$$H(\alpha,p)=\{f\in L_p,\quad 0<p\leqslant\infty\colon \|f(x+h)-f(x)\|_p\leqslant K|h|^{\alpha}\}$$

and introduce the following metric. For $\alpha>0$

$$\|f\|_{(\alpha,p)}=\|f\|_p+\sup\frac{\|f(x+h)-f(x)\|_p}{|h|^{\alpha}}$$

$$\|f\|_{(0,p)}=\|f\|_p. \tag{1.6}$$

It can be easily verified that (1.6) is a norm for $p\geqslant 1$ and p-norm in the case $0<p<1$. Further it can be verified that $H(\alpha,p)$ is a Banach space for $p\geqslant 1$ and a complete p-normed space for $0<p<1$. Note that $H(\alpha,\infty)$ is the familiar H_α space introduced earlier by Prösdorff [15].

Let $A=(a_{n,k})$ be an infinite matrix and let $S_n(x)$ be the nth partial sum of the series (1.1). We denote $T_n(f)$ the A-transform of the Fourier series of f by

$$T_n(f)=T_n(f;x)=\sum_{k=0}^{\infty}a_{n,k}S_k(x) \tag{1.7}$$

provided that the series converges for each $n=0,1,2,\ldots$.

Throughout the present paper we assume that elements of the matrix $A=(a_{n,k})$ satisfy the conditions

$$\|A\|=\sup_n\sum_{k=0}^{\infty}|a_{n,k}|<\infty \tag{1.8}$$

and

$$\sum_{k=0}^{\infty}a_{n,k}=1\quad\text{for each}\quad n=0,1,2,\ldots. \tag{1.9}$$

We write $A \in \tau$ if conditions (1.8) and (1.9) hold. We use the following notations throughout:

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x) \tag{1.10}$$

$$l_n(x) = T_n(f;x) - f(x) \tag{1.11}$$

$$K_n(t) = \sum_{k=0}^{\infty} a_{n,k} \frac{\sin(k+\frac{1}{2})t}{2\sin(t/2)} \tag{1.12}$$

$$\psi(n) = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| \tag{1.13}$$

$$\bar{a}_n(k) = \sum_{r=0}^{k} a_{n,r} \tag{1.14}$$

$$a'_{n,k} = \sum_{r=k}^{n} a_{n,r}. \tag{1.15}$$

2. Introduction

Prösdorff [15] studied the degree of approximation in the Hölder metric and proved the following theorems:

Theorem A [15]. *Let $f \in H_\alpha (0 < \alpha \leqslant 1)$ and $0 \leqslant \beta < \alpha \leqslant 1$. Then*

$$\|\sigma_n(f) - f\|_\beta = O(1) \begin{cases} n^{\beta-\alpha} & (0 < \alpha < 1) \\ n^{\beta-1}(1+\log n)^{1-\beta} & (\alpha = 1) \end{cases}$$

where $\sigma_n(f)$ is the Fejér means of the Fourier series of f.

The case $\beta = 0$ of Theorem A is due to Alexits [2].

Chandra [6] obtained a generalization of Theorem A in the Nörlund or (N,p) and $(\bar{N},p)$ transform set up. Later Mohapatra and Chandra [13] considered the problem by matrix means and obtained the above results as corollaries. Also see Singh ([18], [19]), who claims to have improved the results of Mohapatra and Chandra [13].

With regard to the approximation of functions in L_p norm, the following theorem is due to Quade [16].

Theorem B [16]. *Let $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha \leqslant 1$. Then*

$$\|\sigma_n(f) - f\|_p = O(1) \begin{cases} n^{-\alpha} & (p > 1) \\ n^{-\alpha} & (p = 1, 0 < \alpha < 1) \\ (\log n)/n & (p = 1, \alpha = 1). \end{cases}$$

With a view to generalize the above results in Nörlund transformation set up, attempts were made by Sahney and Rao [17], Chandra [5], [6]. Mohapatra and Russel [14] considered this in generalized Nörlund mean set up.

The object of the present paper is to make a comprehensive study of the above problems and to bring as many corollaries possible within the fold and offer suitable generalization in the following directions:

(a) Hölder metric is to be replaced by generalized Hölder metric,
(b) Transformation is by means of infinite matrix so as to include deferred means introduced by Agnew [1].
(c) Modulus of continuity is to be replaced by more general integral modulus of continuity for $0 < p \leqslant \infty$.

The degree of approximation problem for a class of continuous functions of bounded variation, integral modulus of continuity of order 1 have been extensively studied by Mazhar [10], [11], Mazhar and Totik [12] and Bojanic and Mazhar [3], [4]. However, these studies do not come within the fold of our theorem of the present paper.

3. Main results

We prove the following theorems:

Theorem 1. *Suppose that $A \in \tau$ and let there exist a positive non-decreasing sequence (μ_n) such that*

$$\sum_{k=\mu_n}^{\infty} (k+1)|a_{n,k}| = O(\mu_n). \tag{3.1}$$

Then for $p \geqslant 1$ and $f \in H(\alpha, p)$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < \alpha$

$$\|l_n(x)\|_{(\beta,p)} = O(1)\begin{cases} (1+\log(\mu_n/\lambda_n))^{\beta/\alpha} + \psi(n)\lambda_n^{1-\alpha+\beta}, & (0<\alpha<1) \\ \dfrac{(1+\log(\mu_n/\lambda_n))^{\beta}}{\lambda_n^{1-\beta}} + \psi(n)\lambda_n^{\beta}(\log\lambda_n)^{1-\beta} & (\alpha=1) \end{cases}$$

where $l_n(x)$ and $\psi(n)$ are respectively defined in (1.11) *and* (1.13) *and λ_n is any positive non-decreasing sequence such that $\lambda_n \leqslant \mu_n$.*

Theorem 2. *Let $0<p<1$ and let $A=(a_{n,k})$ satisfy the conditions of Theorem* 1. *Let $f \in H(\alpha, p)$, $0 \leqslant \beta \leqslant \alpha$, $0<\alpha\leqslant 1$. Then*

$$\|l_n(x)\|_{(\beta,p)} = O(1)(\lambda_n^{p-1})^{\beta/\alpha}\lambda_n^{\beta-\alpha} + (\psi(n))^p \begin{cases} O(1) & (\alpha>0, 2p<1) \\ \lambda_n^{(2p-1)\beta/\alpha} & (\alpha>2p-1, 2p>1) \\ (\log\lambda_n)^{\beta/\alpha} & (\alpha>2p-1, 2p=1) \\ \lambda_n^{2p+\beta-\alpha-1} & (0<\alpha<2p-1, 2p>1) \\ (\log\lambda_n)^{1-\beta/\alpha}\lambda_n^{(2p-1)\beta/\alpha} & \\ & (\alpha=2p-1, 2p>1). \end{cases}$$

Theorem 3. *Let $0<\alpha\leqslant 1$, $0\leqslant\beta<\alpha$, $\alpha p>1$. Let $A=(a_{n,k})$ satisfy the same conditions as in Theorem* 1. *Let $f\in H(\alpha,p)$. Then*

$$\|l_n(x)\|_{(\beta-(\beta/\alpha p),p)} = O(1)\left(1+\log\left(\frac{\mu_n}{\lambda_n}\right)\right)^{\beta/\alpha}\lambda_n^{((1/p)-\alpha)(1-(\beta/\alpha))}$$
$$+\,O(1)\psi(n)\begin{cases}\lambda_n^{1+(1/p)+\beta-\alpha-(\beta/\alpha p)} & (\alpha<1+(1/p)) \\ \lambda_n^{\beta/\alpha}(\log\lambda_n)^{(1-\beta/\alpha)} & (\alpha=1+(1/p))\end{cases}$$

Remarks. (i) In case

$$\theta = \liminf \left(\frac{n}{\mu_n}\right) > 0, \tag{3.2}$$

for sufficiently large n

$$\frac{\theta}{2} \sum_{k=\mu_n}^{\infty} |a_{n,k}| \leqslant \frac{n}{\mu_n} \sum_{k=\mu_n}^{\infty} |a_{n,k}| \leqslant \frac{1}{\mu_n} \sum_{k=\mu_n}^{\infty} (k+1)|a_{n,k}| (n \geqslant k)$$

so that hypothesis (3.1) includes (1.8).
(ii) The hypothesis (3.1) seems to be unusual and interesting. It was first introduced by Mohapatra and Chandra [13] in $\mu_n = n+1$. The importance of (3.1) seems to lie in the fact that it moderates the effect of $a_{n,k}$ for large k; in fact (3.1) annihilates it for lower triangular or even deferred matrices. For example if $a_{n,k} = 0$ for $k > \mu_n$, then the hypothesis (3.1) is automatically satisfied. See § 4 for a beautiful application of this in the case of deferred Cesàro mean (Corollaries 4 and 5).

We require the following lemmas for the proof of the theorems.

Lemma 1. *Let* $0 < p \leqslant \infty$. *Then*

(i) $w_p^{(2)}(\delta, f) \leqslant 2w_p(\delta, f)$
(ii) $\|\phi_x - \phi_{x+h}\|_p \leqslant 4K \|f(x) - f(x+h)\|_p$,

where K is some positive constant.

Proof. For $p \geqslant 1$ and by Minkowski's inequality, we have

$$\left(\int_0^{2\pi} |\phi_x(t)|^p \mathrm{d}x\right) \leqslant \left(\int_0^{2\pi} |f(x+t) - f(x)|^p \mathrm{d}x\right)^{1/p} + \left(\int_0^{2\pi} |f(x-t) - f(x)|^p \mathrm{d}x\right)^{1/p} \tag{3.3}$$

and for $0 < p < 1$, we have by the modified Minkowski type inequality

$$\int_0^{2\pi} |\phi_x(t)|^p \mathrm{d}x \leqslant \int_0^{2\pi} |f(x+t) - f(x)|^p \mathrm{d}x + \int_0^{2\pi} |f(x-t) - f(x)|^p \mathrm{d}x. \tag{3.4}$$

Now Lemma 1(i) follows from (3.3) and (3.4). For proving (ii) we first note that

$$\phi_x(t) - \phi_{x+h}(t) = \{f(x+t) - f(x+t+h)\} + \{f(x-t) - f(x+h-t)\} - 2\{f(x) - f(x+h)\}$$

and then apply Minkowski's inequality separately for $p \geqslant 1$ and for $0 < p < 1$.

Lemma 2 [8]. *Suppose that* $f \in \mathrm{Lip}(\alpha, p)$, $p \geqslant 1, 0 < \alpha \leqslant 1$.

(i) *If* $\alpha p \leqslant 1$, $p < q < \dfrac{p}{1 - \alpha p}$, *then* $f \in \mathrm{Lip}\left(\alpha - \dfrac{1}{p} + \dfrac{1}{q}, q\right)$

(ii) *If* $\alpha p > 1$, *then* $f \in \mathrm{Lip}\left(\alpha - \dfrac{1}{p} + \dfrac{1}{q}, q\right)$ *for* $q > p$ *and* f *is equivalent to a function of* $\mathrm{Lip}\left(\alpha - \dfrac{1}{p}\right)$, *i.e., two functions are same almost everywhere.*

Proof of Theorem 1. We know that

$$S_n(x)-f(x)=\frac{1}{\pi}\int_0^{\pi}\phi_x(t)\frac{\sin(n+\frac{1}{2})t}{2\sin t/2}\mathrm{d}t$$

and therefore

$$\begin{aligned} l_n(x) &= \sum_{k=0}^{\infty} a_{n,k}S_k(x)-f(x)\sum_{k=0}^{\infty} a_{n,k} \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty} a_{n,k}\int_0^{\pi}\phi_x(t)\frac{\sin(n+\frac{1}{2})t}{2\sin t/2}\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^{\pi}\phi_x(t)K_n(t)\mathrm{d}t \end{aligned} \tag{3.5}$$

where $K_n(t)$ is defined in (1.12). Note that the change of order of summation and integration is justified provided either side is convergent. We observe that by (1.8) the series for $K_n(t)$ is convergent (even absolutely) and $K_n(t)=O(t^{-1})$ for all $0<t\leqslant\pi$ and hence the integral given in (3.5) exists by (3.6) and by the fact that $f\in H(\alpha,p)$ in which case

$$\|f(x+t)-f(x)\|_p=O(|t|^{\alpha})$$

and by Lemma 1(i)

$$w_p^{(2)}(t,f)=O(|t|^{\alpha}).$$

Now by generalized Minkowski's inequality for $p\geqslant 1$, we have

$$\|l_n(x)-l_n(x+y)\|_p\leqslant\frac{1}{\pi}\int_0^{\pi}\|\phi_x(t)-\phi_{x+y}(t)\|_p|K_n(t)|\mathrm{d}t. \tag{3.6}$$

We split the integral in (3.6) as I_1 and I_2 with limits of integration from 0 to $1/\lambda_n$ and from $1/\lambda_n$ to π respectively.

By making use of the fact that

$$\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|\leqslant\|A\|<\infty$$

and by Lemma 1(i), we obtain

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_0^{1/\lambda_n}\|\phi_x-\phi_{x+y}\|_p|K_n(t)|\mathrm{d}t \\ &= O(1)\int_0^{1/\lambda_n} t^{\alpha}|K_n(t)|\mathrm{d}t \\ &= O(1)\int_0^{1/\lambda_n} t^{\alpha-1}\mathrm{d}t \\ &= O(1)\left(\frac{1}{\lambda_n^{\alpha}}\right). \end{aligned} \tag{3.7}$$

We make use of the fact that, by Abel's transformation

$$\sum_{k=0}^{\infty} a_{n,k} \sin\left(k+\frac{1}{2}\right)t = O(t^{-1}) \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| \tag{3.8}$$

and Lemma 1(i), we obtain

$$\begin{aligned} I_2 &= \frac{1}{\pi}\int_{1/\lambda_n}^{\pi} \|\phi_x(t) - \phi_{x+y}\|_p |K_n(t)| \mathrm{d}t \\ &= O(1)\int_{1/\lambda_n}^{\pi} t^{\alpha}|K_n(t)|\mathrm{d}t \\ &= O(1)\begin{cases} \psi(n)\lambda_n^{1-\alpha} & (0<\alpha<1) \\ \psi(n)\log\lambda_n & (\alpha=1). \end{cases} \end{aligned} \tag{3.9}$$

To obtain alternative set of order estimates for I_1 and I_2 we make use of Lemma 1(ii). Now we split I_1 into I_{11} and I_{12} with limits of integration from 0 to $1/\mu_n$ and from $1/\mu_n$ to $1/\lambda_n$ respectively.

We first note that

$$\begin{aligned} K_n(t) &= \frac{1}{2\sin(\frac{1}{2})t}\left(\sum_{k=0}^{\mu_n} + \sum_{k=\mu_n+1}^{\infty}\right) a_{n,k}\sin\left(k+\frac{1}{2}\right)t \\ &= O(t^{-1})\left[\sum_{k=0}^{\mu_n} |a_{n,k}|(k+1)t + \sum_{k=\mu_n+1}^{\infty} (k+1)|a_{n,k}|t\right] \\ &= O(\mu_n) + O(\mu_n) = O(\mu_n) \end{aligned} \tag{3.10}$$

by (1.8) and (3.1).

Hence by Lemma 1(ii) and (3.10)

$$\begin{aligned} I_{11} &= \left(|y|^{\alpha}\int_0^{1/\mu_n} |K_n(t)|\mathrm{d}t\right) \\ &= O\left(|y|^{\alpha}\mu_n\int_0^{1/\mu_n} \mathrm{d}t\right) = O(|y|^{\alpha}) \end{aligned} \tag{3.11}$$

$$\begin{aligned} I_{12} &= O(|y|^{\alpha})\int_{1/\mu_n}^{1/\lambda_n} \frac{\mathrm{d}t}{t}\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| \\ &= O(|y|^{\alpha})\|A\|\int_{1/\mu_n}^{1/\lambda_n}\frac{\mathrm{d}t}{t} \\ &= O(|y|^{\alpha})\log\frac{\mu_n}{\lambda_n}. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) we obtain

$$I_1 = O(|y|^{\alpha})\left(1+\log\frac{\mu_n}{\lambda_n}\right). \tag{3.13}$$

By Lemma 1(ii) and (3.8) we have

$$\begin{aligned} I_2 &= O(|y|^\alpha)\int_{1/\lambda_n}^{\pi} |K_n(t)|\,dt \\ &= O(|y|^\alpha)\psi(n)\int_{1/\lambda_n}^{\pi} \frac{dt}{t^2} \\ &= O(|y|^\alpha)\psi(n)\lambda_n. \end{aligned} \tag{3.14}$$

Combining the results (3.7) and (3.13) we obtain

$$\begin{aligned} I_1 &= I_1^{\beta/\alpha} I_1^{1-\beta/\alpha} \\ &= O(|y|^\beta)\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha} \lambda_n^{\beta-\alpha} \end{aligned} \tag{3.15}$$

and combining (3.9) and (3.14)

$$\begin{aligned} I_2 &= I_2^{\beta/\alpha} I_2^{1-\beta/\alpha} \\ &= O(|y|^\beta)(\psi(n)\lambda_n)^{\beta/\alpha}\begin{cases}(\psi(n)\lambda_n^{1-\alpha})^{1-\beta/\alpha} & (0<\alpha<1)\\ (\psi(n)\log\lambda_n)^{1-\beta} & (\alpha=1)\end{cases} \\ &= O(|y|^\beta)\psi(n)\begin{cases}\lambda_n^{1+\beta-\alpha} & (0<\alpha<1)\\ \lambda_n^\beta(\log\lambda_n)^{1-\beta} & (\alpha=1).\end{cases} \end{aligned} \tag{3.16}$$

Hence

$$\begin{aligned} &\sup_{y\neq 0}\frac{\|l_n(x+y)-l_n(x)\|_p}{|y|^\beta} \\ &= O(1)\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha}\lambda_n^{\beta-\alpha} + O(1)\psi(n)\begin{cases}\lambda_n^{1-\alpha+\beta} & (0<\alpha<1)\\ \lambda_n^\beta(\log\lambda_n)^{1-\beta} & (\alpha=1).\end{cases} \end{aligned} \tag{3.17}$$

It follows from the analysis of the proofs of (3.7) and (3.9) that

$$\|l_n(x)\|_p = O\left(\frac{1}{\lambda_n^\alpha}\right) + O(1)\psi(n)\begin{cases}\lambda_n^{1-\alpha} & (0<\alpha<1)\\ \log\lambda_n & (\alpha=1).\end{cases} \tag{3.18}$$

Now we combine (3.17) and (3.18) to obtain the degree of approximation for $\|l_n(x)\|_{(\beta,p)}$ as

$$\begin{aligned} \|l_n(x)\|_{(\beta,p)} = O(1)\Bigg[&\frac{1}{\lambda_n^\alpha} + \psi(n)\begin{cases}\lambda_n^{1-\alpha} & (0<\alpha<1)\\ \log\lambda_n & (\alpha=1)\end{cases} \\ &+\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha}\lambda_n^{\beta-\alpha} + \psi(n)\begin{cases}\lambda_n^{1-\alpha+\beta} & (0<\alpha<1)\\ \lambda_n^\beta(\log\lambda_n)^{1-\beta} & (\alpha=1)\end{cases}\Bigg] \end{aligned} \tag{3.19}$$

whence the result follows. This completes the proof of Theorem 1.

We omit the proof of Theorem 2 as it follows the lines of proof of Theorem 1. By Lemma 2(ii) we can make use of the fact that

$$f\in\mathrm{Lip}\left(\alpha-\frac{1}{p}\right) \quad (\alpha p>1) \tag{3.20}$$

for the proof of Theorem 3. Using (3.20) and adopting the argument similar to those used in proving Theorem 1, we can prove Theorem 3.

4. Corollaries

We specialize the matrix A to obtain the following corollaries from Theorem 1.

COROLLARY 1

Let $p \geqslant 1, 0 < \alpha \leqslant 1, 0 \leqslant \beta < \alpha$. Let A be a lower triangular matrix such that

$$a_{n,k} \geqslant 0, \quad a_{n,k} \leqslant a_{n,k+1} (k = 0, 1, 2, \ldots, n-1), \sum_{k=0}^{n} a_{n,k} = 1. \tag{4.1}$$

Then for $f \in H(\alpha, p)$

$$\| l_n(x) \|_{(\beta,p)} = O(1) \begin{cases} \dfrac{1}{n^{\alpha-\beta}} + a_{n,n} n^{1-\alpha+\beta} & (0 < \alpha < 1) \\ \dfrac{1}{n^{1-\beta}} + a_{n,n} n^{\beta} (\log n)^{1-\beta} & (\alpha = 1). \end{cases}$$

Proof. In this case

$$\begin{aligned} \psi(n) = \sum_{k=0}^{n} |a_{n,k} - a_{n,k+1}| &= \sum_{k=0}^{n-1} (a_{n,k+1} - a_{n,k}) + a_{n,n} \\ &= 2a_{n,n} - a_{n,0} \leqslant 2a_{n,n}. \end{aligned}$$

Now, we take $\lambda_n = \mu_n = n + 1$ so that condition (3.1) is automatically satisfied.

COROLLARY 2

Let the conditions of Corollary 1 hold. Then

$$\| l_n(x) \|_{(\beta,p)} = O(1) \begin{cases} a_{n,n}^{\alpha-\beta} (1 + \log(n+1) a_{n,n})^{\beta/\alpha} & (0 < \alpha < 1) \\ a_{n,n}^{1-\beta} \left[(1 + \log(n+1) a_{n,n})^{\beta} + \left(\log \dfrac{1}{a_{n,n}} \right)^{1-\beta} \right] & (\alpha = 1). \end{cases}$$

Proof. Since $1 = \Sigma_{k=0}^{n} a_{n,k} \leqslant (n+1) a_{n,n}$, we choose $\mu_n = n + 1$, $\lambda_n = 1/a_{n,n}$ so as to satisfy condition (3.1). Also we have $\psi(n) \leqslant 2a_{n,n}$.

Remark. The case $p = \infty$ of Corollaries 1 and 2 are respectively Theorems 1 and 2 of Mohapatra and Chandra [13].

COROLLARY 3

Let $p \geqslant 1, 0 < \alpha \leqslant 1, 0 \leqslant \beta < \alpha$. Let A be a lower triangular matrix such that

$$a_{n,k} \geqslant 0, \sum_{k=0}^{n} a_{n,k} = 1, a_{n,k} \geqslant a_{n,k+1} (k = 0, 1, 2, \ldots, n-1). \tag{4.2}$$

Then for $f \in H(\alpha, p)$

$$\| l_n(x) \|_{(\beta,p)} = O(1) \begin{cases} (1 + \log(n+1) a_{n,0})^{\beta/\alpha} a_{n,0}^{\alpha-\beta}, & (0 < \alpha < 1) \\ (1 + \log(n+1) a_{n,0})^{\beta} a_{n,0}^{1-\beta} + a_{n,0}^{1-\beta} \left(\log \dfrac{1}{a_{n,0}} \right)^{1-\beta} & (\alpha = 1). \end{cases} \tag{4.3}$$

Also

$$\|l_n(x)\|_{(\beta,p)} = O(1)\begin{cases} n^{\beta-\alpha} + a_{n,0}n^{1-\alpha+\beta} & (0<\alpha<1) \\ n^{\beta-1} + a_{n,0}n^{\beta}(\log n)^{1-\beta} & (\alpha=1). \end{cases} \tag{4.4}$$

Proof. In this case

$$\psi(n) = \sum_{k=0}^{n} |a_{n,k} - a_{n,k+1}| = \sum_{k=0}^{n} (a_{n,k} - a_{n,k+1}) = a_{n,0}.$$

Since

$$1 = \sum_{k=0}^{n} a_{n,k} \leqslant (n+1)a_{n,0}$$

we choose

$$\mu_n = n+1, \quad \lambda_n = \frac{1}{a_{n,0}}$$

to obtain (4.3) or we may choose

$$\mu_n = \lambda_n = n+1$$

to obtain (4.4).

Remark. In the case $a_{n,k} = (A_{n-k}^{\delta-1}/A_n^{\delta})$ (Cesàro matrix), $a_{n,0} = (A_n^{\delta-1}/A_n^{\delta}) \sim (1/n)$. Cesàro matrix satisfies the condition (4.2) in the case $\delta \geqslant 1$ and it satisfies (4.1) in the case $0<\delta\leqslant 1$. Thus Corollaries 1 (or 2) and 3 together cover the degree of approximation of Cesàro matrix for all $\delta > 0$.

Before we give a few more corollaries, we first describe below a method of summation called 'deferred Cesàro transformation' introduced by Agnew [1].

Deferred Cesàro mean. Let p_n and q_n be sequences of non-negative integers satisfying

$$p_n < q_n \tag{4.5}$$

$$\lim_{n\to\infty} q_n = \infty. \tag{4.6}$$

The deferred Cesàro mean $D(p_n, q_n)$ is defined by ([1], p. 414),

$$D_n(S_n) = \frac{S_{p_n+1} + S_{p_n+2} + \cdots + S_{q_n}}{q_n - p_n}. \tag{4.7}$$

In the notation of matrix transformation

$$D_n(S_n) = \sum_{k=0}^{\infty} a_{n,k}S_k \tag{4.8}$$

where

$$a_{n,k} = \begin{cases} \dfrac{1}{q_n - p_n} & p_n < k \leqslant q_n \\ 0 & \text{(elsewhere)}. \end{cases} \tag{4.9}$$

It is known [1] that $D(p_n, q_n)$ is regular under conditions (4.5) and (4.6). Note that $D_n(n-1, n)$ is the identity transformation and $D(0, n)$ is the $(C, 1)$ transformation.

$D(n, n+k)$ is called the delayed first arithmetic mean (Zygmund [21], p. 80) and it is known [21] that

$$(C, 1) \subset D(n, n+k) \quad \text{if } \frac{n}{k} = O(1) \text{ as } k \to \infty \text{ with } n.$$

In fact, more generally, it is known [1] that

$$(C, 1) \subset D(p_n, q_n) \text{ if and only if } \frac{p_n}{q_n - p_n} = O(1).$$

It is also known [1] that

$$D(p_n, n) \subset (C, 1)$$

and

$$D(p_n, n) \sim (C, 1) \text{ if and only if } \frac{p_n}{q_n - p_n} = O(1).$$

Let $\lambda = \{\lambda_n\}$ be a monotone non-decreasing sequence of positive integers such that $\lambda_1 = 1$ and $\lambda_{n+1} - \lambda_n \leqslant 1$. Then $D(n - \lambda_n, n)$ is same as the nth generalized de la vallee Poussin mean $V_n(\lambda)$ [9] generated by the sequence $\{\lambda_n\}$.

We now obtain the degree of approximation of deferred Cesàro mean.

It may be noted that the condition (3.1) is automatically satisfied by choosing a suitable μ_n for which $a_{n,k}$ vanishes if $k > \mu_n$. For example

$$\mu_n = q_n + 1, \quad \lambda_n = q_n - p_n.$$

Now

$$\begin{aligned} \psi(n) &= \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| \\ &= \sum_{k=p_n}^{q_n} |a_{n,k} - a_{n,k+1}| \\ &= |a_{n,p_n} - a_{n,p_n+1}| + |a_{n,q_n} - a_{n,q_n+1}| \\ &= \left| -\frac{1}{q_n - p_n} \right| + \frac{1}{q_n - p_n} = \frac{2}{q_n - p_n}. \end{aligned} \tag{4.10}$$

Thus from Theorem 1, we obtain Corollary 4.

COROLLARY 4

Let $p \geqslant 1$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < \alpha$ and $f \in H(\alpha, p)$. Then

$$\| l_n(x) \|_{(\beta, p)} =$$

$$O(1) \begin{cases} \left(1 + \log \dfrac{q_n + 1}{q_n + p_n}\right)^{\beta/\alpha} \dfrac{1}{(q_n - p_n)^{\alpha - \beta}} & (0 < \alpha < 1) \\ \dfrac{1}{(q_n - p_n)^{1-\beta}} \left[\left(1 + \log \dfrac{q_n + 1}{q_n - p_n}\right)^{\beta} + (\log(q_n - p_n))^{1-\beta} \right] & (\alpha = 1). \end{cases}$$

Remark. In the case

$$\delta = \limsup_{n \to \infty} \frac{p_n}{q_n} < 1 \tag{4.11}$$

we have

$$\limsup_{n\to\infty} \frac{q_n+1}{q_n-p_n} = \limsup_{n\to\infty} \left(\frac{1+(1/q_n)}{1-(p_n/q_n)}\right) \leqslant \frac{1}{1-\delta}$$

so that from Corollary 4, we obtain Corollary 5.

COROLLARY 5

Let $p \geqslant 1, 0<\alpha\leqslant 1, 0\leqslant\beta<\alpha$. Suppose that (4.11) *holds. Then for $f\in H(\alpha,p)$*

$$\|l_n(x)\|_{(\beta,p)} = O(1)\begin{cases} \dfrac{1}{(q_n-p_n)^{\alpha-\beta}} & (0<\alpha<1) \\ \dfrac{1}{(q_n-p_n)^{1-\beta}}[1+(\log(q_n+p_n))^{1-\beta}] & (\alpha=1). \end{cases}$$

At this stage we remark that the case $p=\infty$ of Corollary 4 for $D(n-\lambda_n, n)$ is due to Stypinski [20] where one log factor seems to have been overlooked. Corollary 4 too covers the case of Chandra [7] in relevant cases.

5. Additional theorems

We establish the following theorems:

Theorem 4. *Let $A=(a_{n,k})$ satisfy the same condition as in Theorem* 1. *Further let*

$$a_{n,k}\geqslant a_{n,k+1}\geqslant 0. \tag{5.1}$$

Then for $f\in L_p, p\geqslant 1$

$$\|l_n(x)\|_p = O(1)w_p^{(2)}\left(\frac{\pi}{\mu_n}\right) + O(1)\sum_{k=1}^{\mu_n}\frac{1}{k}w_p^{(2)}\left(\frac{\pi}{k}\right)\bar{a}_n(k+1).$$

Theorem 5. *Let $A=(a_{n,k})$ satisfy the same conditions as in Theorem* 1. *Then for $p\geqslant 1$ and $f\in H(\alpha,p)$, $0<\alpha\leqslant 1, 0\leqslant\beta<\alpha$*

$$\|l_n(x)\|_{(\beta,p)} = O(1)\left[\frac{1}{\lambda_n^{\alpha-\beta}}\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha} + \left(\sum_{k=1}^{\lambda_n}\frac{\theta(n,k)+\psi(n,k)}{k}\right)^{\beta/\alpha}\left(\sum_{k=1}^{\lambda_n}\frac{\theta(n,k)+\psi(n,k)}{k^{1+\alpha}}\right)^{1-(\beta/\alpha)}\right]$$

where

$$\theta(n,k) = \sum_{m=0}^{k}|a_{n,m}|,\quad \psi(n,k)=(k+1)\sum_{m=k+1}^{\infty}|a_{n,m}-a_{n,m+1}| \tag{5.2}$$

amd μ_n and λ_n are defined in Theorem 1.

Theorem 6. *Let $A=(a_{n,k})$ satisfy condition of Theorem* 4. *For $p\geqslant 1$, suppose that $f\in L_p$ and $w_p^{(2)}(t)/t^\theta$ is monotonic non-increasing as t increases for some θ with $0<\theta\leqslant 1$. Then*

$$\|l_n(x)\|_p = O(1)w_p^{(2)}\left(\frac{\pi}{\mu}\right)\left[1+\mu_n^\theta\sum_{k=1}^{\mu_n}\frac{\bar{a}_n(k+1)}{k^{1+\theta}}\right].$$

We need the following lemma.

Lemma 3. *Let* (5.1) *hold. Then*

$$\sum_{k=0}^{\infty} a_{n,k} \sin\left(k+\frac{1}{2}\right)t = O(\bar{a}_n(T))$$

where $T=[\pi/t]$.

Proof. We have

$$\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| \leqslant \left|\sum_{k=0}^{T-1} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| + \left|\sum_{k=T}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|$$
$$\leqslant \sum_{k=0}^{T-1} a_{n,k} + a_{n,T}O(t^{-1}),$$

by Abel's lemma. Since $(T+1)a_{n,T} \leqslant \Sigma_{k=0}^{T} a_{n,k}$, Lemma 3 follows at once from the above inequality.

Proof of Theorem 4. We know that (see the proof of Theorem 1)

$$l_n(x) = \frac{1}{\pi}\int_0^{\pi} \phi_x(t)K_n(t)\mathrm{d}t.$$

By generalized Minkowski's inequality

$$\begin{aligned}\|l_n(x)\|_p &\leqslant \frac{1}{\pi}\int_0^{\pi} \|\phi_x(t)\|_p |K_n(t)|\mathrm{d}t \\ &= \frac{1}{\pi}\left(\int_0^{\pi/\mu_n} + \int_{\pi/\mu_n}^{\pi}\right)\|\phi_x(t)\|_p|K_n(t)|\mathrm{d}t \\ &= I_1 + I_2. \end{aligned} \tag{5.3}$$

Now

$$\begin{aligned} I_1 &= O(1)\int_0^{\pi/\mu_n} \frac{w_p^{(2)}(t)}{t}\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t \\ &= O(1)\mu_n \int_0^{\pi/\mu_n} w_p^{(2)}(t)\mathrm{d}t \quad \text{(as in (3.10))} \\ &= O(1)\mu_n w_p^{(2)}(\pi/\mu_n)\int_0^{\pi/\mu_n} \mathrm{d}t = O(1)w_p^{(2)}(\pi/\mu_n). \end{aligned} \tag{5.4}$$

By Lemma 3

$$\begin{aligned} I_2 &= O(1)\int_{\pi/\mu_n}^{\pi} \frac{w_p^{(2)}(t)}{t}\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t \\ &= O(1)\int_{\pi/\mu_n}^{\pi} \frac{w_p^{(2)}(t)}{t}\bar{a}_n(T)\mathrm{d}t \\ &= O(1)\sum_{k=1}^{\mu_n-1}\int_{\pi/(k+1)}^{\pi/k} \frac{w_p^{(2)}(t)}{t}\bar{a}_n([\pi/t])\mathrm{d}t \end{aligned}$$

$$= O(1) \sum_{k=1}^{\mu_n} \bar{a}_n(k+1) w_p^{(2)}(\pi/k) \int_{\pi/(k+1)}^{\pi/k} \frac{\mathrm{d}t}{t}$$

$$= O(1) \sum_{k=1}^{\mu_n} \frac{\bar{a}_n(k+1)}{k} w_p^{(2)}(\pi/k). \tag{5.5}$$

Now Theorem 4 follows at once from (5.3), (5.4) and (5.5).

We omit the proof of Theorems 5 and 6 as these are easy and follow the lines of proof of Theorem 4.

6.

In this section, we specialize the matrix $A = (a_{n,k})$ to obtain the following corollary to Theorem 5.

COROLLARY 6

Let $A = (a_{n,k})$ be an infinite matrix satisfying the following.

$$a_{n,k} \geqslant 0, \quad \sum_{k=0}^{\infty} a_{n,k} = 1, \quad a_{n,k} \geqslant a_{n,k+1} \tag{6.1}$$

$$\sum_{k=n+1}^{\infty} (k+1) a_{n,k} = O(n). \tag{6.2}$$

Then for $0 \leqslant \beta < \alpha \leqslant 1$ and $f \in H(\alpha, p)$

$$\|l_n(x)\|_{(\beta,p)} = O(1)\left[\frac{1}{n^{\alpha-\beta}} + \left(\sum_{k=1}^{n} \frac{\bar{a}_n(k+1)}{k}\right)^{\beta/\alpha} \left(\sum_{k=1}^{n} \frac{\bar{a}_n(k+1)}{k^{1+\alpha}}\right)^{1-(\beta/\alpha)}\right].$$

Proof. In this case

$$\begin{aligned}
\theta(n,k) + \psi(n,k) &= \sum_{m=0}^{k} |a_{n,m}| + (k+1) \sum_{m=k+1}^{\infty} |a_{n,m} - a_{n,m+1}| \\
&= \sum_{m=0}^{k} a_{n,m} + (k+1) \sum_{m=k+1}^{\infty} (a_{n,m} - a_{n,m+1}) \\
&= \sum_{m=0}^{k} a_{n,m} + (k+1) a_{n,k+1} \\
&\leqslant \sum_{m=0}^{k} a_{n,m} + \frac{k+1}{k+1} \sum_{m=0}^{k} a_{n,m} \\
&= O(1) \bar{a}_n(k).
\end{aligned}$$

Now taking $\mu_n = \lambda_n = n$ in Theorem 5, we obtain Corollary 6.

Remark. The case $p = \infty$ of Corollary 6 is Theorem 3 of Mohapatra and Chandra [14].

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A note on multidimensional modified fractional calculus operators

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Abstract. In the present investigation some new formulas giving the images under multidimensional modified fractional operators of the celebrated H-function of Fox [*Trans. Am. Math. Soc.* **98** (1961) 395–429] are obtained. Special cases are briefly pointed out and the results are also studied on general spaces of functions $M_\gamma(R^n_+)$.

Keywords. Multidimensional operators; H-function; Mellin transform; fractional calculus operators

1. Introduction and preliminaries

The field of fractional calculus is presently receiving keen attention by many researchers. The latest monographs on the subject by Samko, Kilbas and Marichev [14], Kiryakova [4] and Miller and Ross [6] give fairly good account of the developments in fractional calculus, and consider several aspects of applications to potential problems in analysis. Most of the investigations carried out are for one dimensional case. However, some work relating to two dimensional (and multidimensional) cases have also been considered but in general these happen to be repeated applications of the one dimensional case (see, for example, [3], [8], [9], [14] and [16]).

Our purpose in the present investigation is to obtain new classes of formulas giving the images under the multidimensional modified fractional operators introduced by Brychkov, Glaeske, Prudnikov and Tuan [1] of the celebrated H-function of Fox [2]. The H-function which is usually defined in terms of the Mellin–Barnes contour integral involving the quotients of gamma functions, and as already evidenced in the literature, such single Mellin–Barnes contour integrals are useful in several areas of applied mathematics and statistics ([5] and [15]), our formulas would possess wide applications. Moreover, the H-function includes, as its special cases, mathematically important functions such as the Bessel–Wright function $J^\mu_v(x)$, the Fox–Wright function $p^\Psi q$, the Mittag–Leffler functions E_α and $E_{\alpha,\beta}$, the generalized parabolic function, the generalized hypergeometric function and the Meijer's G-function. The results of this paper may, therefore, be regarded as key formulas. Some of the results given recently by Tuan and Saigo [17] are deducible from our general formulas. The results of the present paper are also studied on general spaces of function $M_\gamma(R^n_+)$.

Throughout the paper we follow the notations and conventions of [1]. Thus, R denotes the field of real numbers and C the field of complex numbers. R^n represents the set of n-tuple real numbers, with R^n_+ denoting the set of non-negative real numbers, and C^n the set of complex numbers. Also, we write x^p for the product $\Pi^n_{i=1} x^p_i$ and x^λ for the product $\Pi^n_{i=1} x_i^{\lambda_i}$ where $x=(x_1,\ldots,x_n)$, $p\in C$ and $\lambda=(\lambda_1,\ldots,\lambda_n)\in C^n$. The symbol $|\lambda|$ means the finite sum $\lambda_1+\cdots+\lambda_n$, and D^1 stands for $(\partial^n/\partial x_1\ldots\partial x_n)$.

By ϕ_+ we mean the positive part of a function ϕ defined by

$$\begin{cases} \phi_+(x) = \phi(x), & \phi(x) > 0 \\ \quad\;\; = 0, & \phi(x) \leqslant 0\,. \end{cases} \tag{1.1}$$

2. Multidimensional operators

Definition: For Re $(\alpha) > 0$, the multidimensional modified fractional integrals of order $\alpha \in C$ of $f\colon R_+^n \to C$ are defined by

$$\begin{aligned} X_+^\alpha f(x) &= \frac{1}{\Gamma(\alpha+1)} \int_{R_+^n} \left[\min\left\{ \frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n} \right\} - 1 \right]_+^\alpha f(t)\,\mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(x_k^{1-n} \int_0^{x_k} (x_k - t_k)^\alpha \right. \\ &\qquad \left. \times\, t_k^{n-\alpha-1} f\left(\frac{x_1 t_k}{x_k}, \ldots, \frac{x_n t_k}{x_k} \right) \mathrm{d}t_k \right), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} X_-^\alpha f(x) &= \frac{(-1)^n}{\Gamma(\alpha+1)} D^1 \int_{R_+^n} \left[1 - \max\left\{ \frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n} \right\} \right]_+^\alpha f(t)\,\mathrm{d}t \\ &= -\frac{1}{\Gamma(\alpha+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(x_k^{1-n} \int_{x_k}^{\infty} (t_k - x_k)^\alpha \right. \\ &\qquad \left. \times\, t_k^{n-\alpha-1} f\left(\frac{x_1 t_k}{x_k}, \ldots, \frac{x_n t_k}{x_k} \right) \mathrm{d}t_k \right). \end{aligned} \tag{2.2}$$

Definitions (2.1) and (2.2) are introduced in [1] and have been invoked recently in [17]. By suitable subdivisions of the region R_+^n (see [17, p. 255]) for fixed $x \in R_+^n$, the multidimensional operators (2.1) and (2.2) are expressed above in terms of finite sums of single integrals.

In order to underline the dimension of the operators, the notations $X_+^{\alpha;\,n}$ and $X_-^{\alpha;\,n}$ are preferred instead of X_+^α and X_-^α. For $n = 1$, we have the following connections.

$$X_+^{\alpha;\,1} f(x) = I_{0+}^\alpha x^{-\alpha} f(x) \tag{2.3}$$

and

$$X_-^{\alpha;\,1} f(x) = I_-^\alpha x^{-\alpha} f(x), \tag{2.4}$$

where I_{0+}^α and I_-^α are the well known one-dimensional Riemann–Liouville and Weyl integral operators, respectively (see [14]).

3. Modified fractional integrals of H-function

In this paper we shall concern ourselves with the widely used class of special functions known as the H-function in literature which is defined as follows ([2, p. 408]; see also [7, Section 8.3] and [15, Chapter 2]).

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \theta(\sigma) z^{-\sigma}\,\mathrm{d}\sigma, \tag{3.1}$$

where

$$\theta(\sigma)=\left\{\prod_{j=1}^{M}\Gamma(b_j+B_j\sigma)\prod_{j=1}^{N}\Gamma(1-a_j-A_j\sigma)\right\}$$
$$\times\left\{\prod_{j=n+1}^{Q}\Gamma(1-b_j-B_j\sigma)\prod_{j=N+1}^{P}\Gamma(a_j+A_j\sigma)\right\}^{-1}, \tag{3.2}$$

with

$$\Delta=\sum_1^N A_j-\sum_{N+1}^P A_j+\sum_1^M B_j-\sum_{M+1}^Q B_j>0,\quad \text{or}\quad \Delta=0,$$

$$\mathrm{Re}\left(\sum_1^P a_j-\sum_1^Q b_j\right)-(P-Q)/2+\rho\left(\sum_1^P A_j-\sum_1^Q B_j\right)>1;$$

$$\mathrm{Re}(b_j)>-\rho B_j(j=1,\ldots,M),\quad \mathrm{Re}(a_j)>1-\rho A_j(j=1,\ldots,N).$$

Incidentally, special forms of H-function kernels have been used to define new fractional calculus operators in [4] (see also [12] and [13]).

Before stating our main formulas, we require the following results:

Lemma 1. *If* $s=(s_1,\ldots,s_n)\in C^n$, $h=(h_1,\ldots,h_n)\in R_+^n$, *and* $g(y)y^{(|s/h|)-1}\in L_1(R_+)$, *then*

$$\int_{R_+^n} x^{s-1}g(\max\{x_1^{h_1},\ldots,x_n^{h_n}\})\mathrm{d}x$$
$$=\frac{\left|\frac{s}{h}\right|}{s_1\ldots s_n}g^*\left(\left|\frac{s}{h}\right|\right), \tag{3.3}$$
$$\mathrm{Re}(s_j)>0\quad (j=1,\ldots,n),$$

and

$$\int_{R_+^n} x^{s-1}g(\min\{x_1^{h_1},\ldots,x_n^{h_n}\})\mathrm{d}x$$
$$=\frac{(-1)^{n-1}\left|\frac{s}{h}\right|}{s_1\ldots s_n}g^*\left(\left|\frac{s}{h}\right|\right), \tag{3.4}$$
$$\mathrm{Re}(s_j)<0\quad (j=1,\ldots,n),$$

where $g^*(u)$ *denotes the one-dimensional Mellin transform of* $g(u)$.

Proof. The results (3.3) and (3.4) follow easily from [17, p. 256]. These can also be derived independently by using the operational rules stated in [1, p. 200, Theorem 3.9 (3.12)].

We begin our derivation of the first formula by recalling the multidimensional Mellin inversion formula [1]:

$$f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s)x^{-s}\mathrm{d}s, \tag{3.5}$$

for the Mellin transform

$$f^*(s)=M\{f(x)\}=\int_{R_+^n} x^{s-1}f(x)\mathrm{d}x. \tag{3.6}$$

Here, and in what follows, we use the notation

$$\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} \quad \text{for} \quad \int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_n-i\infty}^{\gamma_n+i\infty}.$$

Let $\lambda\in R^n$, $\lambda_j<0$ $(j=1,\ldots,n)$ and $h>0$, then in view of (3.5), and by the application of (3.4) of Lemma 1 we can express

$$x^{-1}H_{P,Q}^{M,N}[\min\{x_1^h,\ldots,x_n^h\}]$$
$$=\frac{(-1)^{n-1}}{h(2\pi i)^n}\int_{(\lambda)-i\infty}^{(\lambda)+i\infty}F_h^*(s)\frac{|s|}{s_1\ldots s_n}x^{-s-1}\mathrm{d}s, \tag{3.7}$$

where

$$F_h^*(s)=\left(\prod_{j=1}^{M}\Gamma\left(b_j+B_j\frac{|s|}{h}\right)\prod_{j=1}^{N}\Gamma\left(1-a_j-A_j\frac{|s|}{h}\right)\right)$$
$$\left(\prod_{j=M+1}^{Q}\Gamma\left(1-b_j-B_j\frac{|s|}{h}\right)\prod_{j=N+1}^{P}\Gamma\left(a_j+A_j\frac{|s|}{h}\right)\right)^{-1}. \tag{3.8}$$

Now operating X_+^α on both sides since $\mathrm{Re}(-s_j-1)>-1$ $(j=1,\ldots,n)$, and using the formula [17, p. 257, eq. (3.5)]:

$$X_+^\alpha x^\beta=\frac{\Gamma(n-\alpha+|\beta|)}{\Gamma(n+|\beta|)}x^\beta, \tag{3.9}$$

$$\mathrm{Re}(\beta_j)>-1\ (j=1,\ldots,n),\quad \mathrm{Re}(|\beta|)>\mathrm{Re}(\alpha)-n,$$

we obtain

$$X_+^\alpha x^{-1}H_{P,Q}^{M,N}[\min\{x_1^h,\ldots,x_n^h\}]$$
$$=\frac{(-1)^{n-1}}{h(2\pi i)^n}\int_{(\lambda)-i\infty}^{(\lambda)+i\infty}F_h^*(s)\frac{\Gamma(-\alpha-|s|)}{\Gamma(-|s|)}\cdot\frac{|s|}{s_1\ldots s_n}x^{-s-1}\mathrm{d}s, \tag{3.10}$$

where $\mathrm{Re}(\alpha)+|\lambda|<0$ and $F_h^*(s)$ is given by (3.8).

Interpreting the R.H.S. of (3.10) by means of (3.7), we arrive at the following result:

Formula 1.

$$X_+^\alpha x^{-1}H_{P,Q}^{M,N}[\min\{x_1^h,\ldots,x_n^h\}]$$
$$=x^{-1}H_{P+1,Q+1}^{M,N+1}\left[\min\{x_1^h,\ldots,x_n^h\}\middle|\begin{matrix}(1+\alpha,h),(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q},(1,h)\end{matrix}\right], \tag{3.11}$$

provided that $\mathrm{Re}(\alpha)>0$, $h\in R_+$, $\mathrm{Re}(\alpha)<-|\lambda|$, $\lambda=(\lambda_1,\ldots,\lambda_n)\in R^n$, *with* $\lambda_j<0$ $(j=1,\ldots,n)$, $\mathrm{Re}(a_j)<1-(|\lambda|A_j/h)$ $(j=1,\ldots,N)$, $\mathrm{Re}(b_j)>-|\lambda|B_j/h$ $(j=1,\ldots,M)$, *and the conditions*

(i) $\Delta=\sum_1^N A_j-\sum_{N+1}^P A_j+\sum_1^M B_j-\sum_{M+1}^Q B_j>0$, or

(ii) $\Delta=0, \mathrm{Re}\left(\sum_1^P a_j-\sum_1^Q b_j\right)-(P-Q)/2-|\lambda|\left(\sum_1^Q B_j-\sum_1^P A_j\right)>1$

hold true.

In an analogous manner, by using the formula [17, p. 257, eq. (3.6)] instead of (3.9), we would arrive at the following formula relating to operator (2.2):

Formula 2.

$$X_{-}^{\alpha} x^{-1} H_{P,Q}^{M,N}[\max\{x_1^h,\ldots,x_n^h\}]$$
$$= x^{-1} H_{P+1,Q+1}^{M+1,N}\left[\max\{x_1^h,\ldots,x_n^h\}\left|\begin{matrix}(a_j,A_j)_{1,P},(1+\alpha,h)\\(1,h),(b_j,B_j)_{1,Q}\end{matrix}\right.\right], \tag{3.12}$$

provided that $\mathrm{Re}(\alpha)>0$, $h\in R_+$, $\lambda=(\lambda_1,\ldots,\lambda_n)\in R^n$ *with* $\lambda_j>0$ $(j=1,\ldots,n)$, $\mathrm{Re}(a_j)<1-(|\lambda|A_j/h)(j=1,\ldots,N)$, $\mathrm{Re}(b_j)>-|\lambda|B_j/h(j=1,\ldots,M)$, *and the conditions* (i) and (ii) *of* (3.11) *hold true.*

Remark 1. For $A_j=1\,(j=1,\ldots,P)$, $B_j=1\,(j=1,\ldots,Q)$, and $h=1$, then (3.11) and (3.12) yield the corresponding formulas for the Meijer's G-function [17, p. 260].

Remark 2. It may be observed that for $n=1$, (3.11) and (3.12) reduce precisely to the formulas obtained earlier by Raina and Koul ([10, p. 99, eq. (7)] and [11, p. 277, eq. (2.5)], respectively).

4. Operators on space $M_\gamma(R_+^n)$

Following [17], let $M_\gamma(R_+^n)$ denote the space of functions f which are defined on R_+^n, where $\gamma=(\gamma_1,\ldots,\gamma_n)\in R^n$, satisfying the conditions:

(i) $f(e^{x_1},\ldots,e^{x_n})$ can be analytically continued to an entire function of exponential type on C^n,

(ii) $\left|(1+|\ln z_1|)^{r_1}\cdots(1+|\ln z_n|)^{r_n}\left(z_1\frac{\partial}{\partial z_1}\right)^{k_1}\cdots\left(z_n\frac{\partial}{\partial z_n}\right)^{k_n}(z_1^{\gamma_1}\cdots z_n^{\gamma_n}f(z_1,\ldots,z_n))\right|$
$\leqslant C_{k,r}\exp(a_1|\arg z_1|+\cdots+a_n|\arg z_n|)$, for $k_j,r_j=0,1,2,\ldots;j=1,\ldots,n$; are valid for all $z=(z_1,\ldots,z_n)\in C^n\backslash\{z|z_1,\ldots,z_n=0\}$,

where the real constants $a_1,\ldots,a_n$ and $C_{k,r}=C_{k_1,\ldots,k_n,r_1,\ldots,r_n}$ are independent of $z\in C^n$.

The above inequalities imply that $x^{\gamma-1}f(x)\in L(R_+^n)$ for $f\in M_\gamma(R_+^n)$. We now record the following known results pertaining to the space of functions $M_\gamma(R_+^n)$ involving the operators X_+^α and X_-^α.

Lemma 2. [17, p. 262, Theorem 2]. *If* $\beta\in C^n$ *and* $\gamma\in R^n$ *such that* $\gamma_j+\mathrm{Re}(\beta_j)<1$ $(j=1,\ldots,n)$, *then the operator* $x^\beta X_+^\alpha x^{-\beta}$ $(\mathrm{Re}(\alpha)>0)$ *is a homeomorphism of the space* $M_\gamma(R_+^n)$ *onto itself, and*

$$x^\beta X_+^\alpha x^{-\beta}f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)+i\infty}^{(\gamma)-i\infty}f^*(s)\frac{\Gamma(n-\alpha-|\beta|-|s|)}{\Gamma(n-|\beta|-|s|)}x^{-s}\mathrm{d}s, \tag{4.1}$$

provided that $\mathrm{Re}(\alpha)+\mathrm{Re}|\beta|+|\gamma|<n.$

Lemma 3. [17, p. 263, Theorem 3]. *If* $\beta\in C^n$, $\gamma\in R^n$ *such that* $\gamma_j+\mathrm{Re}(\beta_j)>1$ $(j=1,\ldots,n)$, *then the operator* $x^\beta X_-^\alpha x^{-\beta}$, $\mathrm{Re}(\alpha)>0$, *is a homeomorphism of the space* $M_\gamma(R_+^n)$ *onto itself, and*

$$x^\beta X_-^\alpha x^{-\beta}f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}f^*(s)\frac{\Gamma(1-n+|\beta|+|s|)}{\Gamma(1+\alpha-n+|\beta|+|s|)}x^{-s}\mathrm{d}s. \tag{4.2}$$

Using Lemma 2, we have

$$x^{\beta}X_{+}^{\alpha}x^{-\beta}H_{P,Q}^{M,N}[\min\{x_1^h,\ldots,x_n^h\}]$$
$$=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}H^*(s)\frac{\Gamma(\mathrm{n}-\alpha-|\beta|-|s|)}{\Gamma(n-|\beta|-|s|)}x^{-s}\mathrm{d}s, \tag{4.3}$$

where $H^*(s)$ denotes the Mellin transform of the involved H-function.

Applying (3.4) of Lemma 1 to compute $H^*(s)$, and interpreting the resulting expression by means of H-function defined by (3.7), we get the following result:

Formula 3.

$$x^{\beta}X_{+}^{\alpha}x^{-\beta}H_{P,Q}^{M,N}[\min\{x_1^h,\ldots,x_n^h\}]$$
$$=H_{P+1,Q+1}^{M,N+1}\left[\min\{x_1^h,\ldots,x_n^h\}\left|\begin{matrix}(1-n+\alpha+|\beta|,h),(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q},(1-n+|\beta|,h)\end{matrix}\right.\right], \tag{4.4}$$

provided that $\mathrm{Re}(\alpha)>0$, $\beta\in C^n$, $h\in R_+$, $\mathrm{Re}(\alpha)+\mathrm{Re}|\beta|+|\gamma|<n$, *and the same conditions stated with Formula 1 (with* $|\lambda|$ *replaced by* $|\gamma|$*) are satisfied.*

In an analogous manner (by using Lemma 3), we obtain the following result:

Formula 4.

$$x^{\beta}X_{-}^{\alpha}x^{-\beta}H_{P,Q}^{M,N}[\max\{x_1^h,\ldots,x_n^h\}]$$
$$=H_{P+1,Q+1}^{M+1,N}\left[\max\{x_1^h,\ldots,x_n^h\}\left|\begin{matrix}(a_j,A_j)_{1,P},(1+\alpha-n+|\beta|,h)\\(1-n+|\beta|,h),\ (b_j,B_j)_{1,Q}\end{matrix}\right.\right], \tag{4.5}$$

provided that $\mathrm{Re}(\alpha)>0$, $\beta\in C^n$, $h\in R_+$, $\mathrm{Re}|\beta|+|\gamma|>n-1$, *and the same conditions stated with Formula 2 (with* $|\lambda|$ *replaced by* $|\gamma|$*) are satisfied.*

Remark 3. It is worth noting that for $\beta=1$, Formulas 3 and 4 at once yield the results (3.11) and (3.12), respectively.

5. Fractional differentials on $M_{\gamma}(R_+^n)$

Corresponding to Lemma 2, there exists the inverse operator X_+^{α} which is a homeomorphism of itself, and is defined by ([17])

$$X_{+}^{-\alpha}f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}f^*(s)\frac{\Gamma(n-|s|)}{\Gamma(n-\alpha-|s|)}x^{-s}\mathrm{d}s, \tag{5.1}$$

provided that $\gamma_j<1$ $(j=1,\ldots,n)$, $\mathrm{Re}(\alpha)+|\gamma|<n$, $\mathrm{Re}(\alpha)>0$.

Similarly, there exists the operator $X_{-}^{-\alpha}$ defined by ([17])

$$X_{-}^{-\alpha}f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}f^*(s)\frac{\Gamma(1+\alpha-n+|s|)}{\Gamma(1-n+|s|)}x^{-s}\mathrm{d}s, \tag{5.2}$$

provided that $\gamma_j>1$ $(j=1,\ldots,n)$, $\mathrm{Re}(\alpha)+|\gamma|>n-1$, $\mathrm{Re}(\alpha)>0$.

Using (5.1) and (5.2) in conjuction with the appropriate results of Lemma 1 for computing Mellin transforms (as in the case of (4.4) and (4.5) above), we arrive at the following:

Formula 5.

$$
\begin{aligned}
&X_{+}^{-\alpha} H_{P,Q}^{M,N}[\min\{x_1^h,\ldots,x_n^h\}]\\
&= H_{P+1,Q+1}^{M,N+1}\left[\min\{x_1^h,\ldots,x_n^h\}\,\middle|\,\begin{matrix}(1-n,h),(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q},(1-n+\alpha,h)\end{matrix}\right],
\end{aligned}\tag{5.3}
$$

provided that $\operatorname{Re}(\alpha) > 0$, $h \in R_+$, $\operatorname{Re}(\alpha) + |\gamma| < n$, *and other conditions easily obtainable from* (4.4) *exist.*

Formula 6.

$$
\begin{aligned}
&X_{-}^{-\alpha} H_{P,Q}^{M,N}[\max\{x_1^h,\ldots,x_n^h\}]\\
&= H_{P+1,Q+1}^{M+1,N}\left[\max\{x_1^h,\ldots,x_n^h\}\,\middle|\,\begin{matrix}(a_j,A_j)_{1,P},(1-n,h)\\(1+\alpha-n,h),(b_j,B_j)_{1,Q}\end{matrix}\right],
\end{aligned}\tag{5.4}
$$

provided that $\operatorname{Re}(\alpha) > 0$, $h \in R_+$, $\operatorname{Re}(\alpha) + |\gamma| > n-1$, *and other conditions easily obtainable from* (4.5) *exist.*

Remark 4. In view of the relations

$$
X_{+}^{-r} f(x) = \prod_{j=1}^{r}\left(n-j+x_1\frac{\partial}{\partial x_1}+\cdots+x_n\frac{\partial}{\partial x_n}\right) f(x) \tag{5.5}
$$

and

$$
X_{-}^{-r} f(x) = (-1)^r \prod_{j=1}^{r}\left(n-j+x_1\frac{\partial}{\partial x_1}+\cdots+x_n\frac{\partial}{\partial x_n}\right) f(x), \tag{5.6}
$$

where r is a positive integer, and which are easy consequences of (5.1) and (5.2), respectively (see [17]), one can connect the formulas involving the H-functions from (5.3) and (5.4) with the ordinary derivative operators.

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Changing the variable in convolution of distributions

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Abstract. In this paper the author has extended the concept of changing the variables in distributions to the convolution of distributions. For an infinitely differentiable function $h(x)$, he has first defined the convolution of two distributions $f(h(x))$ and $g(h(x))$ and then proved some of its properties. Finally, he has applied his results to the fractional integral and fractional differential operators.

Keywords. Distributions; changing the variables; convolution; infinitely differentiable functions; fractional; integral; operators.

1. Introduction

Jones [5] and Fisher [3] have given the concept of changing the variable in distributions. For an infinitely differentiable function $h(x)$, Fisher has defined the distribution $F(h(x))$ in the following manner.

Let F be a distribution and h be an infinitely differentiable function. Then the distribution $F(h(x))$ exists and is equal to $f(x)$ on the open interval (a, b), if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(h(x))\phi(x)\mathrm{d}x = \langle f(x), \phi(x)\rangle \tag{1.1}$$

for all test functions $\phi\in K$ (see [4, p. 2)] with compact support contained in (a, b). In (1.1), $\{F_n\}$ is a regular sequence converging to F and is defined by

$$F_n(x) = F * \delta_n = \int_{-\infty}^{\infty} F(x-t)\delta_n(t)\mathrm{d}t, \tag{1.2}$$

where $\delta_n(x) = n\rho(nx)$, $n = 1, 2, 3, \ldots$ and ρ is a fixed infinitely differentiable function having the following properties:

(i) $\rho(x) = 0 \quad \text{for } |x| \geqslant 1.$
(ii) $\rho(x) \geqslant 0.$
(iii) $\rho(x) = \rho(-x).$
(iv) $\int_{-\infty}^{\infty} \rho(x)\mathrm{d}x = 1.$

He has also proved that for $\phi\in K$,

$$\langle F(h(x)), \phi(x)\rangle = \langle F(x), \phi(p(x))|p'(x)|\rangle, \tag{1.3}$$

where $p(x)$ stands for $h^{-1}(x)$.

Since we know that the convolution of two distributions is a distribution, the above study made by Jones and Fisher encourages to see the effect of changing the variable in the convolution of distributions.

In this paper we have defined the convolution of two distributions and studied a few of its properties. We have also applied our results to the fractional integral and differential operators.

2. Convolution of generalized functions and its properties

This section is devoted in defining the convolution of two generalized functions $f(h(x))$ and $f(g(x))$ and studying a few of its properties.

DEFINITION 2.1

Let $h(x)$ be an infinitely differentiable function and $f(h(x))$ and $g(h(x))$ be two generalized functions. The convolution of $f(h(x))$ and $g(h(x))$ is defined as

$$\begin{aligned} f(h(x))*g(h(x)) &= (f*g)h(x) \\ &= \int_{-\infty}^{\infty} f(h(y))g(h(x)-h(y))h'(y)\mathrm{d}y. \end{aligned} \tag{2.1}$$

For $\phi \in K$, we have from (1.3)

$$\begin{aligned} \langle (f*g)h(x)), \phi(x)\rangle &= \langle (f*g)(x), \phi(p(x))|p'(x)|\rangle \\ &= \langle (f*g)(x), \Psi(x)\rangle. \end{aligned} \tag{2.2}$$

Obviously $\Psi(x) = \phi(p(x))|p'(x)| \in K$ with compact support in (a, b). Following Kanwal [6, p. 179], we can write (2.2) as

$$\langle (f*g)(h(x)), \phi(x)\rangle = \langle f(x), \langle g(y), \Psi(x+y)\rangle\rangle. \tag{2.3}$$

We also define the convolution of $f(x)$ and $g(h(x))$ as

$$f(x)*g(h(x)) = \int_{-\infty}^{\infty} f(y)g(h(x)-h(y))h'(y)\mathrm{d}y \tag{2.4}$$

and for $\phi \in K$, we define it as

$$\langle f(x)*g(h(x)), \phi(x)\rangle = \langle f(x), h'(x)\langle g(y), \Psi(h(x)+y)\rangle\rangle. \tag{2.5}$$

Equations (2.3) and (2.5) are meaningful only if (i) f or g has bounded support or (ii) the support of f and g are bounded on the same side.

Now we will study some of the properties of convolution of two generalized functions.

(1) The commutative law holds i.e.

$$f(h(x))*g(h(x)) = g(h(x))*f(h(x)).$$

But

$$f(x)*g(h(x)) \neq g(h(x))*f(x).$$

(2) The associative law holds i.e.

$$(f(h(x))*g(h(x)))*k(h(x)) = f(h(x))*(g(h(x))*k(h(x)))$$

and also

$$(f(x)*g(h(x)))*k(h(x)) = f(x)*(g(h(x))*k(h(x)))$$

provided the support of two of the three distributions are bounded or the support of all the three distributions are bounded on the same side.

(3) Also

$$\left(f*\frac{\mathrm{d}}{\mathrm{d}x}g\right)(h(x))=\left(\frac{\mathrm{d}}{\mathrm{d}x}(f*g)\right)(h(x))=\left(\frac{\mathrm{d}}{\mathrm{d}x}f*g\right)(h(x)). \tag{2.6}$$

For $\phi\in K$, we have

$$\begin{aligned}\left\langle\left(\frac{\mathrm{d}}{\mathrm{d}x}(f*g)\right)h(x),\phi(x)\right\rangle&=\left\langle\frac{\mathrm{d}}{\mathrm{d}x}(f*g)(x),\Psi(x)\right\rangle\quad(\text{by (2.2)}).\\&=-\langle(f*g)(x),\Psi'(x)\rangle\\&=-\langle f(x),\langle g(y),\Psi'(x+y)\rangle\rangle\\&=\langle f(x),\langle g'(y),\Psi(x+y)\rangle\rangle\\&=\left\langle\left(f*\frac{\mathrm{d}}{\mathrm{d}x}g\right)(h(x)),\phi(x)\right\rangle.\end{aligned}$$

Similarly we can show that

$$\left\langle\left(\frac{\mathrm{d}}{\mathrm{d}x}(f*g)\right)(h(x)),\phi(x)\right\rangle=\left\langle\left(\frac{\mathrm{d}}{\mathrm{d}x}f*g\right)(h(x)),\phi(x)\right\rangle.$$

(4) Lastly, let us consider the Dirac-delta distribution of $h(x)$, which is defined as (see [2, p. 74])

$$\begin{aligned}\langle\delta(h(x)),\phi(x)\rangle&=\langle\delta(x),\phi(p(x))|p'(x)|\rangle\\&=\phi(p(0))|p'(0)|,\end{aligned}$$

and its convolution with $f(h(x))$ is given as

$$\begin{aligned}\langle f(h(x))*\delta(h(x)),\phi(x)\rangle&=\langle f(x),\langle\delta(y),\Psi(x+y)\rangle\rangle\quad(\text{by (2.2) and (2.3)})\\&=\langle f(x),\Psi(x)\rangle\\&=\langle f(h(x)),\phi(x)\rangle.\end{aligned}$$

Similarly we can show that

$$\langle\delta(h(x))*f(h(x)),\phi(x)\rangle=\langle f(h(x)),\phi(x)\rangle.$$

This implies that

$$(f*\delta)(h(x))=f(h(x))=(\delta*f)(h(x)). \tag{2.7}$$

The convolution of $\delta(x)$ with $f(h(x))$ is defined by

$$\begin{aligned}\langle\delta(x)*f(h(x)),\phi(x)\rangle&=\langle\delta(x),h'(x)\langle f(y),\Psi(h(x)+y)\rangle\rangle\quad(\text{by (2.5)})\\&=h'(0)\langle f(y),\Psi(y)\rangle\\&\qquad(\text{provided } h(0)=0 \text{ but } h'(0)\neq 0)\\&=\langle h'(0)f(h(x)),\phi(x)\rangle.\end{aligned}$$

i.e.

$$\delta(x)*f(h(x))=h'(0)f(h(x)). \tag{2.8}$$

By (2.6) and (2.7), we can easily see that

$$\left(f*\frac{d}{dx}\delta\right)h(x)=\left(\frac{d}{dx}f*\delta\right)h(x)=\left(\frac{d}{dx}f\right)h(x), \tag{2.9}$$

and even for higher derivatives

$$\left(f*\frac{d^k}{dx^k}\delta\right)h(x)=\left(\frac{d^k}{dx^k}f*\delta\right)h(x)=\left(\frac{d^k}{dx^k}f\right)h(x). \tag{2.10}$$

If $L=\Sigma_{|k|\leqslant m}a_kD^k$ is a linear differential operator, then

$$(f*L\delta)h(x)=(Lf*\delta)h(x)=(Lf)h(x), \tag{2.11}$$

which shows that any linear differential operator with constant coefficient can be represented as a convolution.

3. Application

In this section we have tried to apply our results to fractional integral and differential operators.

McBride [7, p. 38] has given the following form of the Riemann–Liouville fractional integral operator

$$I^{\alpha}_{h(x)}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(h(x)-h(t))^{\alpha-1}f(t)h'(t)dt. \tag{3.1}$$

By making use of (2.4), we can write (3.1) as

$$I^{\alpha}_{h(x)}f(x)=f(x)*\phi_\alpha(h(x)), \tag{3.2}$$

where

$$\phi_\alpha(h(x))=\frac{h_+^{\alpha-1}(x)}{\Gamma(\alpha)} \tag{3.3}$$

and

$$h_+(x)=\begin{cases}h(x) & x>0\\ 0, & x<0.\end{cases} \tag{3.4}$$

It is well-known, when α is replaced by $-\alpha$, then the operator $I^{-\alpha}_{h(x)}f(x)$ represents the fractional differential operator i.e.

$$I^{-\alpha}_{h(x)}f(x)=D^{\alpha}_{h(x)}f(x)=f(x)*\phi_{-\alpha}(h(x)), \tag{3.5}$$

where $D=(d/dx)$. This operator was also the subject of study by the author [1].

We will now prove the following results:

Result (I). If $h(0)=0$ then

$$\phi_\alpha(h(x))*\phi_\beta(h(x))=\phi_{\alpha+\beta}(h(x)). \tag{3.6}$$

Proof. By (2.1) we have

$$\begin{aligned}\phi_\alpha(h(x))*\phi_\beta(h(x))&=\int_{-\infty}^{\infty}\phi_\alpha(h(t))\phi_\beta(h(x)-h(t))h'(t)dt.\\&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x h^{\alpha-1}(t)\{h(x)-h(t)\}^{\beta-1}h'(t)dt.\end{aligned}$$

$$= \frac{h^{\alpha+\beta-1}(x)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} \mathrm{d}y.$$

$$= \frac{h_+^{\alpha+\beta-1}(x)}{\Gamma(\alpha+\beta)}.$$

$$= \phi_{\alpha+\beta}(h(x)).$$

Result (II). If $h(0)=0$ and $h'(0)\neq 0$, then

$$I_{h(x)}^{-\alpha}\delta(x) = h'(0)\phi_{-\alpha}(h(x)), \tag{3.7}$$

and

$$I_{h(x)}^{-\alpha}f(x) = \frac{1}{h'(0)}\{f(x) * I_{h(x)}^{-\alpha}\delta(x)\}. \tag{3.8}$$

Proof. Equation (3.7) is obtained directly from (2.8) and (3.5). For (3.8) we have from (3.7)

$$\phi_{-\alpha}(h(x)) = \frac{1}{h'(0)} I_{h(x)}^{-\alpha}\delta(x),$$

which on substituting in (3.5) gives the desired result.

It may be noted that (3.7) reduces to the result obtained by Kanwal [6, p. 96 eq. (4.4.50)], when $h(x)=x$ and α is taken to be an integer.

Result (III). Consider the fractional differential equation

$$I_{h(x)}^{-\alpha}f(x) = D_{h(x)}^{\alpha}f(x) = g(x). \tag{3.9}$$

By (3.5) we can write (3.9) as

$$g(x) = D_{h(x)}^{\alpha}f(x) = f(x) * \phi_{-\alpha}(h(x)),$$

which implies that

$$\begin{aligned} g(x) * \phi_\alpha(h(x)) &= (f(x) * \phi_{-\alpha}(h(x))) * \phi_\alpha(h(x)) \\ &= f(x) * (\phi_{-\alpha}(h(x)) * \phi_\alpha(h(x))) \quad \text{(by Associative law)} \\ &= f(x) * \phi_{-\alpha+\alpha}(h(x)) \quad \text{(by (3.6))} \\ &= f(x). \end{aligned}$$

This shows that

$$f(x) = g(x) * \phi_\alpha(h(x))$$

is the only solution of the fractional differential equation (3.9).

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Linearized oscillations for higher order neutral differential equations

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Abstract. In this paper, the oscillation of certain nonlinear neutral differential equations has been studied with the help of the oscillation of associated linear neutral differential equations.

Keywords. Linearization; oscillation; neutral differential equations.

1. Introduction

In the literature we find two types of definitions for an oscillatory function, viz.,

DEFINITION 1 (D_1)

A function $y \in C([t_0, \infty), R)$, $t_0 \in R$, is said to be oscillatory if there exists a sequence $\langle t_m \rangle \subset [t_0, \infty)$ such that $t_m \to \infty$ as $m \to \infty$ and $y(t_m) = 0$ for $m = 1, 2, \ldots$.

DEFINITION 2 (D_2)

A function $y \in C^1([t_0, \infty), R)$, $t_0 \in R$, is said to be oscillatory if there exists a sequence $\langle t_m \rangle \subset [t_0, \infty)$ such that $\lim_{m \to \infty} t_m = \infty$ and $y(t_m)y'(t_m) = 0$ for each m.

We frequently encounter the first definition and its various forms in oscillation theory of differential equations. However, we find the use of the second definition in the studies of Gopalsamy [2] and Ruan [8]. It may be noted that if a continuously differentiable function is oscillatory in the sense of D_1, then it is oscillatory in the sense of D_2 but the converse is not necessarily true. For example, we may take $y(t) = 2 + \sin t$. Further, every continuously differentiable periodic function is also oscillatory in the sense of D_2. People working on mathematical problems related to biology or ecology consider the oscillation of a function about some line $y = k$, $k \in R^+$ (not necessarily $y = 0$) (see [3, 4]). We may note that $y(t) = 2 + \sin t$ oscillates in the sense of D_1 about the line $y = 2$. However, there are functions which do not oscillate in the sense of D_1 about any line but oscillates in the sense of D_2. For example, $y(t) = (t/2) + \sin t$, $t > 4$, is such a function. If we plot a smooth curve of the price of a commodity against time, then usually we get such a function. The price of the commodity fluctuates as the time evolves but it seldom oscillates about a fixed price line. Such a consideration makes definition 2 more realistic than definition 1. In the present work we follow the second definition. As in the oscillation theory of differential equations we often deal with continuously differentiable functions, the difference of two spaces in these two definitions does not matter.

There exist several examples (see [5]) where the study of appropriate linear equations predicts the dynamics of nonlinear ones. In stability theory of differential and difference equations, linearization technique plays an important role. By linearized oscillations we mean that certain nonlinear differential equations, under appropriate

hypotheses, have the same oscillatory behaviour as associated linear equations [6, 7]. In § 2 we consider this problem for equations of the form

$$[x(t)-p(t)g(x(t-\tau))]^{(n)}+q(t)h(x(t-\sigma))=0, \tag{1}$$

where p and $q\in C([t_0,\infty),R^+)$, $t_0\in R$ and $R^+=[0,\infty)$, g and $h\in C(R,R)$, $\tau\in(0,\infty)$ and $\sigma\in(0,\infty)$ such that

$$\limsup_{t\to\infty} p(t)=p_0\in[1,\infty) \quad \text{and} \quad \lim_{t\to\infty} q(t)=q_0\in(0,\infty), \tag{2}$$

$$\frac{g(u)}{u}\geqslant 1 \quad \text{for } u\neq 0 \quad \text{and} \quad \lim_{|u|\to\infty}\frac{g(u)}{u}=1 \tag{3}$$

and

$$uh(u)>0 \quad \text{for } u\neq 0, \quad |h(u)|\geqslant h_0>0 \quad \text{for } |u|\geqslant \lambda_0>0, \tag{4}$$

and

$$\lim_{|u|\to\infty}\frac{h(u)}{u}=1.$$

The linear equation associated with (1) is given by

$$[y(t)-p_0y(t-\tau)]^{(n)}+q_0y(t-\sigma)=0. \tag{5}$$

This equation is called the limiting equation. The oscillatory behaviour of solutions of (5) predicts the oscillatory behaviour of solutions of (1). We may note that (1) is highly nonlinear and hence it is difficult to predict the behaviour of its solutions.

Suppose $T=\max\{\tau,\sigma\}$. By a solution of (1) on $[t_1,\infty)$, for some $t_1\geqslant t_0$, we mean a function $x\in C([t_1-T,\infty),R)\cap C^1([t_1,\infty),R)$ such that $[x(t)-p(t)g(x(t-\tau))]$ is n-times continuously differentiable on $[t_1,\infty)$ and (1) is satisfied for $t\geqslant t_1$. We consider those solutions of (1) which are not identically equal to zero together with their first derivatives in any neighbourhood of infinity. A solution of (1) is said to be oscillatory if it is oscillatory in the sense of D_2. Equation (1) is said to be oscillatory if each of its solution oscillates.

2. Linearized oscillations

We need the following lemmas for the proof of our main result.

Lemma 1. *Equation* (5) *is oscillatory if and only if its characteristic equation*

$$F(\lambda)\equiv\lambda^n[1-p_0e^{-\lambda\tau}]+q_0e^{-\lambda\sigma}=0 \tag{6}$$

has no real root.

Proof. Suppose that the characteristic equation (6) of (5) has no real root. Then every solution of (5) oscillates in the sense of D_1 (see [1]) and hence in the sense of D_2. On the other hand, if (6) has a real root λ (note that $\lambda=0$ is not a root of (6) because $q_0>0$), then $\lambda\neq 0$ and $y(t)=e^{\lambda t}$ is a non-oscillatory solution of (5). Hence the lemma is proved.

Lemma 2. *Suppose that* (5) *is oscillatory. Then there exists an* $\varepsilon_0>0$ *such that for each* $\varepsilon, 0\leqslant\varepsilon\leqslant\varepsilon_0$, *every solution of*

$$[z(t)-(p_0+2\varepsilon)z(t-\tau)]^{(n)}+(q_0-\varepsilon)z(t-\sigma)=0 \tag{7}$$

oscillates if n is odd or even and $\sigma>\tau$.

Proof. Since (5) is oscillatory, then from Lemma 1, it follows that (6) has no real root. As $F(0)=q_0>0$ and $F(\lambda)$ is a continuous function of λ, we have $F(\lambda)>0$ for all $\lambda\in R$. Moreover, $F(\lambda)\to\infty$ as $\lambda\to\infty$.

Let $0<\varepsilon^*<\frac{1}{3}q_0$. Set, for $\lambda\in R$,

$$G^*(\lambda)=2\varepsilon^*\lambda^n e^{-\lambda\tau}+\varepsilon^* e^{-\lambda\sigma}.$$

Thus

$$F(\lambda)-G^*(\lambda)=\lambda^n\{1-(p_0+2\varepsilon^*)e^{-\lambda\tau}\}+(q_0-\varepsilon^*)e^{-\lambda\sigma}$$

and hence $F(\lambda)-G^*(\lambda)\to\infty$ as $\lambda\to\infty$. It is possible to find $\lambda_0>0$ such that $\lambda\geqslant\lambda_0$ implies that $F(\lambda)-G^*(\lambda)>(m/2)$, where $m=\inf_{\lambda\geqslant 0}F(\lambda)>0$. Further, writing

$$F(\lambda)-G^*(\lambda)=\lambda^n e^{-\lambda\tau}\left[e^{\lambda\tau}-(p_0+2\varepsilon^*)+(q_0-\varepsilon^*)\frac{e^{-\lambda(\sigma-\tau)}}{\lambda^n}\right]$$

and noting that

$$\lim_{\lambda\to-\infty}\frac{e^{-\lambda(\sigma-\tau)}}{\lambda^n}=\infty$$

if $\sigma>\tau$ and n is even, we have $F(\lambda)-G^*(\lambda)\to\infty$ as $\lambda\to-\infty$. Since $F(\lambda)\to\infty$ as $\lambda\to-\infty$ when n is even and $\sigma>\tau$, then $\min_{\lambda\leqslant 0}F(\lambda)=m_1>0$. Consequently, there exists $\lambda_1>0$ such that $F(\lambda)-G^*(\lambda)>(m_1/2)$ for $\lambda\leqslant-\lambda_1$ when n is even and $\sigma>\tau$.

Suppose that

$$\varepsilon_0=\min\left\{\varepsilon^*,\frac{m}{2(2\lambda_0^n+1)},\frac{m_1}{2(2\lambda_1^n e^{\lambda_1\tau}+e^{\lambda_1\sigma})}\right\}.$$

Let $0<\varepsilon\leqslant\varepsilon_0$. The characteristic equation of (7) is given by

$$G_\varepsilon(\lambda)\equiv\lambda^n\{1-(p_0+2\varepsilon)e^{-\lambda\tau}\}+(q_0-\varepsilon)e^{-\lambda\sigma}=0. \tag{8}$$

In view of Lemma 1, to complete the proof of the lemma, it is sufficient to show that (8) does not admit a real solution. Clearly, $G_\varepsilon(0)=(q_0-\varepsilon)>0$ and, for $\lambda>0$,

$$\begin{aligned}G_\varepsilon(\lambda)&=F(\lambda)-2\varepsilon\lambda^n e^{-\lambda\tau}-\varepsilon e^{-\lambda\sigma}\\&\geqslant F(\lambda)-G^*(\lambda).\end{aligned}$$

If $\lambda\geqslant\lambda_0$, then $G_\varepsilon(\lambda)>(m/2)$. For $\lambda\in(0,\lambda_0)$,

$$\begin{aligned}G_\varepsilon(\lambda)&=F(\lambda)-\varepsilon(2\lambda^n e^{-\lambda\tau}+e^{-\lambda\sigma})\\&\geqslant m-\varepsilon(2\lambda_0^n+1)\\&>\frac{m}{2}.\end{aligned}$$

Thus $G_\varepsilon(\lambda)>0$ for $\lambda\geqslant 0$. Suppose that n is odd. We claim that $G_\varepsilon(\lambda)>0$ for $\lambda<0$. If not, there exists a $\lambda^*<0$ such that $G_\varepsilon(\lambda^*)=0$. This leads to the contradiction

$$1>e^{\lambda^*\tau}>p_0+2\varepsilon>p_0\geqslant 1.$$

Hence our claim holds. Next let n be even and $\sigma > \tau$. For $\lambda \in (-\infty, -\lambda_1]$, $G_\varepsilon(\lambda) > F(\lambda) - G^*(\lambda) > (m_1/2)$ and for $\lambda \in (-\lambda_1, 0)$,

$$G_\varepsilon(\lambda) > F(\lambda) - \varepsilon(2\lambda_1^n e^{\lambda_1 \tau} + e^{\lambda_1 \sigma}) \geqslant m_1 - \frac{m_1}{2} = \frac{m_1}{2}.$$

Thus $G_\varepsilon(\lambda) > 0$ for $\lambda < 0$ in the case n is even and $\sigma > \tau$.

This completes the proof of the lemma.

Theorem 3. *Suppose that* (5) *is oscillatory. Then every solution of* (1) *oscillates or tends to zero as* $t \to \infty$ *if* n *is odd and oscillates or tends to zero as* $t \to \infty$ *if* n *is even and* $\sigma > \tau$.

Proof. Let $x(t)$ be a non-oscillatory solution of (1) on $[t_1, \infty)$ for some $t_1 \geqslant t_0$. Let $x(t) > 0$ for $t \geqslant t_2 > t_1$. The case when $x(t) < 0$ for $t \geqslant t_2$ may similarly be dealt with. Hence from the definition D_2 it follows that $x'(t) > 0$ or < 0 for large t. If $x'(t) < 0$ eventually, then $\lim_{t\to\infty} x(t) = \lambda$, $0 \leqslant \lambda < \infty$. If $\lambda = 0$, then there is nothing to prove. On the other hand, $\lambda > 0$ implies that $x(t) > K > 0$ for large t. If $x'(t) > 0$ eventually, then there exists a $K > 0$ such that $x(t) > K$ for large t. Hence in either case $h(x(t)) \geqslant h_0$ for $t \geqslant t_3 > t_2$. Let $\sigma_0 = \max\{\sigma, \tau\}$. Setting, for $t \geqslant t_3 + \sigma_0$,

$$z(t) = x(t) - p(t)g(x(t-\tau)) \tag{9}$$

we obtain

$$z^{(n)}(t) = -q(t)h(x(t-\sigma)) < 0. \tag{10}$$

Thus $z^{(n-1)}(t) > 0$ or < 0 for $t \geqslant t_4 > t_3$. If $z^{(n-1)}(t) > 0$ for $t \geqslant t_4$, then integrating (10) yields

$$h_0 \int_{t_4}^{t} q(s)\mathrm{d}s \leqslant \int_{t_4}^{t} q(s)h(x(s-\sigma))\mathrm{d}s < z^{(n-1)}(t_4),$$

that is, $\int_{t_4}^{\infty} q(t)\mathrm{d}t < \infty$.

This contradicts the fact that

$$\int_{t_4}^{\infty} q(t)\mathrm{d}t = \infty,$$

because $q_0 > 0$. Hence $z^{(n-1)}(t) < 0$ for $t \geqslant t_4$. Consequently, $z^{(i)}(t) < 0$ for $i = 0, 1, \ldots, (n-2)$ and $t \geqslant t_5 > t_4$ and $\lim_{t\to\infty} z(t) = -\infty$ for $n \geqslant 2$. If $n = 1$, then $z'(t) < 0$ and $z(t) < 0$ for $t \geqslant t_5$. If $\lim_{t\to\infty} z(t) = \mu$, $-\infty < \mu < 0$, then integrating (10) for $n = 1$ as above, we arrive at a contradiction. Thus $\lim_{t\to\infty} z(t) = -\infty$ for $n \geqslant 1$.

We claim that $\lim_{t\to\infty} \inf x(t) = \infty$. If not, $\lim_{t\to\infty} \inf x(t) = \alpha$, $0 \leqslant \alpha < \infty$. Hence there exists a sequence $\langle t_m \rangle$ such that $t_m \to \infty$ as $m \to \infty$ and $x(t_m) \to \alpha$ as $m \to \infty$. From (9) we get for $t_m \geqslant t_5$,

$$-z(t_m + \tau) < p(t_m + \tau)g(x(t_m)).$$

Taking lim sup as $m \to \infty$, we have

$$\begin{aligned}
\infty &\leqslant \limsup_{m\to\infty} [p(t_m + \tau)g(x(t_m))] \\
&\leqslant \limsup_{m\to\infty} p(t_m + \tau) \lim_{m\to\infty} g(x(t_m)) \\
&\leqslant p_0 g(\alpha) < \infty,
\end{aligned}$$

a contradiction. Thus our claim holds. Hence $\lim_{t\to\infty} x(t) = \infty$.

Equation (1) may be written as, for $t \geqslant t_5$

$$[x(t)-P(t)x(t-\tau)]^{(n)}+Q(t)x(t-\sigma)=0,$$

that is

$$z^{(n)}(t)+Q(t)x(t-\sigma)=0, \tag{11}$$

where

$$P(t)=\frac{p(t)g(x(t-\tau))}{x(t-\tau)} \text{ and } Q(t)=\frac{q(t)h(x(t-\sigma))}{x(t-\sigma)}.$$

Thus, by (3),

$$\limsup_{t\to\infty} P(t) \leqslant \limsup_{t\to\infty} p(t) \lim_{t\to\infty} \frac{g(x(t-\tau))}{x(t-\tau)} = p_0$$

and

$$\limsup_{t\to\infty} P(t) \geqslant \limsup_{t\to\infty} p(t) = p_0.$$

Hence $\lim_{t\to\infty} \sup P(t)=p_0$. Further, by (4), $\lim_{t\to\infty} Q(t)=q_0$. Integrating (11) n-times from t_5 to t ($t_5<t$), we obtain

$$z(t)+\frac{1}{(n-1)!}\int_{t_5}^{t}(t-s)^{n-1}Q(s)x(s-\sigma)\mathrm{d}s<0,$$

that is

$$x(t)-P(t)x(t-\tau)+\frac{1}{(n-1)!}\int_{t_5}^{t}(t-s)^{n-1}Q(s)x(s-\sigma)\mathrm{d}s<0,$$

$$x(t)>\frac{1}{P(t+\tau)}\left[x(t+\tau)+\frac{1}{(n-1)!}\int_{t_5}^{t+\tau}(t+\tau-s)^{n-1}Q(s)x(s-\sigma)\mathrm{d}s\right]. \tag{12}$$

From Lemma 2, it follows that there exists an $\varepsilon_0>0$ such that for each $\varepsilon, 0 \leqslant \varepsilon \leqslant \varepsilon_0$, every solution of (7) oscillates if n is odd or even and $\sigma>\tau$. In what follows we show that (7) admits a positive solution leading to a contradiction which completes the proof of the theorem. Clearly, $P(t)<p_0+\varepsilon$ for $t \geqslant T_1$ and $Q(t)>q_0-\varepsilon$ for $t \geqslant T_2$. Let $T_0=\max\{t_5, T_1, T_2\}$. Choose

$$1<\beta<1+\frac{\varepsilon}{p_0+\varepsilon}.$$

Thus, for $t \geqslant T_0$, $P(t)<(p_0+2\varepsilon)/\beta$ and (12) yields

$$\begin{aligned} x(t) &> \frac{\beta}{p_0+2\varepsilon}\left[x(t+\tau)+\frac{1}{(n-1)!}\int_{T_0}^{t+\tau}(t+\tau-s)^{n-1}Q(s)x(s-\sigma)\mathrm{d}s\right] \\ &> \frac{\beta}{p_0+2\varepsilon}\left[x(t+\tau)+\frac{(q_0-\varepsilon)}{(n-1)!}\int_{T_0}^{t+\tau}(t+\tau-s)^{n-1}x(s-\sigma)\mathrm{d}s\right]. \end{aligned} \tag{13}$$

Let $X=BC([T_0-\sigma-\tau,\infty), R)$, the Banach space of real-valued bounded continuous functions on $[T_0-\sigma-\tau,\infty)$ with usual 'sup' norm. Let

$$B=\{y\in X: 0\leqslant y(t)\leqslant 1, t\geqslant T_0-\sigma-\tau \text{ but } y(t)\not\equiv 0 \text{ on any sub-interval of } [T_0-\sigma-\tau,\infty)\}.$$

Clearly, B is a closed, bounded and convex subset of X. Define $T:B\to X$ as follows: for $y\in B$,

$$(Ty)(t)=\begin{cases}(Ty)(T_0), \ t\in[T_0-\sigma-\tau, T_0]\\ \dfrac{1}{(p_0+2\varepsilon)x(t)}\Big[x(t+\tau)y(t+\tau)\\ \quad +\dfrac{(q_0-\varepsilon)}{(n-1)!}\displaystyle\int_{T_0}^{t+\tau}(t+\tau-s)^{n-1}x(s-\sigma)y(s-\sigma)\mathrm{d}s\Big], \quad t\geqslant T_0.\end{cases}$$

Clearly, for $t\geqslant T_0-\sigma-\tau$, $Ty(t)>0$ and

$$\begin{aligned}Ty(t)&\leqslant\frac{1}{(p_0+2\varepsilon)x(t)}\left[x(t+\tau)+\frac{(q_0-\varepsilon)}{(n-1)!}\int_{T_0}^{t+\tau}(t+\tau-s)^{n-1}x(s-\sigma)\mathrm{d}s\right]\\ &<\frac{1}{\beta}<1.\end{aligned}$$

Thus $T:B\to B$. Further, for y_1 and $y_2\in B$

$$\begin{aligned}&|Ty_1(t)-Ty_2(t)|\\ &=\begin{cases}|Ty_1(T_0)-Ty_2(T_0)|, t\in[T_0-\sigma-\tau, T_0]\\ \dfrac{1}{(p+2\varepsilon)x(t)}\Big|\Big[x(t+\tau)\{y_1(t+\tau)-y_2(t+\tau)\}\\ \quad +\dfrac{(q_0-\varepsilon)}{(n-1)!}\displaystyle\int_{T_0}^{t+\tau}(t+\tau-s)^{n-1}x(s-\sigma)\{y_1(s-\sigma)-y_2(s-\sigma)\}\mathrm{d}s\Big]\Big|,\end{cases}\end{aligned}$$

for $t\geqslant T_0$. Thus, for $t\geqslant T_0-\sigma-\tau$,

$$\begin{aligned}|Ty_1(t)-Ty_2(t)|&\leqslant\frac{\|y_1-y_2\|}{(p_0+2\varepsilon)x(t)}\\ &\quad\times\left[x(t+\tau)+\frac{(q_0-\varepsilon)}{(n-1)!}\int_{T_0}^{t+\tau}(t+\tau-s)^{n-1}x(s-\sigma)\mathrm{d}s\right]\\ &\leqslant\frac{1}{\beta}.\end{aligned}$$

Consequently,

$$\|Ty_1-Ty_2\|\leqslant\frac{1}{\beta}\|y_1-y_2\|$$

which implies that T is a contraction. Hence T has a unique fixed point in B, that is, there exists $y_0\in B$ such that $Ty_0(t)=y_0(t)$, that is

$$y_0(t)=\begin{cases}y_0(T_0), t\in[T_0-\tau-\sigma, T_0],\\ \dfrac{1}{(p_0+2\varepsilon)+x(t)}\Big[x(t+\tau)y_0(t+\tau)+\dfrac{(q_0-\varepsilon)}{(n-1)!}\displaystyle\int_{T_0}^{t+\tau}(t+\tau-s)^{n-1}\\ \quad\times x(s-\sigma)y_0(s-\sigma)\mathrm{d}s\Big],\end{cases}$$

for $t \geqslant T_0$. Setting $w(t) = x(t)y_0(t)$ for $t \geqslant T_0 - \sigma - \tau$, we note that $w(t) > 0$ and

$$w(t) = \frac{1}{(p_0 + 2\varepsilon)}\left[w(t+\tau) + \frac{(q_0 - \varepsilon)}{(n-1)!}\int_{T_0}^{t+\tau}(t+\tau-s)^{n-1}w(s-\sigma)\mathrm{d}s\right]$$

for $t \geqslant T_0$. Thus

$$w(t) - (p_0 + 2\varepsilon)w(t-\tau) = -\frac{(q_0 - \varepsilon)}{(n-1)!}\int_{T_0}^{t}(t-s)^{n-1}w(s-\sigma)\mathrm{d}s.$$

Differentiating n-times we obtain

$$[w(t) - (p_0 + 2\varepsilon)w(t-\tau)]^{(n)} + (q_0 - \varepsilon)w(t-\sigma) = 0.$$

Hence $w(t)$ is a positive solution of (7), a contradiction. This completes the proof of the theorem.

The following example illustrates the theorem.

Example. Consider

$$\left[x(t) - \left(1 + \frac{1}{t}\right)x(t-1)(1 + \mathrm{e}^{-|x(t-1)|})\right]'$$
$$+\left(1 - \frac{1}{t}\right)x(t-2)(1 + \mathrm{e}^{-|x(t-2)|}) = 0, \quad t \geqslant 3. \tag{14}$$

Thus $n = 1$, $p(t) = 1 + (1/t)$, $q(t) = 1 - (1/t)$, $g(u) = h(u) = u(1 + \mathrm{e}^{-|u|})$ for $u \in R$, $\tau = 1$ and $\sigma = 2$. Clearly, g and h satisfy conditions (2) and (3) respectively. Further, $\lim_{t\to\infty} p(t) = 1$ and $\lim_{t\to\infty} q(t) = 1$ imply that the limiting equation (5) takes the form

$$[y(t) - y(t-1)]' + y(t-2) = 0 \tag{15}$$

and the characteristic equation associated with it is given by

$$F(\lambda) = \lambda(1 - \mathrm{e}^{-\lambda}) + \mathrm{e}^{-2\lambda} = 0. \tag{16}$$

Clearly, $F(\lambda) = \lambda - (\lambda/\mathrm{e}^{\lambda}) + (1/\mathrm{e}^{2\lambda}) \to +\infty$ as $\lambda \to +\infty$ and $F(\lambda) = \mathrm{e}^{-2\lambda}[1 + \lambda\mathrm{e}^{2\lambda} - \lambda\mathrm{e}^{\lambda}] \to +\infty$ as $\lambda \to -\infty$. Further, $F'(\lambda) = 1 - \mathrm{e}^{-\lambda} + \lambda\mathrm{e}^{-\lambda} - 2\mathrm{e}^{-2\lambda} < 0$ and $\lambda \leqslant 0$ implies that $F(\lambda)$ is decreasing in $(-\infty, 0]$ and $F'(\lambda) > 0$ for $\lambda \geqslant 1$ shows that $F(\lambda)$ is increasing in $[1, \infty)$. Moreover, $F(\lambda)$ is continuous and $F(\lambda) > 0$ for $0 \leqslant \lambda \leqslant 1$ imply that there exists a $\lambda_0 \in [0, 1]$ such that $0 < F(\lambda_0) \leqslant F(\lambda)$ for $\lambda \in [0, 1]$. Hence (16) does not admit a real root. From Lemma 1 it follows that (15) is oscillatory. Consequently, every solution of (14) is oscillatory or tends to zero as $t \to \infty$ by Theorem 3.

Remark. We could have taken $\tau = \sigma = 1$ in the above example.

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A note on a class of singular integro-differential equations

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Abstract. A simplified analysis is employed to handle a class of singular integro-differential equations for their solutions.

Keywords. Integro-differential equation.

1. Introduction

The problem of solving the singular integro-differential equation, as given by

$$u(x) = \frac{1}{\pi}\int_0^\infty \frac{v(s)\mathrm{d}s}{s-x} + g(x), \quad (0 < x < \infty), \tag{1.1}$$

where $u(x)$ and $v(x)$ represent two linear differential expressions of the forms

$$u(x) = \sum_{k=0}^{n} a_k f^k(x) \tag{1.2}$$

and

$$v(x) = \sum_{k=0}^{n} b_k f^k(x), \tag{1.3}$$

in which a_k's and b_k's are complex constants, $f^k(x)$ represents the kth derivative of the unknown function $f(x)$ with prescribed initial values $f^k(0)$, along with the conditions that $f^k(\infty) = 0$, for $k = 0, 1, 2, \ldots, n$, and $g(x)$ represents a known differentiable function arises in a natural way while solving a class of mixed boundary value problems of mathematical physics (see [3, 5, 6]).

Varley and Walker [7] have discussed a general method of solving (1.1) by converting it to a singular integral equation of second kind with a Cauchy Kernel, and have discussed a method of solution of the resulting singular integral equation which avoids the complication (see [4]) of calculating various singular integrals appearing in the final form of the solution.

In the present investigation, we have employed a straightforward analysis which simplifies the work of Varley and Walker [7] slightly and demonstrates clearly the underlying difficulties.

A detailed procedure is explained to examine (1.1) for its solution under the assumptions that

$$v(x) \approx O(\mathbf{x}^\alpha), \quad \text{as } \mathbf{x} \to 0 \tag{1.4}$$

and

$$v(x) \approx O(\mathbf{x}^\beta), \quad \text{as } \mathbf{x} \to \infty \tag{1.5}$$

with $-1 < \mathrm{real}(\beta) < 0$ and $\mathrm{real}(\alpha) > -1$.

2. The detailed method of solution

With the aid of the Laplace transform $V(p)$ of the function $v(x)$ as defined by

$$V(p)=\int_0^\infty v(x)\exp(-px)\mathrm{d}x, \quad (p>0) \tag{2.1}$$

equation (1.1) can be shown to be equivalent to the following singular integral equation (as in the work of Varley and Walker (1989))

$$U(p)+\frac{1}{\pi}\int_0^\infty \frac{V(q)\mathrm{d}q}{(q-p)}=G(p), \tag{2.2}$$

where, $V(p)$, $U(p)$ and $G(p)$ are the Laplace transforms of the functions $v(x)$, $u(x)$ and $g(x)$ respectively.

The expressions for $V(p)$ and $U(p)$ are given by

$$U(p)=A(p)F(p)-A_1(p), \quad V(p)=B(p)F(P)-B_1(p), \tag{2.3}$$

with

$$A(p)=a_np^n+a_{n-1}p^{n-1}+\cdots+a_1+a_0$$

$$A_1(p)=(a_np^{n-1}+a_{n-1}p^{n-2}+\cdots+a_1)f_0$$
$$+(a_np^{n-2}+a_{n-1}p^{n-3}+\cdots+a_2)f_1+\cdots+a_nf_{n-1},$$

$$B(p)=b_np^n+b_{n-1}p^{n-1}+\cdots+b_1+b_0$$

and

$$B_1(p)=(b_np^{n-1}+b_{n-1}p^{n-2}+\cdots+b_1)f_0$$
$$+(b_np^{n-2}+b_{n-1}p^{n-3}+\cdots+b_2)f_1+\cdots b_nf_{n-1},$$

where $f_k=f^k(0)$ for $k=0,1,2,\ldots,n$.

Using the relation (2.3), equation (2.2) can be cast into the form

$$A(p)\,V(p)+\frac{B(p)}{\pi}\int_0^\infty \frac{V(q)\mathrm{d}q}{(q-p)}=C(p), \quad p>0 \tag{2.4}$$

where

$$C(p)=A_1(p)B(p)-B_1(p)A(p)+G(p)B(p).$$

By the Abelian theorem on Laplace transforms (see [1]) and using the relations (1.4) and (1.5), we obtain

$$V(p)\approx O\left(\frac{1}{p^{1+\alpha}}\right), \quad \text{as } p\to\infty \tag{2.5}$$

and

$$V(p)\approx O\left(\frac{1}{p^{1+\beta}}\right), \quad \text{as } p\to 0. \tag{2.6}$$

The above behaviours of $V(p)$ assure that the integral in (2.4) exists and is finite.

We then discuss, in detail, the method of determination of the function $V(p)$.

We consider the general case of (1.1) for which the pair (a_0, b_0) has the property that either a_0 or b_0 or both of them are non-zero; the same is assumed to hold good for the pair (a_n, b_n) also.

Defining

$$\psi(z)=\frac{1}{2\pi i}\int_0^\infty \frac{V(t)\mathrm{d}t}{t-z} \quad (z=p+ip', i^2=-1) \tag{2.7}$$

and using the Plemelj's formulae (see [2]) from (2.7), we obtain that

$$\psi_+(p)-\psi_-(p)=V(p) \tag{2.8}$$

and

$$\psi_+(p)+\psi_-(p)=\frac{1}{i\pi}\int_0^\infty \frac{V(t)\mathrm{d}t}{(t-p)}, \tag{2.9}$$

where $\psi_+(p)$, $\psi_-(p)$ are the limiting values of $\psi(z)$ as $z=p+ip'$ approaches a point p of the positive real axis, from above and from below respectively.

From (2.4) and the relations (2.8) and (2.9) we then obtain that

$$\frac{A(p)+iB(p)}{A(p)-iB(p)}\psi_+(p)-\psi_-(p)=\frac{C(p)}{A(p)-iB(p)}, \quad p\geqslant 0, \tag{2.10}$$

which represents a Riemann Hilbert problem for the determination of $\psi(z)$, which can be solved as described below.

We first construct a sectionally analytic function $n(z)$ in the z-plane cut along the positive real axis $(0,\infty)$ satisfying the relation

$$\frac{n_+(p)}{n_-(p)}=\frac{A(p)+iB(p)}{A(p)-iB(p)}, \quad p\in(0,\infty), \tag{2.11}$$

where $n_+(p)$, $n_-(p)$ are the limiting values of $n(z)$ on the two sides of the positive real axis.

Assuming that $A(p)+iB(p)$ possesses zeros at the points $\mu_1,\mu_2,\ldots,\mu_n$ and $A(p)-iB(p)$ possesses zeros at different points $\lambda_1,\lambda_2,\ldots,\lambda_n$ none of which lie on the positive real axis and setting $n(z)=z^\delta m(z)$, with $\delta=(1/2\pi i)\ln((a_n+ib_n)/(a_n-ib_n))$, we can recast the problem (2.11) into the form

$$\frac{m_+(p)}{m_-(p)}=\frac{(p-\mu_1)(p-\mu_2)\ldots(p-\mu_n)}{(p-\lambda_1)(p-\lambda_2)\ldots(p-\lambda_n)}. \tag{2.12}$$

The solution of the problem posed by relation (2.12) can be easily written down as (see [2])

$$\ln m(z)=\frac{1}{2\pi i}\int_0^\infty \frac{\ln\dfrac{(t-\mu_1)(t-\mu_2)\ldots(t-\mu_n)}{(t-\lambda_1)(t-\lambda_2)\ldots(t-\lambda_n)}}{t-z}\mathrm{d}t+\ln E_1(z) \tag{2.13}$$

where $E_1(z)$ is an entire function, and, using Plemelj's formulae we obtain that

$$\ln\frac{E_1(p)}{m_+(p)}=\sum_{j=1}^{n}\left\{\frac{1}{2}\ln\left(\frac{p-\lambda_j}{p-\mu_j}\right)+\frac{1}{2\pi i}\int_0^\infty \frac{\ln\left(\dfrac{t-\lambda_j}{t-\mu_j}\right)}{(t-p)}\mathrm{d}t\right\}. \tag{2.14}$$

After simplification this gives

$$\frac{E_1(p)}{m_+(p)}=\prod_{j=1}^{n} p^{\gamma_j}(p-\lambda_j)\, V_j(p), \tag{2.15}$$

where

$$V_j(p)=\left(\frac{p-\lambda_j}{p-\mu_j}\right)^{\theta_{2j}/2\pi}(p-\lambda_j)^{-(1+\gamma_j)}C_j\exp\left(-\frac{(\lambda_j-\mu_j)}{2\pi i}\int_0^p\frac{\ln(t/|\mu_j|)\mathrm{d}t}{(t-\lambda_j)(t-\mu_j)}\right),\tag{2.16}$$

with $\gamma_j=-(1/2\pi i)\ln(\lambda_j/\mu_j)$, $\theta_{1j}=\arg(\lambda_j)$, $\theta_{2j}=\arg(\mu_j)$,

$$C_j=\exp\left(-\frac{(\lambda_j-\mu_j)}{2\pi i}I(0)+\gamma_j(\ln|\lambda_j|-i(2\pi-\theta_{1j}-\theta_{2j}))\right)$$

and

$$I(0)=\int_0^\infty\frac{\ln t\mathrm{d}t}{(t-\lambda_j)(t-\mu_j)}.$$

We thus determine the limiting value of $(1/n_+(p))$, as given by

$$\frac{1}{n_+(p)}=\frac{p^{\delta+\gamma_1+\gamma_2+\cdots+\gamma_n}(p-\lambda_1)(p-\lambda_2)\ldots(p-\lambda_n)\,V_1(p)\,V_2(p)\ldots V_n(p)}{E_1(p)}.\tag{2.17}$$

Next, using the relation (2.11), we can recast (2.10) as

$$n_+(p)\psi_+(p)-n_-(p)\psi_-(p)=\frac{C(p)n_-(p)}{A(p)-iB(p)}.\tag{2.18}$$

giving

$$n(z)\psi(z)=\frac{1}{2\pi i}\int_0^\infty\frac{C(t)n_-(t)\mathrm{d}t}{[A(t)-iB(t)](t-z)}+E_2(z),\tag{2.19}$$

where $E_2(z)$ is an entire function.

By Plemelj's formulae from (2.19) we get

$$\psi_+(p)=\frac{1}{2}\frac{C(p)}{A(p)-iB(p)}\frac{n_+(p)}{n_-(p)}+\frac{1}{2\pi in_+(p)}\int_0^\infty\frac{C(t)n_-(t)\mathrm{d}t}{(A(t)-iB(t))(t-p)}+\frac{E_2(p)}{n_+(p)},\quad p>0\tag{2.20}$$

and

$$\psi_-(p)=-\frac{1}{2}\frac{C(p)}{A(p)-iB(p)}+\frac{1}{2\pi in_-(p)}\int_0^\infty\frac{C(t)n_-(t)\mathrm{d}t}{(A(t)-iB(t))(t-p)}+\frac{E_2(p)}{n_-(p)},\quad p>0.\tag{2.21}$$

The function $V(p)$ is finally determined, by using the relations (2.20) and (2.21) in the relation (2.8), and we find that

$$V(p)=\frac{1}{2}\frac{C(p)}{A(p)-iB(p)}\left(\frac{n_+(p)}{n_-(p)}+1\right)+\frac{1}{2\pi i}\left(\frac{1}{n_+(p)}-\frac{1}{n_-(p)}\right)\times\int_0^\infty\frac{C(t)n_-(t)\mathrm{d}t}{(A(t)-iB(t))(t-p)}+E_2(p)\left(\frac{1}{n_+(p)}-\frac{1}{n_+(p)}\right).\tag{2.22}$$

The integral in the relation (2.22) can be easily evaluated by considering the contour integral

$$\int_\Gamma \frac{C(\xi)n(\xi)\mathrm{d}\xi}{B(\xi)(\xi - z)},$$

where Γ is a closed contour comprising of a circle of large radius along with a loop around the positive real axis in the complex plane, with the assumption that $\xi_1, \xi_2, \ldots, \xi_n$ are the n distinct zeros of $B(\xi)$, which do not lie on the positive real axis and $B'(\xi_j) \neq 0$, for $j = 1, 2, \ldots, n$ (the case of multiple zeros can also be dealt with). Then, by application of the Cauchy residue theorem we obtain that (using the relation (2.11) also)

$$\frac{2}{\pi}\int_0^\infty \frac{C(t)n_-(t)\mathrm{d}t}{(A(t) - iB(t))(t-p)} = \frac{C(p)}{B(p)}n_+(p)\left[1 + \frac{n_-(p)}{n_+(p)}\right] + 2\sum_{i=1}^{n}\frac{n(\xi_i)C(\xi_i)}{B'(\xi_i)(\xi_i - p)}. \tag{2.23}$$

Using the result (2.22) in (2.23) we derive that

$$V(p) = \frac{B(p)\left[\sum_{j=1}^{n}\dfrac{n(\xi_j)C(\xi_j)}{B'(\xi_j)(p-\xi_j)} - 2iE_2(p)\right]}{(a_n - ib_n)(p-\lambda_1)(p-\lambda_2)\ldots(p-\lambda_n)n_+(p)}, \tag{2.24}$$

which on substituting for $n_+(p)$ from the relation (2.17) gives

$$V(p) = H(p)p^{\delta+\gamma}\, V_1(p)\, V_2(p)\ldots V_n(p), \tag{2.25}$$

where

$$H(p) = \frac{B(p)[\Sigma_{j=1}^{n} n(\xi_j)C(\xi_j) - 2i(p-\xi_j)E_2(p)]}{(p-\xi_j)E_1(p)}, \tag{2.26}$$

which is a rational function, with $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$.

The analysis that has been explained above is applicable to (1.1) for the general case when a_k's and b_k's are complex constants. But, for the particular case, when a_k's, b_k's are real constants, we have $0 < \delta < 1$. Setting $\theta_{1j} = \theta_j$, $\theta_{2j} = 2\pi - \theta_j$ and $\mu_j = \bar{\lambda}_j$, we obtain $\gamma_j = (1 - (\theta_j/\pi))$, $C_j = 1$ and using the relation (2.16) we find that

$$V_j(p) = \exp\left[-\frac{\sin\theta_j}{\pi}\int_0^{p/|\lambda_j|}\frac{\ln t\mathrm{d}t}{t^2 - 2t\cos\theta_j + 1}\right]\Big/[(p-\lambda_j)(p-\bar{\lambda}_j)]^{1-(\theta_j/2\pi)}. \tag{2.27}$$

This is equivalent to the result as obtained by Varley and Walker (1989).

Ultimately, by using the form (2.25), we find that

$$V(p) \approx \frac{H(p)}{p^{n-\delta}}, \quad \text{as } p \to \infty \tag{2.28}$$

and

$$V(p) \approx H(p)p^{\gamma+\delta}, \quad \text{as } p \to 0, \tag{2.29}$$

which help in expressing $H(p)$ as

$$H(p) = \sum_{j=-k}^{n-1} h_j p^j, \tag{2.30}$$

where $k - 1 < \delta + \gamma < k$, $k = 0, 1, \ldots, n-1$, when the relations (2.5) and (2.6) are also utilized.

The determination of h_j's can be completed by considering two separate cases as described next. These cases take care of even different degrees of the two polynomials $A(p)$ and $B(p)$.

Case 1. *If* $A(p)$ and $B(p)$ *are of the same degree n*, the constants $h_0, h_1, \ldots, h_n$ can be obtained by using the relation (2.4) along with the fact that $\xi_1, \xi_2, \ldots, \xi_n$ are the simple zeros of $B(\xi)$. The other constants $h_{-1}, h_{-2}, \ldots, h_{-n}$ will remain arbitrary.

Case 2. *If* $n > m$ *where, n is the degree of* $A(\xi)$ *and m is the degree of* $B(\xi)$, the m constants $h_1, h_2, \ldots, h_m$ can be determined by using the m zeros of $B(\xi)$ and the relation (2.4). The other $n - m$ constants $h_{m+1} \ldots h_{n-1}$ can be determined by using the asympototic behaviour of $m(z)$ as $z \to \infty$, as has been discussed by Varley and Walker [7]. The other constants will remain arbitrary.

The function $v(x)$ can finally be determined by using the Laplace inversion formula and we find that

$$v(x) = 2 \sum_{i=1}^{n} R_i \cos(x|\lambda_i| \sin\theta_i + \alpha_i) \mathrm{e}^{x|\lambda_i|\cos\theta_i} + \int_0^\infty l(s) \mathrm{e}^{-xs} \mathrm{d}s, \tag{2.31}$$

where $R_i \mathrm{e}^{i\alpha_i}$ denotes the residue of $V(p)$ at $p = \lambda_i$ and

$$l(s) = (2\pi i)^{-1} [V(s\mathrm{e}^{-i\pi}) - V(s\mathrm{e}^{i\pi})], \tag{2.32}$$

with the expression for $V(p)$ as given by the relation (2.25). The unknown function $f(x)$ can then be determined successfully, by using the the second relation in (2.3) along with the convolution theorem for Laplace transforms.

Following are some observations on the method that has been presented above. (i) We have avoided a lot of complication by calculating the Cauchy type integral in the relation (2.22) directly. (ii) In our analysis, the determination of $H(p)$ is simpler as compared to that explained in Varley and Walker's [7] paper. (iii) The behaviour of the function $V(p)$ at the end points $p = 0$ and $p = \infty$ along with the analyticity property of $V(p)$ has helped us in arriving at the expression for $H(p)$, as given by the relation (2.26).

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Wave propagation in a micropolar generalized thermoelastic body with stretch

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Abstract. In the present investigation, we discuss two different problems namely, (i) Rayleigh–Lamb problem in micropolar generalized thermoelastic layer with stretch and (ii) Rayleigh wave in a micropolar generalized thermoelastic half-space with stretch. The frequency and wave velocity equations for symmetric and anti-symmetric vibrations are obtained for the first problem. The frequency equation has also been derived for the second problem. The special cases of the above said problems of micropolar generalized thermoelasticity with stretch for Green–Lindsay and Lord–Shulman theory have been discussed in detail. Results of these analysis reduce to those without thermal and stretch effects.

Keywords. Wave propagation; micropolar; generalized thermoelastic; stretch; frequency equation.

1. Introduction

A theory of micropolar continua was proposed by Eringen and Suhubi [2] and Eringen [4] to describe the continuum behaviour of materials possessing microstructure. Basically, the difference between classical and micropolar theories is that the layer admits independent rotations of the material structure; that is, the local intrinsic rotations (microrotation), which are taken to be kinematically independent of the linear displacement. It is believed that such a theory is applicable in the treatment of granular and fibrous composite material.

In the classical theory of thermoelasticity, the constitutive relations are assumed to be dependent on the time rate of change of the absolute temperature and this idea was advanced by Müller [11] and later the coupled theory of thermoelasticity has been extended by including the thermal relaxation time in the constitutive equations by Lord and Shulman [10] and Green and Lindsay [6]. These new theories eliminate the paradox of infinite velocity of heat propagation and are termed as generalized theory of thermoelasticity.

The linear theory of micropolar thermoelasticity was developed by extending the theory of micropolar continua to include thermal effect by Eringen [5] and Nowacki [12] and is known as micropolar coupled thermoelasticity. Dost and Tabarrok [1] have presented the generalized micropolar thermoelasticity by using Green–Lindsay theory.

Eringen [5] extended his work to include the effect of axial stretch during the rotation of molecules and developed the theory of micropolar elastic solid with stretch. The mechanical model underlying the theory of micropolar elastic solids with stretch can be envisioned as an elastic medium composed of a large number of short springs. These springs possess average inertia moments and can deform in axial direction. The material points in this continuum possess not only classical translation degree of freedom of the deformation vector field but also intrinsic rotations and intrinsic axial stretch.

We extend the micropolar thermoelasticity with stretch given by Eringen [3, 5] and Nowacki [12], in the context of thermoelastic theory given by Lord and Shulman [10] and Green–Lindsay [6] and termed it as micropolar generalized thermoelasticity with stretch. There exists the following differences between L–S (Lord–Shulman) and G–L (Green–Lindsay) theory:

(i) L–S theory involves one relaxation time of thermoelastic process and that of G–L involves two relaxation times.
(ii) The energy equation of L–S theory depends both on the strain-velocity and strain-acceleration whereas the corresponding equation of G–L theory depends only on the strain velocity.
(iii) In the linearized case, according to the approach of G–L, heat cannot propagate with a finite speed unless the stresses depend on the temperature velocity.

Due to theoretical and practical importance, many problems of waves and vibrations of micropolar elasticity have been investigated by Nowacki [12] and Sengupta and De [14] and Rao and Rao [13].

The problem of propagation of waves in micropolar elastic layer with stretch immersed in an infinite liquid has been investigated by Kumar and Gogna [7]. Kumar *et al* [9] have investigated the Lamb's plane problem in a thermoelastic micropolar medium with stretch. Kumar and Chadha [8] discuss the Lamb's problem in micropolar elastic half-space with stretch. In the present investigation, we discuss two problems namely (i) Rayleigh–Lamb problem in micropolar generalized thermoelastic plate with stretch and (ii) Rayleigh waves in micropolar generalized thermoelastic half-space with stretch. Some special cases have been discussed.

2. Problem I: Rayleigh–Lamb problem in micropolar generalized thermoelastic layer with stretch

2.1 *Formulation of the problem and solution*

We consider a homogeneous micropolar generalized thermoelastic body with stretch. Suppose that the body considered is a plate occupying the cartesian space $-\infty < x < \infty$, $-H \leqslant z \leqslant H$, $-\infty < y < \infty$ and the free plane wave propagate in the plate in the positive x-direction causing plane deformation parallel to the xz-plane, then the displacement vector lies completely parallel to xz-plane and all the field variables depend only on x, z and t.

Following Eringen [3, 5] and Nowacki [12], the field equations of micropolar generalized thermoelasticity with stretch can be written as

$$(\mu+\alpha)\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}\mathbf{u}+2\alpha\operatorname{rot}\boldsymbol{\omega}-\nu\operatorname{grad}(\theta+t_1\dot{\theta})=\rho\frac{\partial^2\mathbf{u}}{\partial t^2},$$

$$(\gamma+\varepsilon)\nabla^2\boldsymbol{\omega}+(\gamma+\beta-\varepsilon)\operatorname{grad}\operatorname{div}\boldsymbol{\omega}-4\alpha\boldsymbol{\omega}+2\alpha\operatorname{rot}\mathbf{u}=J\frac{\partial^2\boldsymbol{\omega}}{\partial t^2},$$

$$K\nabla^2\theta=\rho C^*(\dot{\theta}+t_0\ddot{\theta})+\nu\theta_0[\dot{u}_{i,j}+\delta_{1k}\ddot{u}_{i,j}],$$

$$\alpha_0\nabla^2\phi^*-\eta_0\phi^*=\frac{J}{2}\frac{\partial^2\phi^*}{\partial t^2}, \tag{1}$$

where $\lambda, \mu, \gamma, \alpha, \beta, \varepsilon, \alpha_0, \eta_0$ are material constants, ρ the density, J the rotational inertia, K the thermal conductivity, $v = (3\lambda + 2\mu)\alpha_t, \alpha_t$ the coefficient of linear thermal expansion, C^* the specific heat at constant strain, θ_0 the initial uniform temperature, t_0, t_1, are the thermal relaxation times, δ_{lk} the Kronecker delta, $\mathbf{u}$ the displacement vector, $\boldsymbol{\omega}$ the microrotation vector and ϕ^* the scalar microstretch. The dot denotes the derivative with respect to time.

For the L–S theory $t_1 = 0, \delta_{lk} = 1$ and for the G–L theory $t_1 > 0$ and $\delta_{lk} = 0$ ($k = 1$ for L–S and 2 for G–L theory). The thermal relaxations t_0 and t_1 satisfy the inequality $t_1 \geqslant t_0 \geqslant 0$ for the G–L theory only.

Taking $\mathbf{u} = (u_1, 0, u_3)$ and $\boldsymbol{\omega} = (0, \omega_2, 0)$, the system of equations (1) reduce to

$$(\mu + \alpha)\nabla^2 u_1 + (\lambda + \mu - \alpha)\frac{\partial e}{\partial x} - 2\alpha\frac{\partial \omega_2}{\partial z} - v\frac{\partial}{\partial x}(\theta + t_1\dot{\theta}) = \rho\frac{\partial^2 u_1}{\partial t^2},$$

$$(\mu + \alpha)\nabla^2 u_3 + (\lambda + \mu - \alpha)\frac{\partial e}{\partial z} + 2\alpha\frac{\partial \omega_2}{\partial x} - v\frac{\partial}{\partial z}(\theta + t_1\dot{\theta}) = \rho\frac{\partial^2 u_3}{\partial t^2},$$

$$(\gamma + \varepsilon)\nabla^2 \omega_2 - 4\alpha\omega_2 - 2\alpha\left(\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right) = J\frac{\partial^2 \omega_2}{\partial t^2},$$

$$K\nabla^2\theta = \rho C^*(\dot{\theta} + t_0\ddot{\theta}) + v\theta_0\left\{\frac{\partial e}{\partial t} + \delta_{lk}t_0\frac{\partial^2 e}{\partial t^2}\right\},$$

$$\alpha_0\nabla^2\phi^* = \eta_0\phi^* = \frac{J}{2}\frac{\partial^2\phi^*}{\partial t^2}, \tag{2}$$

where

$$e = \frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z},$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

The displacement components u_1 and u_3 may be written in terms of potential functions as

$$u_1 = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z},$$

$$u_3 = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}. \tag{4}$$

Substituting (4) in (2), we get

$$\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\phi - q(\theta + t_1\dot{\theta}) = 0, \tag{5}$$

$$\left(\nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\psi + p\omega_2 = 0, \tag{6}$$

$$\left(\nabla^2 - r - \frac{1}{c_3^2}\frac{\partial^2}{\partial t^2}\right)\omega_2 - r_0\nabla^2\psi = 0, \tag{7}$$

$$\left\{\nabla^2 - \frac{1}{c_4^2}\left(\frac{\partial}{\partial t} + t_0\frac{\partial^2}{\partial t^2}\right)\right\}\theta - s\left(\frac{\partial}{\partial t} + \delta_{lk}t_0\frac{\partial^2}{\partial t^2}\right)\nabla^2\phi = 0, \tag{8}$$

$$\left(\nabla^2 - \frac{1}{c_5^2}\frac{\partial^2}{\partial t^2} - v_1^2\right)\phi^* = 0, \tag{9}$$

where

$$\begin{aligned}
&c_1^2 = (\lambda + 2\mu)/\rho, \quad c_2^2 = (\mu + \alpha)/\rho,\\
&c_3^2 = (\gamma + \varepsilon)/J, \quad c_4^2 = K/\rho C^*,\\
&c_5^2 = 2\alpha_0/J, \quad p = 2\alpha/(\mu + \alpha),\\
&q = v/(\lambda + 2\mu), \quad r_0 = 2\alpha/(\gamma + \varepsilon),\\
&r = 4\alpha/(\gamma + \varepsilon), \quad v_1^2 = \eta_0/\alpha_0.
\end{aligned} \tag{10}$$

Eliminating ϕ and θ from (5) and (8), we obtain

$$\left[\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\left\{\nabla^2 - \frac{1}{c_4^2}\left(\frac{\partial}{\partial t} + t_0\frac{\partial^2}{\partial t^2}\right)\right\} - sq\left(\frac{\partial}{\partial t} + \delta_{lk}t_0\frac{\partial^2}{\partial t^2}\right)\right.$$
$$\left.\times\nabla^2\left(1 + t_1\frac{\partial}{\partial t}\right)\right](\phi, \theta) = 0. \tag{11}$$

Also, eliminating ψ and ω_2 from (6) and (7), we obtain

$$\left[\left(\nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - r - \frac{1}{c_3^2}\frac{\partial^2}{\partial t^2}\right) + r_0 p\nabla^2\right](\psi, \omega_2) = 0. \tag{12}$$

To obtain the solutions of (9), (11) and (12), let

$$\begin{aligned}
\phi &= f(z)\exp[i(kx - \omega t)],\\
\theta &= g(z)\exp[i(kx - \omega t)],\\
\psi &= h(z)\exp[i(kx - \omega t)],\\
\omega_2 &= e(z)\exp(i(kx - \omega t)],\\
\phi^* &= I(z)\exp[i(kx - \omega t)].
\end{aligned} \tag{13}$$

Substituting (13) in (9), (11) and (12), after simplification, we obtain

$$\phi = [A\sinh\lambda_1 z + B\cosh\lambda_1 z + C\sinh\lambda_2 z + D\cosh\lambda_2 z] \times \exp[i(kx - \omega t)], \tag{14}$$

$$\theta = [A_1\sinh\lambda_1 z + B_1\cosh\lambda_1 z + C_1\sinh\lambda_2 z + D_1\cosh\lambda_2 z] \times \exp[i(kx - \omega t)], \tag{15}$$

$$\psi = [E\sinh\lambda_3 z + F\cosh\lambda_3 z + R\sinh\lambda_4 z + S\cosh\lambda_4 z] \times \exp[i(kx - \omega t)], \tag{16}$$

$$\omega_2 = [E_1\sinh\lambda_3 z + F_1\cosh\lambda_3 z + R_1\sinh\lambda_4 z + S_1\cosh\lambda_4 z] \times \exp[i(kx - \omega t)], \tag{17}$$

$$\phi^* = [M\sinh\lambda_5 z + N\cosh\lambda_5 z]\exp[i(kx - \omega t)], \tag{18}$$

where

$$\lambda_1^2+\lambda_2^2=2k^2-\frac{\omega^2}{c_1^2}-\frac{i\omega}{c_4^2}(1-i\omega t_0)-i\omega sq(1-i\omega t_1)(1-i\omega t_0\delta_{lk})$$

$$\lambda_1^2\lambda_2^2=\left(\frac{\omega^2}{c_1^2}-k^2\right)\left\{\frac{i\omega}{c_4^2}(1-i\omega t_0)-k^2\right\}-i\omega sqk^2(1-i\omega t_0\delta_{lk})(1-i\omega t_1),$$

$$\lambda_3^2+\lambda_4^2=2k^2-\omega^2\left(\frac{1}{c_2^2}+\frac{1}{c_3^2}\right)+r-r_0p,$$

$$\lambda_3^2\lambda_4^2=\left(k^2-\frac{\omega^2}{c_2^2}\right)\left(k^2+r-\frac{\omega^2}{c_3^2}\right)-r_0pk^2,$$

$$\lambda_5^2=k^2+v_1^2-(\omega^2/c_5^2),$$

$$A_1=aA,\quad B_1=aB,\quad C_1=bC,$$

$$D_1=bD,\quad E_1=a_1E,\quad F_1=a_1F,$$

$$R_1=b_1R,\quad S_1=b_1S,$$

and

$$a=\left(\lambda_1^2-k^2+\frac{\omega^2}{c_1^2}\right)\frac{1}{q(1-i\omega t_1),}$$

$$b=\left(\lambda_2^2-k^2+\frac{\omega^2}{c_1^2}\right)\frac{1}{q(1-i\omega t_1)},$$

$$a_1=\frac{1}{p}\left(k^2-\lambda_3^2-\frac{\omega^2}{c_2^2}\right),$$

$$b_1=\frac{1}{p}\left(k^2-\lambda_4^2-\frac{\omega^2}{c_2^2}\right). \tag{19}$$

2.2 *Boundary conditions*

Following Eringen [3, 5] and Nowacki [12], the stresses are given by

$$t_{ji}=\lambda\mu_{k,k}\delta_{ji}+(\mu-\alpha)(u_{i,j}+u_{j,i})+2\alpha(u_{i,j}-\varepsilon_{kji}\omega_k)-v(\theta+t_1\dot{\theta})\delta_{ij},$$

$$m_{ji}=\beta_0\varepsilon_{kji}\phi^*_{,k}+\beta\omega_{k,k}\delta_{ji}+(\gamma+\varepsilon)\omega_{i,j}+(\gamma-\varepsilon)\omega_{j,i},$$

$$\lambda_j=\alpha_0\phi^*_{,j}+\frac{\beta_0}{3}\varepsilon_{kji}\omega_{k,i}. \tag{20}$$

The above equations in terms of potentials are

$$t_{zz}=\lambda\nabla^2\phi+2\mu\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\psi}{\partial x\partial z}\right)-v(\theta+t_1\dot{\theta}),$$

$$t_{zx}=\mu\left[2\frac{\partial^2\phi}{\partial x\partial z}-\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial x^2}\right]-\alpha(\nabla^2\psi+2\omega_2),$$

$$m_{zy}=(\gamma+\varepsilon)\frac{\partial\omega_2}{\partial z}-\beta_0\frac{\partial\phi^*}{\partial x},$$

$$\lambda_z=\alpha_0\frac{\partial\phi^*}{\partial z}+\frac{\beta_0}{3}\frac{\partial\omega_2}{\partial x}. \tag{21}$$

The boundary conditions are

$$t_{zz}=0, \quad t_{zx}=0, \quad m_{zy}=0, \quad \frac{\partial\theta}{\partial z}=0, \quad \lambda_z=0, \quad \text{at } z=\pm H \tag{22}$$

where t_{zz}, t_{zx} are the components of force stress, m_{zy} and λ_z are the components of couple stress and vector first moment.

Now we discuss two cases, one by considering the symmetric vibrations and other by anti-symmetric vibrations.

2.3 *Symmetric vibrations*

We consider the symmetry of the displacement u_1, the force stress t_{xx} and t_{zz}, the couple stress m_{zx} and the first moment λ_z with respect to the plane $z=0$, we obtain the following equations from (14)–(18).

$$\phi=[B\cosh\lambda_1 z+D\cosh\lambda_2 z]\exp[i(kx-\omega t)], \tag{23}$$

$$\theta=[B_1\cosh\lambda_1 z+D_1\cosh\lambda_2 z]\exp[i(kx-\omega t)], \tag{24}$$

$$\psi=[E\sinh\lambda_3 z+R\sinh\lambda_4 z]\exp[i(kx-\omega t)], \tag{25}$$

$$\omega_2=[E_1\sinh\lambda_3 z+R_1\sinh\lambda_4 z]\exp[i(kx-\omega t)], \tag{26}$$

$$\phi^*=N\cosh(\lambda_5 z)\exp[i(kx-\omega t)]. \tag{27}$$

Substituting the values of $\phi, \theta, \psi, \omega_2$ and ϕ^* from (23)–(27) in the boundary conditions (22) and using (21), we obtain

$$d\cosh(\lambda_1 H)B+d_1\cosh(\lambda_2 H)D+2i\mu k\lambda_3\cosh(\lambda_3 H)E$$
$$+2i\mu k\lambda_4\cosh(\lambda_4 H)R=0, \tag{28}$$

$$2i\mu k\lambda_1\sinh(\lambda_1 H)B+2i\mu k\lambda_2\sinh(\lambda_2)D-d_3\sinh(\lambda_3 H)E$$
$$-d_4\sinh(\lambda_4 H)R=0, \tag{29}$$

$$(\gamma+\varepsilon)\lambda_3 a_1\cosh(\lambda_3 H)E+(\gamma+\varepsilon)\lambda_4 b_1\cosh(\lambda_4 H)R$$
$$-i\beta_0 k\cosh(\lambda_5 H)N=0, \tag{30}$$

$$\lambda_1 a\sinh(\lambda_1 H)B+\lambda_2 b\sinh(\lambda_2 H)D=0, \tag{31}$$

$$i\beta_0 ka_1\sinh(\lambda_3 H)E+i\beta_0 kb_1\sinh(\lambda_4 H)R$$
$$+3\alpha_0\lambda_5\sinh(\lambda_5 H)N=0, \tag{32}$$

where

$$d=(\lambda+2\mu)\lambda_1^2-\lambda k^2-\nu(1-i\omega t_1)a,$$
$$d_1=(\lambda+2\mu)\lambda_2^2-\lambda k^2-\nu(1-i\omega t_1)b,$$
$$d_3=(\mu+\alpha)\lambda_3^2+(\mu-\alpha)k^2+2\alpha a_1,$$
$$d_4=(\mu+\alpha)\lambda_4^2+(\mu-\alpha)k^2+2\alpha b_1. \tag{33}$$

Eliminating the constants B, D, E, R and N from (28)–(32), we obtain the period equation as

$$(\gamma+\varepsilon)\lambda_4 b_1\ T_5[4\mu^2k^2\lambda_1\lambda_2\lambda_3\ T_1\ T_2 d(a-b)+dd_3\ T_3(\lambda_2 T_2 db-\lambda_1\ T_1 d_1 a)]$$
$$-(\gamma+\varepsilon)\lambda_3 a_1\ T_5[4\mu^2k^2\lambda_1\lambda_2\lambda_4\ T_1\ T_2 d(a-b)+dd_4\ T_4(\lambda_2\ T_2 db-\lambda_1\ T_1 d_1 a)]$$

$$+\varepsilon_3 a_1 T_3 [4\mu^2 k^2 \lambda_1 \lambda_2 \lambda_4 T_1 T_2 d(a-b) + dd_4 T_4(\lambda_2 T_2 db - \lambda_1 T_1 d_1 a)]$$
$$-\varepsilon_3 b_1 T_4[4\mu^2 k^2 \lambda_1 \lambda_2 \lambda_3 T_1 T_2 d(a-b) + dd_3 T_3(\lambda_2 db - \lambda_1 T_1 d_1 a)] = 0, \tag{34}$$

where

$$T_i = \tanh(\lambda_i H), \quad (i = 1, \dots, 5)$$
$$\varepsilon_3 = \beta_0^2 k^2 / 3\alpha_0 \lambda_5.$$

(For L–S theory):

$$(\gamma+\varepsilon)\lambda_4 b_1 T_5[4\mu^2 k^2 m_1 m_2 \lambda_3 T_1^* T_2^* d^*(a^*-b^*) + d^* d_3 T_3(m_2 T_2^* d^* b^*$$
$$- m_1 T_1^* d_1^* a^*)] - (\gamma+\varepsilon)\lambda_3 a_1 T_5[4\mu^2 k^2 m_1 m_2 \lambda_4 T_1^* T_2^* d^*(a^*-b^*)$$
$$+ d^* d_4 T_4(m_2 T_2^* d^* b^* - m_1 T_1^* d_1^* a^*)]$$
$$+\varepsilon_3 a_1 T_3[4\mu^2 k^2 m_1 m_2 \lambda_4 T_1^* T_2^* d^*(a^*-b^*)$$
$$+ d^* d_4 T_4(m_2 T_2^* d^* b^* - m_1 T_1^* d_1^* a^*)]$$
$$-\varepsilon_3 b_1 T_4 [4\mu^2 k^2 m_1 m_2 \lambda_3 T_1^* T_2^* d^*(a^*-b^*)$$
$$+ d^* d_3 T_3(m_2 T_2^* d^* b^* - m_1 T_1^* d_1^* a^*)] = 0, \tag{35}$$

where

$$T_i^* = \tanh(m_i H), \quad (i = 1, 2)$$
$$m_1^2 + m_2^2 = 2k^2 - \frac{\omega^2}{c_1^2} - i\omega(1 - i\omega t_0)\left(\frac{1}{c_4^2} + sq\right),$$
$$m_1^2 m_2^2 = \left(\frac{\omega^2}{c_1^2} - k^2\right)\left\{\frac{i\omega}{c_4^2}(1 - i\omega t_0) - k^2\right\} - i\omega s q k^2 (1 - i\omega t_0),$$
$$d^* = (\lambda + 2\mu) m_1^2 - \lambda k^2 - \nu a^*,$$
$$d_1^* = (\lambda + 2\mu) m_2^2 - \lambda k^2 - \nu b^*,$$
$$a^* = \left(m_1^2 - k^2 + \frac{\omega^2}{c_1^2}\right)\frac{1}{q},$$
$$b^* = \left(m_2^2 - k^2 + \frac{\omega^2}{c_1^2}\right)\frac{1}{q}. \tag{36}$$

(For G–L theory):

$$(\gamma+\varepsilon)\lambda_4 b_1 T_5[4\mu^2 k^2 n_1 n_2 \lambda_3 T'_1 T'_2 d'(a'-b') + d' d_3 T_3(n_2 T'_2 d' b'$$
$$- n_1 T'_1 d'_1 a')] - (\gamma+\varepsilon)\lambda_3 a_1 T_5[4\mu^2 k^2 n_1 n_2 \lambda_4 T'_1 T'_2 d'(a'-b')$$
$$+ d' d_4 T_4 (n_2 T'_2 d' b' - n_1 T'_1 d'_1 a')]$$
$$+\varepsilon_3 a_1 T_3[4\mu^2 k^2 n_1 n_2 \lambda_4 T'_1 T'_2 d(a'-b')$$
$$+ d' d_4 T_4(n_2 T'_2 d' b' - n_1 T'_1 d'_1 a')]$$
$$-\varepsilon_3 b_1 T_4[4\mu^2 k^2 n_1 n_2 \lambda_3 T'_1 T'_2 d'(a'-b')$$
$$+ d' d_3 T_3(n_2 T'_2 d' b' - n_1 T'_1 d'_1 a')] = 0. \tag{37}$$

where

$$T_i' = \tanh(n_i H), \quad (i = 1, 2),$$

$$n_1^2 + n_2^2 = 2k^2 - \omega^2 \left(\frac{1}{c_1^2} + \frac{i}{c_4^2} \right) - i\omega sq(1 - i\omega t_1),$$

$$n_1^2 n_2^2 = \left(\frac{\omega^2}{c_1^2} - k^2 \right) \left(\frac{i\omega}{c_4^2} - k^2 \right) + i\omega sqk^2(1 - i\omega t_1),$$

$$d' = (\lambda + 2\mu)n_1^2 - \lambda k^2 - \nu(1 - i\omega t_1)a',$$

$$d_1' = (\lambda + 2\mu)n_2^2 - \lambda k^2 - \nu(1 - i\omega t_1)b',$$

$$a' = \left(n_1^2 - k^2 + \frac{\omega^2}{c_1^2} \right) \frac{1}{q(1 - i\omega t_1)},$$

$$b' = \left(n_2^2 - k^2 + \frac{\omega^2}{c_1^2} \right) \frac{1}{q(1 - i\omega t_1)}. \tag{38}$$

2.4 *Antisymmetric vibrations*

Here, considering that the displacement u_1, force stresses t_{xx}, t_{zz}, couple stress m_{zy} and first moment λ_z are antisymmetric with respect to the plane $z = 0$; we obtain from (14)–(18), the following equations

$$\phi = [A \sinh \lambda_1 z + C \sinh \lambda_2 z] \exp[i(kx - \omega t)],$$

$$\theta = [A_1 \sinh \lambda_1 z + C_1 \sinh \lambda_2 z] \exp[i(kx - \omega t)],$$

$$\psi = [F \cosh \lambda_3 z + S \cosh \lambda_4 z] \exp[i(kx - \omega t)],$$

$$\omega_2 = [F_1 \cosh \lambda_3 z + S_1 \cosh \lambda_4 z] \exp[i(kx - \omega t)],$$

$$\phi^* = M \sinh(\lambda_5 z) \exp[i(kx - \omega t)]. \tag{39}$$

Substituting the values of potentials from (39) in the boundary conditions (22), and making use of (21), we obtain

$$d \sinh(\lambda_1 H)A + d_1 \sinh(\lambda_2 H)C + 2i\mu k\lambda_3 \sinh(\lambda_3 H)F$$
$$+ 2i\mu k\lambda_4 \sinh(\lambda_4 H)S = 0, \tag{40}$$

$$2i\mu k\lambda_1 \cosh(\lambda_1 H)A + 2i\mu k\lambda_2 \cosh(\lambda_2 H)C - d_3 \cosh(\lambda_3 H)F$$
$$- d_4 \cosh(\lambda_4 H)S = 0, \tag{41}$$

$$(\gamma + \varepsilon)\lambda_3 a_1 \sinh(\lambda_3 H)F + (\gamma + \varepsilon)\lambda_4 b_1 \sinh(\lambda_4 H)S$$
$$- ik\beta_0 \sinh(\lambda_5 H)M = 0, \tag{42}$$

$$i\beta_0 ka_1 \cosh(\lambda_3 H)F + i\beta_0 kb_1 \cosh(\lambda_4 H)S + 3\alpha_0 \lambda_5 \cosh(\lambda_5 H)M = 0, \tag{43}$$

$$\lambda_1 a \cosh(\lambda_1 H)A + \lambda_2 b \cosh(\lambda_2 H)C = 0. \tag{44}$$

Eliminating the constants A, C, F, S and M from (40)–(44), we obtain the wave velocity equation for the antiysymmetric vibrations of a micropolar generalized thermoelastic layer with stretch as

$$
\begin{aligned}
&(\gamma+\varepsilon)\lambda_4 b_1 T_4[4\mu^2k^2\lambda_1\lambda_2\lambda_3 T_3 d(a-b)+dd_3(\lambda_2 T_1 ab-\lambda_1 T_2 d_1 a)]\\
&\quad-(\gamma+\varepsilon)\lambda_3 a_1 T_3[4\mu^2k^2\lambda_1\lambda_2\lambda_4 T_4 d(a-b)+dd_4(\lambda_2 T_1 db-\lambda_1 T_2 d_1 a)]\\
&\quad+\varepsilon_3 a_1 T_5[4\mu^2k^2\lambda_1\lambda_2\lambda_4 T_4 d(a-b)+dd_4(\lambda_2 T_1 db-\lambda_1 T_2 d_1 a)]\\
&\quad-\varepsilon_3 b_1 T_5[4\mu^2k^2\lambda_1\lambda_2\lambda_3 T_3 d(a-b)\\
&\quad+dd_3(\lambda_2 T_1 db-\lambda_1 T_2 d_1 a)]=0. \qquad (45)
\end{aligned}
$$

(For L–S theory):

$$
\begin{aligned}
&(\gamma+\varepsilon)\lambda_4 b_1 T_4[4\mu^2k^2 m_1 m_2\lambda_3 T_3 d^*(a^*-b^*)+d^* d_3(m_2 T_1^* d^* b^*\\
&\quad-m_1 T_2^* d_1^* a^*)]-(\gamma+\varepsilon)\lambda_3 a_1 T_3[4\mu^2k^2 m_1 m_2\lambda_4 T_4 d^*(a^*-b^*)\\
&\quad+d^* d_4(m_2 T_1^* d^* b^*-m_1 T_2^* d_1^* a^*)]\\
&\quad+\varepsilon_3 a_1 T_5[4\mu^2k^2 m_1 m_2\lambda_4 T_4 d^*(a^*-b^*)\\
&\quad+d^* d_4(m_2 T_1^* d^* b^*-m_1 T_2^* d_1^* a^*)]\\
&\quad-\varepsilon_3 b_1 T_5[4\mu^2k^2 m_1 m_2\lambda_3 T_3 d^*(a^*-b^*)\\
&\quad+d^* d_3(m_2 T_1^* d^* b^*-m_1 T_2^* d_1^* a^*)]=0. \qquad (46)
\end{aligned}
$$

(For G–L theory):

$$
\begin{aligned}
&(\gamma+\varepsilon)\lambda_4 b_1 T_4[4\mu^2k^2 n_1 n_2\lambda_3 T_3 d'(a'-b')\\
&\quad+d' d_3(n_2 T_1' d' b'-n_1 T_2' d_1' a')]\\
&\quad-(\gamma+\varepsilon)\lambda_3 a_1 T_3[4\mu^2k^2 n_1 n_2\lambda_4 T_4 d'(a'-b')\\
&\quad+d' d_4(n_2 T_1' d' b'-n_1 T_2' d_1' a')]\\
&\quad+\varepsilon_3 a_1 T_5[4\mu^2k^2 n_1 n_2\lambda_4 T_4 d'(a'-b')\\
&\quad+d' d_4(n_2 T_1' d' b'-n_1 T_2' d_1' a')]\\
&\quad-\varepsilon_3 b_1 T_5[4\mu^2k^2 n_1 n_2\lambda_3 T_3 d'(a'-b')\\
&\quad+d' d_3(n_2 T_1' d' b'-n_1 T_2' d_1' a')]=0. \qquad (47)
\end{aligned}
$$

2.5 *Discussions of period equations*

Case 2.1.1. When the wavelength is large as compared to the thickness of the layer, then the quantities $\lambda_1 H$, $\lambda_2 H$, $\lambda_3 H$, $\lambda_4 H$ and $\lambda_5 H$ are small. Equations (34), (35), (37) (for symmetric vibrations) and (45)–(47) (for antisymmetric vibrations) reduce to:

(a) *Symmetric case*

$$
\begin{aligned}
&d(\gamma+\varepsilon)(b_1 d_3-a_1 d_4)\frac{\lambda_2}{\lambda_1}b+(\gamma+\varepsilon)d_1(a_1 d_4-b_1 d_3)\frac{\lambda_1}{\lambda_2}a\\
&\quad+4\mu^2k^2(\gamma+\varepsilon)(a-b)(b_1-a_1)\lambda_1\lambda_2\\
&\quad-\frac{\varepsilon_3}{\lambda_5}\left[4\mu^2k^2(a-b)(a_1-b_1)\lambda_1\lambda_2+db(a_1 d_4-b_1 d_3)\frac{\lambda_2}{\lambda_1}\right.\\
&\qquad\left.+ad_1(b_1 d_3-a_1 d_4)\frac{\lambda_1}{\lambda_2}\right]=0. \qquad (48)
\end{aligned}
$$

(For L–S theory):

$$d^*(\gamma+\varepsilon)(b_1d_3-a_1d_4)\frac{m_2}{m_1}b^*+d_1^*(\gamma+\varepsilon)(a_1d_4-b_1d_3)\frac{m_1}{m_2}a^*$$
$$+4\mu^2k^2(\gamma+\varepsilon)(a^*-b^*)(b_1-a_1)m_1m_2$$
$$+\frac{\varepsilon_3}{\lambda_5}\left[4\mu^2k^2(a^*-b^*)(a_1-b_1)m_1m_2+d^*b^*(a_1d_4-b_1d_3)\frac{m_2}{m_1}\right.$$
$$\left.+a^*d_1^*(b_1d_3-a_1d_4)\frac{m_1}{m_2}\right]=0. \tag{49}$$

(For G–L theory):

$$d'(\gamma+\varepsilon)(b_1d_3-a_1d_4)\frac{n_2}{n_1}b'+d_1'(\gamma+\varepsilon)(a_1d_4-b_1d_3)\frac{n_1}{n_2}a'$$
$$+4\mu^2k^2(\gamma+\varepsilon)(a'-b')(b_1-a_1)n_1n_2$$
$$+\frac{\varepsilon_3}{\lambda_5}\left[4\mu^2k^2(a'-b')(a_1-b_1)n_1n_2\right.$$
$$\left.+d'b'(a_1d_4-b_1d_3)\frac{m_2}{m_1}+a'd_1'(b_1b_3-a_1d_4)\frac{m_1}{m_2}\right]=0. \tag{50}$$

(b) *Anti-symmetric case*

$$4\mu^2k^2(a-b)(b_1-a_1)(\gamma+\varepsilon)+(\gamma+\varepsilon)(bdS_1-ad_1S_2)\left(\frac{b_1d_3}{S_3\lambda_3^2}-\frac{a_1d_4}{S_4\lambda_4^2}\right)$$
$$+4\mu^2k^2\varepsilon_3\lambda_5(a-b)\left(\frac{a_1}{S_3\lambda_3^2}-\frac{b_1}{S_4\lambda_4^2}\right)$$
$$+\frac{\varepsilon_3\lambda_5}{S_3S_4\lambda_3^2\lambda_4^2}(bdS_1-ad_1S_2)(a_1d_4-b_1d_3)=0 \tag{51}$$

where

$$S_i=\left(1-\frac{\lambda_i^2H^2}{3}\right),\quad i=(1,2,\ldots,4). \tag{52}$$

(For L–S theory):

$$4\mu^2k^2(a^*-b^*)(b_1-a_1)(\gamma+\varepsilon)+(\gamma+\varepsilon)\left(b^*d^*S_1^*-a^*d_1^*S_2^*\left(\frac{b_1d_3}{S_3\lambda_3^2}-\frac{a_1d_4}{S_4\lambda_4^2}\right)\right.$$
$$+4\mu^2k^2\varepsilon_3\lambda_5(a^*-b^*)\left(\frac{a_1}{S_3\lambda_3^2}-\frac{b_1}{S_4\lambda_4^2}\right)$$
$$+\frac{\varepsilon_3\lambda_5}{S_3S_4\lambda_3^2\lambda_4^2}(b^*d^*S_1^*-a^*d_1^*S_2^*)(a_1d_4-b_1d_3)=0, \tag{53}$$

where

$$S_1^*=\left(1-\frac{m_1^2H^2}{3}\right),$$
$$S_2^*=\left(1-\frac{m_2^2H^2}{3}\right). \tag{54}$$

(For G–L theory):

$$4\mu^2 k^2 (a' - b')(b_1 - a_1)(\gamma + \varepsilon) + (\gamma + \varepsilon)(b'd'S'_1 - a'd'_1 S'_2)\left(\frac{b_1 d_3}{S_3 \lambda_3^2} - \frac{a_1 d_4}{S_4 \lambda_4^2}\right)$$
$$+ 4\mu^2 k^2 \varepsilon_3 \lambda_5 (a' - b')\left(\frac{a_1}{S_3 \lambda_3^2} - \frac{b_1}{S_4 \lambda_4^2}\right)$$
$$+ \frac{\varepsilon_3 \lambda_5}{S_3 S_4 \lambda_3^2 \lambda_4^2}(b'd'S'_1 - a'd'_1 S'_2)(a_1 d_4 - b_1 d_3) = 0 \tag{55}$$

where

$$S'_1 = \left(1 - \frac{n_1^2 H^2}{3}\right),$$
$$S'_2 = \left(1 - \frac{n_2^2 H^2}{3}\right). \tag{56}$$

Equations (48)–(50), (51), (53) and (55) represent the wave velocity equation for the longitudinal vibration of micropolar generalized thermoelastic layer with stretch.

Special Cases

(i) When we neglect stretch effect, the above equations (48)–(50), (51), (53) and (55) for symmetric and antisymmetric vibrations reduce in micropolar generalized thermoelastic layer.

(a) *Symmetric case*

$$(b_1 d_3 - a_1 d_4)(db\lambda_2^2 - d_1 a\lambda_1^2) + 4\mu^2 k^2 (a - b)(b_1 - a_1)\lambda_1^2 \lambda_2^2 = 0. \tag{57}$$

(For L–S theory):

$$(b_1 d_3 - a_1 d_4)(d^* b^* m_2^2 - d_1^* a^* m_1^2) + 4\mu^2 k^2 (a^* - b^*)(b_1 - a_1) m_1^2 m_2^2 = 0. \tag{58}$$

(For G–L theory):

$$(b_1 d_3 - a_1 d_4)(d'b'n_2^2 - d'_1 a' n_1^2) + 4\mu^2 k^2 (a' - b')(b_1 - a_1) n_1^2 n_2^2 = 0. \tag{59}$$

(b) *Antisymmetric case*

$$4\mu^2 k^2 (a - b)(b_1 - a_1) + (bdS_1 - ad_1 S_2)\left(\frac{b_1 d_3}{S_3 \lambda_3^2} - \frac{a_1 d_4}{S_4 \lambda_4^2}\right) = 0. \tag{60}$$

(For L–S theory):

$$4\mu^2 k^2 (a^* - b^*)(b_1 - a_1) + (b^* d^* S_1^* - a^* d_1^* S_2^*)\left(\frac{b_1 d_3}{S_3 \lambda_3^2} - \frac{a_1 d_4}{S_4 \lambda_4^2}\right) = 0. \tag{61}$$

(For G–L theory):

$$4\mu^2 k^2 (a' - b')(b_1 - a_1) + (b'd'S'_1 - a'd'_1 S'_2)\left(\frac{b_1 d_3}{S_3 \lambda_3^2} - \frac{a_1 d_4}{S_4 \lambda_4^2}\right) = 0. \tag{62}$$

(ii) If we let $t_1 = 0$, $\delta_{1k} = 1$ in L–S and $t_0 = 0$ in G–L theory, then (57)–(62) reduce to the following period equations in micropolar coupled thermoelastic layer as

$$(b_1 d_3 - a_1 d_4)(d_5 b_2 \lambda_2^{*2} - d_6 a_2 \lambda_1^{*2}) + 4\mu^2 k^2 (a_2 - b_2)(b_1 - a_1)\lambda_1^{*2}\lambda_2^{*2} = 0 \tag{63}$$

$$4\mu^2 k^2 (a_2 - b_2)(b_1 - a_1) + (b_2 d_5 S_5 - a_2 d_6 S_6)\left(\frac{b_1 d_3}{S_3 \lambda_3^2} - \frac{a_1 d_4}{S_4 \lambda_4^2}\right) = 0 \tag{64}$$

where

$$S_5 = \left(1 - \frac{\lambda_1^{*2} H^2}{3}\right),$$

$$S_6 = \left(1 - \frac{\lambda_2^{*2} H^2}{3}\right),$$

$$d_5 = (\lambda + 2\mu)\lambda_1^{*2} - \lambda k^2,$$

$$d_6 = (\lambda + 2\mu)\lambda_2^{*2} - \lambda k^2,$$

$$a_2 = \left(\lambda_1^{*2} - k^2 + \frac{\omega^2}{c_1^2}\right)\frac{1}{q},$$

$$b_2 = \left(\lambda_2^{*2} - k^2 + \frac{\omega^2}{c_1^2}\right)\frac{1}{q},$$

$$\lambda_1^{*2} + \lambda_2^{*2} = 2k^2 - \omega\left(\frac{1}{c_1^2} + \frac{i}{c_4^2}\right) - i\omega s q,$$

$$\lambda_1^{*2}\lambda_2^{*2} = \left(\frac{\omega^2}{c_1^2} - k^2\right)\left(\frac{i\omega}{c_4^2} - k^2\right) - i\omega s c k^2. \tag{65}$$

(iii) If we neglect thermal effect, then (63) and (64) reduce to

$$d_7(b_1 d_3 - a_1 d_4) = 4\mu^2 k^2 \lambda_6^2 (b_1 - a_1), \tag{66}$$

$$d_7 S_7 \left(\frac{b_1 d_3}{S_3 \lambda_3^2} - \frac{a_1 d_4}{S_4 \lambda_4^2}\right) = 4\mu^2 k^2 (b_1 - a_1), \tag{67}$$

where

$$d_7 = (\lambda + 2\mu)\lambda_6^2 - \lambda k^2,$$

$$S_7 = \left(1 - \frac{\lambda_6^2 H^2}{3}\right),$$

$$\lambda_6^2 = \left(k^2 - \frac{\omega^2}{c_1^2}\right). \tag{68}$$

The above results agree with those obtained by Nowacki [12].

(iv) If we let $t_1 = 0$, $\delta_{1k} = 1$ in L–S and $t_0 = 0$ in G–L theory, (49)–(51), (52), (54) and (56) reduce to two period equations for symmetric and antisymmetric vibrations in micropolar coupled thermoelastic layer with stretch.

$$\left[(\gamma + \varepsilon) - \frac{\varepsilon_3}{\lambda_5}\right][4\mu^2 k^2 (a_2 - b_2)(b_1 - a_1)\lambda_1^{*2}\lambda_2^{*2} + (b_1 d_3 - a_1 d_4)(b_2 d_5 \lambda_2^{*2}) - a_2 d_6 \lambda_1^{*2}] = 0, \tag{69}$$

$$4\mu^2k^2(a_2-b_2)\left[(\gamma+\varepsilon)(b_1-a_1)+\varepsilon_3\lambda_5\left(\frac{a_1}{S_3\lambda_3^2}-\frac{b_1}{S_4\lambda_4^2}\right)\right]$$
$$+(b_2d_5S_5-a_2d_6S_6)\left[(\gamma+\varepsilon)\left(\frac{b_1d_3}{S_3\lambda_3^2}-\frac{a_1d_4}{S_4\lambda_4^2}\right)\right.$$
$$\left.+\frac{\varepsilon_3\lambda_5}{S_3S_4\lambda_3^2\lambda_4^2}(a_1d_4-b_1d_3)\right]=0. \tag{70}$$

(v) If we neglect thermal effect, from (69) and (70), then we have

$$\left[(\gamma+\varepsilon)-\frac{\varepsilon_3}{\lambda_5}\right][d_7(b_1d_3-a_1d_4)-4\mu^2k^2(b_1-a_1)\lambda_6^2]=0 \tag{71}$$

$$(\gamma+\varepsilon)d_7S_7\left(\frac{b_1d_3}{S_3^2\lambda_3^2}-\frac{a_1d_4}{S_4^2\lambda_4^2}\right)-4\mu^2k^2\,(\gamma+\varepsilon)(b_1-a_1)$$
$$-\frac{d_7\varepsilon_3S_7\lambda_5}{S_3S_4\lambda_3^2\lambda_4^2}(b_1d_3-a_1d_4)-4\mu^2k^2\varepsilon_3\lambda_5\left(\frac{a_1}{S_3\lambda_3^2}-\frac{b_1}{S_4\lambda_4^2}\right)=0. \tag{72}$$

Equations (71) and (72) represent the wave velocity equation for longitudinal vibration of micropolar elastic layer with stretch. The above results agree with those obtained by Kumar and Gogna [7].

Case 2.1.2. If wavelength is small as compared to the thickness of the layer, then (34), (35) and (37) (for symmetric vibrations) and (45)–(47) (for antisymmetric vibrations) reduce to a period equation as

$$(\gamma+\varepsilon)(b_1d_3\lambda_4-a_1d_4\lambda_3)(bd\lambda_2-ad_1\lambda_1)$$
$$+4\mu^2k^2(\gamma+\varepsilon)\lambda_1\lambda_2\lambda_3\lambda_4(a-b)(b_1-a_1)$$
$$+4\mu^2k^2\varepsilon_3\lambda_1\lambda_2(a-b)(a_1\lambda_4-b_1\lambda_3)$$
$$+\varepsilon_3(b_1d_3-a_1d_4)(ad_1\lambda_1-bd\lambda_2)=0. \tag{73}$$

(For L–S theory):

$$(\gamma+\varepsilon)(b_1d_3\lambda_4-a_1d_4\lambda_3)(b^*d^*m_2-a^*d_1^*m_1)$$
$$+4\mu^2k^2(\gamma+\varepsilon)m_1m_2\lambda_3\lambda_4(a^*-b^*)(b_1-a_1)$$
$$+4\mu^2k^2\varepsilon_3m_1m_2(a^*-b^*)(a_1\lambda_4-b_1\lambda_3)$$
$$+\varepsilon_3(b_1d_3-a_1d_4)(a^*d_1^*m_1-b^*d^*m_2)=0. \tag{74}$$

(For G–L theory):

$$(\gamma+\varepsilon)(b_1d_3\lambda_4-a_1d_4\lambda_3)(b'd'n_2-a'd_1'n_1)$$
$$+4\mu^2k^2n_1n_2\lambda_3\lambda_4(a'-b')(b_1-a_1)(\gamma+\varepsilon)$$
$$+4\mu^2k^2\varepsilon_3n_1n_2(a'-b')(a_1\lambda_4-b_1\lambda_3)$$
$$+\varepsilon_1(b_1d_3-a_1d_4)(a'd_1'n_1-b'd'n_2)=0. \tag{75}$$

The expressions (73)–(75) represent the modified Rayleigh wave in a micropolar generalized thermoelastic layer with stretch.

Special cases

(i) When we neglect stretch effect, the above equations (73)–(75) reduce for the case micropolar generalized thermoelastic layer as

$$4\mu^2 k^2 \lambda_1 \lambda_2 \lambda_3 \lambda_4 (a-b)(b_1-a_1) + (b_1 d_3 \lambda_4 - a_1 d_4 \lambda_3)(bd\lambda_2 - ad_1 \lambda_1) = 0. \tag{76}$$

(For L–S theory):

$$4\mu^2 k^2 m_1 m_2 \lambda_3 \lambda_4 (a^* - b^*)(b_1 - a_1) + (b_1 d_3 \lambda_4 - a_1 d_4 \lambda_3)(b^* d^* m_2 - a^* d_1^* m_1) = 0. \tag{77}$$

(For G–L theory):

$$4\mu^2 k^2 n_1 n_2 \lambda_3 \lambda_4 (a' - b')(b_1 - a_1) + (b_1 d_3 \lambda_4 - a_1 d_4 \lambda_3)(b' d' n_2 - a' d_1' n_1) = 0. \tag{78}$$

(ii) If we put $t_1 = 0$, $\delta_{lk} = 1$ in L–S and $t_0 = 0$ in G–L theory, then (76)–(78) reduces to the wave velocity equation for the micropolar coupled thermoelastic layer as

$$4\mu^2 k^2 \lambda_1^* \lambda_2^* \lambda_3 \lambda_4 (a_2 - b_2)(b_1 - a_1) + (b_1 d_3 \lambda_4 - a_1 d_4 \lambda_3)(b_2 d_5 \lambda_2^* - a_2 d_6 \lambda_1^*) = 0. \tag{79}$$

(iii) By neglecting the thermal effect from (79), we obtain

$$d_7 (b_1 d_3 \lambda_4 - a_1 d_4 \lambda_3) = 4\mu^2 k^2 (b_1 - a_1) \lambda_6 \lambda_3 \lambda_4. \tag{80}$$

Equation (80) is a wave velocity equation for micropolar elastic layer and it is in agreement with those obtained by Nowacki [12].

(iv) By putting $t_1 = 0$, $\delta_{lk} = 1$ in L–S and $t_0 = 0$ in G–L theory, we obtain from (73)–(75), the period equations for micropolar coupled thermoelastic layer with stretch

$$\begin{aligned}&(\gamma + \varepsilon)(b_1 d_3 \lambda_4 - a_1 d_4 \lambda_3)(b_2 d_5 \lambda_2^* - a_2 d_6 \lambda_1^*) \\ &\quad + 4\mu^2 k^2 (\gamma + \varepsilon) \lambda_1^* \lambda_2^* \lambda_3 \lambda_4 (a_2 - b_2)(b_1 - a_1) \\ &\quad + 4\mu^2 k^2 \varepsilon_3 \lambda_1^* \lambda_2^* (a_2 - b_2)(a_1 \lambda_4 - b_1 \lambda_3) \\ &\quad + \varepsilon_3 (b_1 d_3 - a_1 d_4)(a_2 d_6 \lambda_1^* - b^2 d_5 \lambda_2^*) = 0.\end{aligned} \tag{81}$$

(v) Neglecting the thermal effect from (81), we obtain the period equation for micropolar elastic layer with stretch

$$\begin{aligned}&d_7 (\gamma + \varepsilon)(b_1 d_3 \lambda_4 - a_1 d_4 \lambda_3) - 4\mu^2 k^2 (\gamma + \varepsilon)(b_1 - a_1) \lambda_6 \lambda_3 \lambda_4 \\ &\quad - d_7 \varepsilon_3 (b_1 d_3 - a_1 d_4) - 4\mu^2 k^2 \varepsilon_3 \lambda_6 (a_1 \lambda_4 - b_1 \lambda_3) = 0.\end{aligned} \tag{82}$$

The results (82) agree with that obtained by Kumar and Gogna [7].

3. Problem II. Rayleigh waves in micropolar generalized thermoelastic half-space with stretch

Here, we discuss the possibility of the existence of Rayleigh wave in a micropolar generalized thermoelastic half-space with stretch. For this purpose the plane boundary

is considered to be stress free surface. Therefore, the appropriate boundary conditions are as

$$t_{zz} = t_{zx} = m_{zy} = \lambda_z = 0 \quad \text{at } z = 0$$

$$\frac{\partial \theta}{\partial z} + h\theta = 0 \quad \text{at } z = 0. \tag{83}$$

We assume the solutions of (5)–(9) of the form

$$\begin{aligned}
\phi &= (A\mathrm{e}^{-\lambda_1 z} + B\mathrm{e}^{-\lambda_2 z})\mathrm{e}^{ik(x-ct)},\\
\theta &= (A_1\mathrm{e}^{-\lambda_1 z} + B_1\mathrm{e}^{-\lambda_2 z})\mathrm{e}^{ik(x-ct)},\\
\psi &= (C\mathrm{e}^{-\lambda_3 z} + D\mathrm{e}^{-\lambda_4 z})\mathrm{e}^{ik(x-ct)},\\
\omega_2 &= (C_1\mathrm{e}^{-\lambda_3 z} + D_1\mathrm{e}^{-\lambda_4 z})\mathrm{e}^{ik(x-ct)},\\
\phi^* &= E\mathrm{e}^{-\lambda_5 z}\mathrm{e}^{ik(x-ct)},
\end{aligned} \tag{84}$$

where A_1, A; $B_1 B$; C_1, C and D_1, D are related by

$$A_1 = aA, \quad B_1 = bB, \quad C_1 = a_1 C \quad \text{and } D_1 = b_1 D \tag{85}$$

where a, b, a_1 and b_1 are given by (19).

Substituting the values of $\phi, \psi, \omega_2, \theta, \phi^*$ from (84) in the boundary conditions (83) and making use of (21), we obtain the following equations

$$dA + d_1 B - 2i\mu k\lambda_3 C - 2i\mu k\lambda_4 D = 0, \tag{86}$$

$$2i\mu k\lambda_1 A + 2i\mu k\lambda_2 B + d_3 C + d_4 D = 0, \tag{87}$$

$$(\gamma + \varepsilon)\lambda_3 a_1 C + (\gamma + \varepsilon)\lambda_4 b_1 D + i\beta_0 kE = 0, \tag{88}$$

$$ika_1\beta_0 C + ikb_1\beta_0 D - 3\alpha_0\lambda_5 E = 0, \tag{89}$$

$$(h - \lambda_1)aA + (h - \lambda_2)bB = 0. \tag{90}$$

Solving (86)–(90), we obtain the frequency equation in a micropolar generalized thermoelastic half-space with stretch

$$\begin{aligned}
&4\mu^2 k^2 \lambda_1\lambda_2\varepsilon_3(a - b)(\lambda_3 b_1 - \lambda_4 a_1)\\
&\quad - 4\mu^2 k^2 \lambda_1\lambda_2\lambda_3\lambda_4(\gamma + \varepsilon)(a - b)(b_1 - a_1)\\
&\quad - 4\mu^2 k^2 h\varepsilon_3(\lambda_3 b_1 - \lambda_4 a_1)(\lambda_2 a - \lambda_1 b)\\
&\quad + 4\mu^2 k^2 h\lambda_3\lambda_4(b_1 - a_1)(\lambda_2 a - \lambda_1 b)\\
&\quad + [bd(\lambda_2 - h) - ad_1(\lambda_1 - h)][\varepsilon_3(a_1 d_4 - b_1 d_3)\\
&\quad + (\gamma + \varepsilon)(b_1 d_3\lambda_4 - a_1 d_4\lambda_3)] = 0.
\end{aligned} \tag{91}$$

(For L–S theory):

The above equation (91) becomes

$$\begin{aligned}
&4\mu^2 k^2 m_1 m_2\varepsilon_3(a^* - b^*)(\lambda_3 b_1 - \lambda_4 a_1)\\
&\quad - 4\mu^2 k^2 m_1 m_2\lambda_3\lambda_4(\gamma + \varepsilon)(a^* - b^*)(b_1 - a_1)\\
&\quad - 4\mu^2 k^2 h\varepsilon_3(\lambda_3 b_1 - \lambda_4 a_1)(m_2 a^* - m_1 b^*)
\end{aligned}$$

$$+4\mu^2k^2h\lambda_3\lambda_4(b_1-a_1)(m_2a^*-m_1b^*)$$
$$+[b^*d^*(m_2-h)-a^*d_1^*(m_1-h)][\varepsilon_3(a_1d_4-b_1d_3)$$
$$+(\gamma+\varepsilon)(b_1d_3\lambda_4-a_1d_4\lambda_3)]=0. \tag{92}$$

(For G–L theory)

Equation (91) takes the form

$$4\mu^2k^2n_1n_2\varepsilon_3(a'-b')(\lambda_3b_1-\lambda_4a_1)$$
$$-4\mu^2k^2n_1n_2\lambda_3\lambda_4(\gamma+\varepsilon)(a'-b')(b_1-a_1)$$
$$-4\mu^2k^2h\varepsilon_3(\lambda_3b_1-\lambda_4a_1)(n_2a'-n_1b')$$
$$+4\mu^2k^2h\lambda_3\lambda_4(b_1-a_1)(n_2a'-n_1b')$$
$$+[b'd'(n_2-h)-a'd_1'(n_1-h)][\varepsilon_3(a_1d_4-b_1d_3)$$
$$+(\gamma+\varepsilon)(b_1d_3\lambda_4-a_1d_4\lambda_3)]=0. \tag{93}$$

Equations (91), (92) and (93) determine the wave velocities of Rayleigh type waves in a micropolar generalized thermoelastic medium with stretch.

3.1.1 *Special cases*

(i) If we neglect thermal effect, then the frequency equations (91)–(93) reduce to

$$4\mu^2k^2\lambda_6\lambda_3\lambda_4(\gamma+\varepsilon)(b_1-a_1)-4\mu^2k^2\lambda_6\varepsilon_3(\lambda_3b_1-\lambda_4a_1)$$
$$+d_7[\varepsilon_3(a_1d_4-b_1d_3)+(\gamma+\varepsilon)(b_1d_3\lambda_4-a_1d_4\lambda_3)]=0. \tag{94}$$

Equation (94) is a wave velocity equation in a micropolar elastic half-space with stretch.

(ii) By neglecting stretch effect, (91)–(93) reduce to the wave velocity equations in a micropolar generalized thermoelastic half-space

$$4\mu^2k^2h\lambda_3\lambda_4(b_1-a_1)(\lambda_2a-\lambda_1b)$$
$$-4\mu^2k^2\lambda_1\lambda_2\lambda_3\lambda_4(\gamma+\varepsilon)(a-b)(b_1-a_1)$$
$$+(\gamma+\varepsilon)(b_1d_3\lambda_4-a_1d_4\lambda_3)[bd(\lambda_2-h)-ad_1(\lambda_1-h)]=0. \tag{95}$$

(For L–S theory):

$$4\mu^2k^2h\lambda_3\lambda_4(b_1-a_1)(m_2a^*-m_1b^*)$$
$$-4\mu^2k^2m_1m_2\lambda_3\lambda_4(\gamma+\varepsilon)(a^*-b^*)(b_1-a_1)$$
$$+(\gamma+\varepsilon)(b_1d_3\lambda_4-a_1d_4\lambda_3)[b^*d^*(m_2-h)-a^*d_1^*(m_1-h)]=0. \tag{96}$$

(For G–L theory):

$$4\mu^2k^2h\lambda_3\lambda_4(b_1-a_1)(n_2a'-n_1b')$$
$$-4\mu^2k^2n_1n_2\lambda_3\lambda_4(\gamma+\varepsilon)(a'-b')(b_1-a_1)$$
$$+(\gamma+\varepsilon)(b_1d_3\lambda_4-a_1d_4\lambda_3)[b'd'(n_2-h)-a'd_1'(n_1-h)]=0. \tag{97}$$

(iii) By neglecting the thermal effect from the above equation, we obtain

$$4\mu^2 k^2 \lambda_6 \lambda_3 \lambda_4 (b_1 - a_1) = d_7 (a_1 \lambda_3 d_4 - b_1 \lambda_4 d_3). \tag{98}$$

Equation (98) is the frequency equation for Rayleigh waves in a micropolar elastic half-space and it is in agreement with the result obtained by Sengupta and De [14].

(iv) If we let $t_1 = 0$, $\delta_{lk} = 1$ in L–S and $t_0 = 0$ in G–L theory, eqs (91)–(93) reduce to a frequency equation in micropolar coupled thermoelasticity with stretch

$$\begin{aligned}
&4\mu^2 k^2 \lambda_1^* \lambda_2^* \varepsilon_3 (a_2 - b_2)(\lambda_3 b_1 - \lambda_4 a_1) \\
&- 4\mu^2 k^2 \lambda_1^* \lambda_2^* \lambda_3 \lambda_4 (\gamma + \varepsilon)(a_2 - b_2)(b_1 - a_1) \\
&- 4\mu^2 k^2 h \varepsilon_3 (\lambda_3 b_1 - \lambda_4 a_1)(\lambda_2^* \mathrm{a}_2 - \lambda_1^* \mathrm{b}_2) \\
&+ 4\mu^2 k^2 h \lambda_3 \lambda_4 (b_1 - a_1)(\lambda_2^* a_2 - \lambda_1^* b_2) \\
&+ [b_2 d_5 (\lambda_2^* - h) - a_2 d_6 (\lambda_1^* - h)][\varepsilon_3 (a_1 d_4 - b_1 d_3) \\
&+ (\gamma + \varepsilon)(b_1 d_3 \lambda_4 - a_1 d_4 \lambda_3)] = 0.
\end{aligned} \tag{99}$$

(v) If we neglect stretch effect, from (99), we obtain

$$\begin{aligned}
&4\mu^2 k^2 h \lambda_3 \lambda_4 (b_1 - a_1)(\lambda_2^* a_2 - \lambda_1^* b_2) \\
&- 4\mu^2 k^2 \lambda_1^* \lambda_2^* \lambda_3 \lambda_4 (\gamma + \varepsilon)(a_2 - b_2)(b_1 - a_1) \\
&+ [b_2 d_5 (\lambda_2^* - h) - a_2 d_6 (\lambda_1^* - h)](\gamma + \varepsilon)(b_1 d_3 \lambda_4 - a_1 d_4 \lambda_3) = 0.
\end{aligned} \tag{100}$$

Equation (100) gives the phase velocity in micropolar coupled thermoelastic half-space.

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New families of graceful banana trees

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Abstract. Consider a family of stars. Take a new vertex. Join one end-vertex of each star to this new vertex. The tree so obtained is known as a *banana tree*. It is proved that the banana trees corresponding to the family of stars

i) $(K_{1,1}, K_{1,2}, \ldots, K_{1,t-1}, (\alpha+1)K_{1,t}, K_{1,t+1}, \ldots, K_{1,n})$, $\alpha \geqslant 0$,

ii) $(2K_{1,1}, 2K_{1,2}, \ldots, 2K_{1,t-1}, (\alpha+2)K_{1,t}, 2K_{1,t+1}, \ldots, 2K_{1,n})$, $0 \leqslant \alpha < t$ and

iii) $(3K_{1,1}, 3K_{1,2}, \ldots, 3K_{1,n})$

are graceful.

Keywords. Trees; graceful.

1. Introduction

A graph G with p vertices and q edges is *graceful* if there is an injective map ϕ from the vertex set V of G into $(0, 1, \ldots, q)$ such that the induced map $\bar{\phi}$ from edge set E of G into $(1, 2, \ldots, q)$ defined by $\bar{\phi}(e) = |\phi(u) - \phi(v)|$ where $e = uv$, is surjective. Such a ϕ is known as a *graceful labeling of G*, $\bar{\phi}(e)$ is known as *weight of e induced by* ϕ. A *tree* is a connected acyclic graph. For a tree $q = p - 1$. A well-known conjecture due to Ringel and Kotzig [3, 2] is that all trees are graceful. This conjecture is still unsettled. A *banana tree* [1] is one obtained from a family of stars by joining one end-vertex of each star to a new vertex.

We prove that a banana tree corresponding to the family of stars $(K_{1,1}, K_{1,2}, \ldots, K_{1,t-1}, (\alpha+1)K_{1,t}, K_{1,t+1}, \ldots, K_{1,n})$, $0 \leqslant \alpha$ is graceful when $\alpha < t$. Using this result we show that a banana tree obtained from the family of stars $(2K_{1,1}, 2K_{1,2}, \ldots, 2K_{1,t-1}, (\alpha+2)K_{1,t}, 2K_{1,t+1}, \ldots, 2K_{1,n})$ is graceful when $0 \leqslant \alpha < t$. Chen, Lü and Yeh [1] have proved that a banana tree obtained from the family of stars $(2K_{1,1}, 2K_{1,2}, \ldots, 2K_{1,n})$ is graceful. We give a different graceful labeling for this banana tree and use it to establish that a banana tree obtained from the family of stars $(3K_{1,1}, 3K_{1,2}, \ldots, 3K_{1,n})$ is graceful. The banana tree obtained from the family of stars $(K_{1,1}, K_{1,2}, \ldots, K_{1,n})$ is known as a *standard banana tree* [1] and we denote it by SB^n. We also establish that a banana tree corresponding to the family of stars $(K_{1,1}, K_{1,2}, \ldots, K_{1,t-1}, (\alpha+1)K_{1,t}, K_{1,t+1}, \ldots, K_{1,n})$, $\alpha \geqslant t$, is graceful. However, our graceful labeling here is not amenable to combining this banana tree with the standard banana tree SB^n to get a gracefully labeled banana tree which has $(2K_{1,1}, 2K_{1,2}, \ldots, 2K_{1,t-1}, (\alpha+2)K_{1,t}, 2K_{1,t+1}, \ldots, 2K_{1,n})$ as the family of stars when $\alpha \geqslant t$ unlike the case when $\alpha < t$.

We use the following terminology. Consider a banana tree obtained from the family of stars $(K_{1,x_1}, K_{1,x_2}, \ldots, K_{1,x_m})$. The new vertex to which these stars are attached is called the *apex*. The end-vertices of these stars which are joined to the apex are called

Table 1. Graceful labeling of $SB''(t,\alpha)$, $\alpha < t$.

Star	$K_{1,1}$	$K_{1,2}$	$K_{1,3}$	$\cdots$	$K_{1,n-i}$	$\cdots$	$K_{1,n-1}$	$K_{1,n}$	$K^{*_1}_{1,t}$	$K^{*_2}_{1,t}$	$\cdots$	$K^{*_{t-1}}_{1,t}$	$K^{*_t}_{1,t}$
Central vertex label	1	2	3	$\cdots$	$n-i$	$\cdots$	$n-1$	$2n+\alpha t$	$2n+\alpha t+1$	$2n+\alpha t+2$	$\cdots$	$2n+\alpha t$ $+(\alpha-1)$	$2n+\alpha t+\alpha$
Range of end vertex labels		β_1+5	β_1+2 to β_1+3	$\cdots$	$(i+1)n+\alpha t+\alpha$ $-i(i-1)/2+2$ to $(i+2)n+\alpha t+\alpha$ $-i(i+1)/2$	$\cdots$	$2n+\alpha t$ $+\alpha+2$ to $3n+\alpha t$ $-1+\alpha$	$n+\alpha t$ to $2n+\alpha t-1$ except	$n+(\alpha-1)t$ to $n+\alpha t-1$ except	$n+(\alpha-2)t$ to $n+(\alpha-1)t-1$ except	$\cdots$	$n+t$ to $n+2t-1$ except	n to $n+t-1$ except
Link vertex labels	β_1+6	β_1+4	β_1+1	$\cdots$	$(i+1)n+\alpha t+\alpha$ $-(i(i-1)/2)+1$	$\cdots$	$2n+\alpha t$ $+\alpha+1$	$n+\alpha t$ $+(\alpha+1)$	$n+(\alpha-1)t$ $+\alpha$	$n+(\alpha-2)t$ $+(\alpha-1)$	$\cdots$	$n+t+2$	$n+1$
Star edge weight range	β_1+5	β_1+2 to β_1+3	β_1-2 to β_1	$\cdots$	$in+\alpha t+\alpha-(i(i-1)/2)$ $+i+1$ to $(i+1)n+\alpha t+\alpha$ $-(i(i+1)/2)+i$	$\cdots$	$n+\alpha t$ $+\alpha+2$ to $2n+\alpha t+\alpha$	1 to n	$n+2$ to $n+t+1$	$n+t+3$ to $n+2t+2$	$\cdots$	$n+(\alpha-1)t$ $+(\alpha-1)$ to $n+(\alpha-2)t$ $+(\alpha-1)-1$	$n+\alpha t+\alpha$ to $n+(\alpha-1)t$ $+(\alpha+1)$

Note: i) Link vertex label is the same as link edge weight; ii) $\beta_1 = q-6 = (n(n+1)/2)+\alpha t+(n+\alpha)-6$; iii) Except means except the link vertex label.

link vertices. The vertex of K_{1,x_i} of degree x_i is called its *central vertex*. Note that all vertices of K_{1,x_i} other than the central vertex and the link vertex are end-vertices of the banana tree.

2. Graceful labeling of $\mathrm{SB}^n(t, \alpha)$, $\alpha < t$

The banana tree obtained from the family of stars $(K_{1,1}, K_{1,2}, \ldots, K_{1,t-1}, (\alpha+1)K_{1,t}, K_{1,t+1}, \ldots, K_{1,n})$ is denoted by $\mathrm{SB}^n(t,\alpha)$, where $\alpha < t$. Clearly, $\mathrm{SB}^n(t,\alpha)$ is a tree with $q = (n(n+1)/2) + n + \alpha t + \alpha$ edges. The star $K_{1,t}$ is repreated $\alpha + 1$ times in the family. We denote them by $K_{1,t}, K^{*_1}_{1,t}, K^{*_2}_{1,t}, \ldots, K^{*_\alpha}_{1,t}$.

Table 1 indicates the labeling of $\mathrm{SB}^n(t,\alpha)$, $\alpha < t$ and also the induced weights of the q edges and clearly points out that this labeling is graceful. Label 0 is given to the apex.

Figure 1 shows a graceful labeling of $\mathrm{SB}^7(5,3)$ given as per table 1. For convenience, the apex and the link edges are not drawn. The topmost vertex of each star is its link vertex.

3. Graceful labeling of $2 - \mathrm{SB}^n(t, \alpha)$, $\alpha < t$

The banana tree corresponding to the family of stars $(2K_{1,1}, 2K_{1,2}, \ldots, 2K_{1,t-1}, (\alpha+2)K_{1,t}, 2K_{1,t+1}, \ldots, 2K_{1,n})$ is denoted by $2 - \mathrm{SB}^n(t,\alpha)$ where $\alpha < t$. The banana tree $2 - \mathrm{SB}^n(t,\alpha)$ can be thought of as the one obtained from the standard banana tree SB^n and the banana tree $\mathrm{SB}^n(t,\alpha)$, $\alpha < t$, by identifying their apex vertices. Chen, Lü and Yeh [1] have given a graceful (in fact interlaced) labeling of SB^n which is given in table 2. Figure 2 gives a graceful labeling of SB^7.

Now we give an algorithm for obtaining a graceful labeling of $2 - \mathrm{SB}^n(t,\alpha)$, $\alpha < t$, using the graceful labelings of $\mathrm{SB}^n(t,\alpha)$ and SB^n as given in tables 1 and 2.

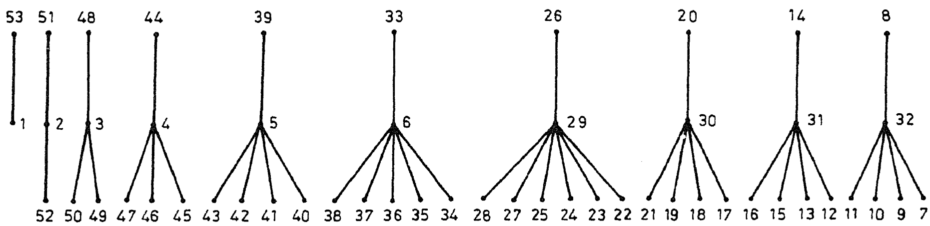

Figure 1. Graceful labeling of $\mathrm{SB}^7(5,3)$.

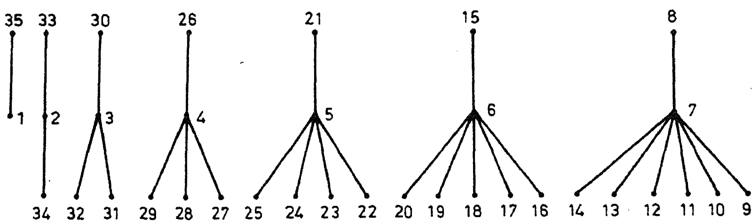

Figure 2. Graceful labeling of SB^7.

Table 2. Graceful labeling of SB″.

Star	$K_{1,1}$	$K_{1,2}$	$K_{1,3}$	$\cdots$	$K_{1,n-i}$	$\cdots$	$K_{1,n-2}$	$K_{1,n-1}$	$K_{1,n}$
Central vertex label	1	2	3	$\cdots$	$n-i$	$\cdots$	$n-2$	$n-1$	n
Range of end vertex labels		$((n^2+3n)/2)-1$	$((n^2+3n)/2)-3$ to $((n^2+3n)/2)-4$	$\cdots$	$n+1+n'+1$ to $(n+1)+n'+n-(i+1)$	$\cdots$	$3n+1$ to $4n-3$	$2n+2$ to $3n-1$	$n+2$ to $2n$
Link vertex labels	$((n^2+3n)/2)$	$((n^2+3n)/2)-2$	$((n^2+3n)/2)-5$	$\cdots$	$n+1+n'$	$\cdots$	$3n$	$2n+1$	$n+1$
Edge weight range	$((n^2+3n)/2)-1$	$((n^2+3n)/2)-3$ to $((n^2+3n)/2)-4$	$((n^2+3n)/2)-6$ to $((n^2+3n)/2)-8$	$\cdots$	$n'+i+1$ to $n'+n$	$\cdots$	$2n+2$ to $3n-1$	$n+2$ to $2n$	1 to n

Note: i) Link vertex label is the same as link edge weight; ii) Here $n'=n'(i)=n+(n-1)+(n-2)+\cdots+(n-(i-1))$; iii) $q(\mathrm{SB}'')=(n^2+3n)/2$.

Algorithm I

Step 1. Identify the apex vertices of SB^n and $\mathrm{SB}^n(t,\alpha)$, $\alpha < t$. Give label n to the apex of $2-\mathrm{SB}^n(t,\alpha)$ thus obtained.
Step 2. Add n to each label of $\mathrm{SB}^n(t,\alpha)$ except to the apex which is now the apex of $2-\mathrm{SB}^n(t,\alpha)$ and is covered by step 1.
Step 3. For all the central vertices of SB^n change label x to $n-x$.
Step 4. For all other vertices of SB^n change label x to $n^2+4n+\alpha(t+1)+1-x$.

Remark. Note that $q(2-\mathrm{SB}^n(t,\alpha)) = n^2+3n+\alpha(t+1)$. Also note that $x \geqslant n+1$.

Table 3 gives the edge weights of $2-\mathrm{SB}^n(t,\alpha)$, $\alpha < t$, when this algorithm is executed. Table 3 does not include the edge weights of edges from $\mathrm{SB}^n(t,\alpha)$ part of $2-\mathrm{SB}^n(t,\alpha)$ since they are exactly the same as those of $\mathrm{SB}^n(t,\alpha)$ and these are already listed in table 1. These account for weights 1 to $((n^2+3n)/2)+\alpha(t+1) = q(\mathrm{SB}^n(t,\alpha))$.

Let $\beta_2 = ((n^2+3n)/2)+\alpha(t+1) = q(\mathrm{SB}^n(t,\alpha))$. The edge weights (β_2+1) to $(q(2-\mathrm{SB}^n(t,\alpha) = \beta_2((n^2+3n)/2)$ are covered by the edges of SB^n part of $2-\mathrm{SB}^n(t,\alpha)$.

Figure 3 gives a graceful labeling of $2-\mathrm{SB}^7(5,3)$ obtained by our technique. Figure 3 is to be read with figures 1 and 2. As before, the apex and link edges are not shown. The apex gets label 7.

4. Graceful labeling of $3-\mathrm{SB}^n$

The banana tree corresponding to the family of stars $(3K_{1,1}, 3K_{1,2}, \ldots, 3K_{1,n})$ is denoted by $3-\mathrm{SB}^n$. The banana tree $3-\mathrm{SB}^n$ can be thought of as the one obtained from SB^n and $2-\mathrm{SB}^n$ by identifying their apex vertices. Here $2-\mathrm{SB}^n$ denotes the banana tree corresponding to the family of stars $(2K_{1,1}, 2K_{1,2}, \ldots, 2K_{1,n})$. Chen, Lü and Yeh [1] have given a graceful (in fact interlaced) labeling of $2-\mathrm{SB}^n$. Interlaced labeling is a specialized graceful labeling. However, their graceful labeling of $2-\mathrm{SB}^n$ and their technique of combining interlaced banana trees to get a larger interlaced, and hence graceful, banana tree does not give a graceful labeling of $3-\mathrm{SB}^n$. The reason is, in their interlaced labeling of $2-\mathrm{SB}^n$, the apex receives the label n (and not 0).

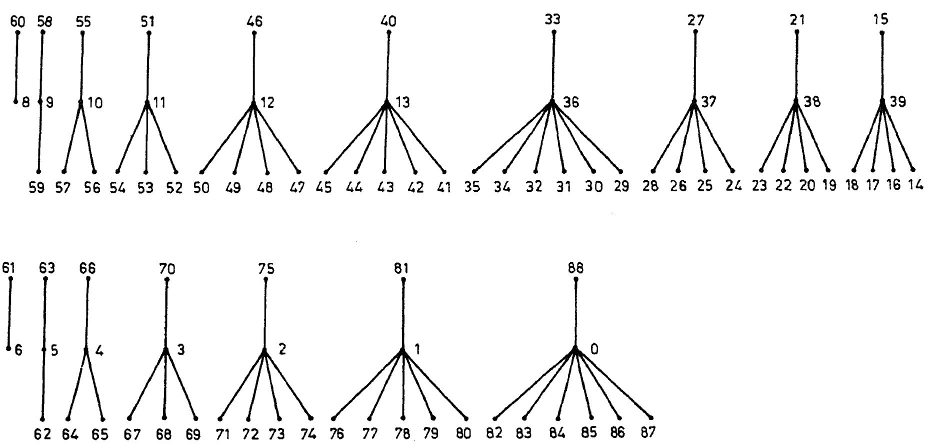

Figure 3. Graceful labeling of $2-\mathrm{SB}^7(5,3)$. The apex gets label 7.

Table 3. Edge weights of edges in SB$''$ part of $2 - \mathrm{SB}''(t, \alpha)$.

Stars	$K_{1,1}$	$K_{1,2}$	$K_{1,3}$	$\cdots$	$K_{1,n-i}$	$\cdots$	$K_{1,n-2}$	$K_{1,n-1}$	$K_{1,n}$
Link edge weight	$\beta_2 + 1$	$\beta_2 + 3$	$\beta_2 + 6$	$\cdots$	$\beta_2(n^2 + 3n)/2 - n' - n$	$\cdots$	$\beta_2 + (n^2 + 3n)/2 - 3n + 1$	$\beta_2 + (n^2 + 3n)/2 - 2n$	$\beta_2 + (n^2 + 3n)/2 - n$
Range of star edge weights	$\beta_2 + 2$	$\beta_2 + 4$ to $\beta_2 + 5$	$\beta_2 + 7$ to $\beta_2 + 9$	$\cdots$	$\beta_2 + (n^2 + 3n)/2 - n' - n + i - 1$ to $\beta_2 + (n^2 + 3n)/2 - n' - n + 1$	$\cdots$	$\beta_2 + (n^2 + 3n)/2 - 2n - 1$ to $\beta_2 + (n^2 + 3n)/2 - 3n + 2$	$\beta_2 + (n^2 + 3n)/2 - n - 1$ to $\beta_2 + (n^2 + 3n)/2 - 2n + 1$	$\beta_2 + (n^2 + 3n)/2$ to $\beta_2 + (n^2 + 3n)/2 - n + 1$

Note: i) $\beta_2 = ((n^2 + 3n)/2) + \alpha(t + 1)$; ii) $n' = n'(i) = n + (n - 1) + (n - 2) + \cdots + (n - (i - 1))$.

Table 4.1. Graceful labeling of $2 - \mathrm{SB}''$.

Star	$K_{1,1}$	$K_{1,2}$	$K_{1,3}$	$\cdots$	$K_{1,i}$	$\cdots$	$K_{1,n-1}$	$K_{1,n}$
Central vertex label	1	2	3	$\cdots$	i	$\cdots$	$n - 1$	n
Range of end vertex labels		$q - 1$	$q - 3$ to $q - 4$	$\cdots$	$q - (i(i - 1)/2)$ to $q - (i(i - 1)/2) - (i - 2)$	$\cdots$	$q - ((n - 1)(n - 2)/2)$ to $q - ((n - 1)(n - 2)/2) - (n - 3)$	$q - (n(n - 1)/2)$ to $q - (n(n - 1)/2) - (n - 2)$
Link vertex label	q	$q - 2$	$q - 5$	$\cdots$	$q - (i(i - 1)/2 - (i - 1)$	$\cdots$	$q - ((n - 1)(n - 2)/2) - (n - 2)$	$q - (n(n - 1)/2) - (n - 1)$
Star edge weight range	$q - 1$	$q - 3$ to $q - 4$	$q - 6$ to $q - 8$	$\cdots$	$q - (i(i - 1)/2) - i$ to $q - (i(i - 1)/2) - 2i + 1$	$\cdots$	$q - ((n - 1)(n - 2)/2) - (n - 1)$ to $q - ((n - 1)(n - 2)/2) - 2(n - 1) - 1$	$q - (n(n - 1)/2) - n$ to $q - (n(n - 1)/2) - 2n + 1$

Note: Link vertex label is the same as link edge weight.

Table 4·2. (even)

Star	$K_{1,2}$	$K_{1,4}$	$K_{1,6}$	…	$K_{1,n-i}$	…	$K_{1,n-4}$	$K_{1,n-2}$	$K_{1,n}$
Central vertex label	$\beta_3+((n-2)/2)$	$\beta_3+((n-4)/2)$	$\beta_3+((n-6)/2)$	…	$\beta_3+(i/2)$	…	β_3+2	β_3+1	β_3
Range of end vertex labels	$\beta_3-(n(n-2)/2)$ $+((n-2)(n-4)/4)$ -2	$\beta_3-(n(n-4)/2)$ $+((n-4)(n-6)/4)-2$ to $\beta_3-(n(n-2)/2)$ $+((n-2)(n-4)/4)$	$\beta_3-(n(n-6)/2)$ $+((n-6)(n-8)/4)-2$ to $\beta_3-(n(n-4)/2)$ $+((n-4)(n-6)/4)$	…	$\beta_3-(i/2)n$ $+(i(i-2)/4)-2$ to $\beta_3-((i/2)+1)n$ $+(i(i+2)/4)$	…	β_3-2n to β_3-3n+6	β_3-n-2 to β_3-2n+2	β_3-2 to β_3-n
Link vertex label	$\beta_3-(n(n-2)/2)$ $+((n-2)(n-4)/4)$ -1	$\beta_3-(n(n-4)/2)$ $+((n-4)(n-6)/4)$ -1	$\beta_3-(n(n-6)/2)$ $+((n-6)(n-8)/4)$ -1	…	$\beta_3-in/2$ $+(i(i-2)/4)-1$	…	β_3-2n+1	β_3-n-1	β_3-1
Star edge weight range	$n(n-2)/2+(n-2)/2$ $-((n-2)(n-4)/4)+1$ to $n(n-2)/2+(n-2)/2$ $-((n-2)(n-4)/4)+2$	$n(n-4)/2+(n-4)/2$ $-((n-4)(n-6)/4)+1$ to $n(n-2)/2+(n-4)/2$ $-((n-4)(n-2)/4)$	$n(n-6)/2+(n-6)/2$ $-((n-6)(n-8)/4)+1$ to $n(n-4)/2+(n-6)/2$ $-((n-4)(n-6)/4)$	…	$(i/2)n-i(i-2)/4$ $+(i/2)+1$ to $(i/2+1)n+(i/2)$ $-(i(i+2)/4)$	…	$2n+1$ to $3n-4$	$n+2$ to $2n-1$	1 to n

Note: i) Link vertex label is the same as link edge weight; ii) $\beta_3=q-(n(n-1)/2)-2n+1$.

Table 4.2. (Odd)

Stars	$K_{1,1}$	$K_{1,3}$	$K_{1,5}$	$\cdots$	$K_{1,n-i-1}$	$K_{1,n-i+1}$	$\cdots$	$K_{1,n-1}$
Central vertex label	β_5	β_5+1	β_5+2	$\cdots$	$\beta_5+((n-i-2)/2)$	$\beta_5+((n-i)/2)$	$\cdots$	$\beta_5+(n-2)/2$
Range of end vertex labels		β_4-4 to β_4-5	β_4-7 to β_4-10	$\cdots$	$\beta_4-h(n-i-3)-2$ to $\beta_4-h(n-i-1)$	$\beta_4-(h(n-i-1)-2$ to $\beta_4-h(n-i+1)$	$\cdots$	$\beta_4-h(n-3)-2$ to $\beta_4-h(n-1)$
Link vertex label	β_4-3	β_4-6	β_4-11	$\cdots$	$\beta_4-h(n-i-1)-1$	$\beta_4-h(n-i+1)-1$	$\cdots$	$\beta_4-h(n-1)-1$
Star edge weight range	$\beta_5-\beta_4+3$	$\beta_5-\beta_4+5$ to $\beta_5-\beta_4+7$	$\beta_5-\beta_4+9$ to $\beta_5-\beta_4+13$	$\cdots$	$\beta_5-\beta_4+((n-i-2)/2)$ $+h(n-i-3)+2$ to $\beta_5-\beta_4+((n-i-2)/2)$ $+h(n-i-1)+1$	$\beta_5-\beta_4+((n-i)/2)$ $+h(n-i-1)+2$ to $\beta_5-\beta_4+(n-i)/2$ $+h(n-i+1)+1$	$\cdots$	$\beta_5-\beta_4+(n-2)/2$ $+h(n-3)+2$ to $\beta_5-\beta_4+h(n-1)+1$

Note: i) Link vertex label is the same as link edge weight; ii) $\beta_4=\beta_3-(n(n-2)/2)+((n-2)(n-4)/4)$; iii) $\beta_5=\beta_3+((n-2)/2)+1$; iv) $h(r)=2+(3+5+7+\cdots+r)$.

Table 4.3. (even)

Stars	$K_{1,2}$	$K_{1,4}$	$K_{1,6}$	$K_{1,8}$	...	$K_{1,n-i-1}$	$K_{1,n-i+1}$	...	$K_{1,n-1}$
Central vertex label	β_7	β_7+1	β_7+2	β_7+3	...	$\beta_7+(n-i-3)/2$	$\beta_7+(n-i-1)/2$	...	$\beta_7+(n-3)/2$
Range of end vertex labels	β_6-1	β_3-3 to β_6-5	β_6-7 to β_6-11	β_6-13 to β_6-19	...	$\beta_6-h(n-i-3)-1$ to $\beta_6-h(n-i-1)+1$	$\beta_6-h(n-i-1)-1$ to $\beta_6-h(n-i+1)+1$	...	$\beta_6-h(n-3)-1$ to $\beta_6-h(n-1)+1$
Link vertex label	β_6-2	β_6-6	β_6-12	β_6-20	...	$\beta_6-h(n-i-1)$	$\beta_6-h(n-i+1)$	...	$\beta_6-h(n-1)$
Star edge weight range	$\beta_7-\beta_6+1$ to $\beta_7-\beta_6+2$	$\beta_7-\beta_6+4$ to $\beta_7-\beta_6+7$	$\beta_7-\beta_6+9$ to $\beta_7-\beta_6+14$	$\beta_7-\beta_6+16$ to $\beta_7-\beta_6+23$	... ...	$\beta_7-\beta_6+(n-i-3)/2$ $+h(n-i-3)+1$ to $\beta_7-\beta_6+(n-i-3)/2$ $+h(n-i-1)$	$\beta_7-\beta_6+(n-i-1)/2$ $+h(n-i-1)+1$ to $\beta_7-\beta_6+h(n-i+1)$ $+(n-i-1)/2$	... ...	$\beta_7-\beta_6+h(n-3)$ $+(n-3)/2+1$ to $\beta_7-\beta_6+h(n-1)$ $+(n-3)/2$

Note: i) Link vertex label is the same as link edge weight; ii) $\beta_6=\beta_3-(n(n-3)/2)+((n-3)(n-5)/4)-4$; iii) $\beta_7=\beta_3+((n-1)/2)+1$; iv) $h(r)=2+4+6+8+\cdots+r$.

Table 4.3. (Odd)

Stars	$K_{1,1}$	$K_{1,3}$	$K_{1,5}$	…	$K_{1,n-i}$	…	$K_{1,n-2}$	$K_{1,n}$
Central vertex label	$\beta_3+(n-1)/2$	$\beta_3+(n-3)/2$	$\beta_3+(n-5)/2$	…	$\beta_3+(i/2)$	…	β_3+1	β_3
Range of end vertex labels		$\beta_3-(n(n-3)/2)$ $+((n-2)(n-5)/4)-2$ to $\beta_3-(n(n-3)/2)$ $+((n-2)(n-5)/4)-3$	$\beta_3-(n(n-5)/2)$ $+((n-5)(n-7)/4)-2$ to $\beta_3-(n(n-3)/2)$ $+((n-3)(n-5)/4)$	…	$\beta_3-(in/2)$ $-(i(i-2)/4)-2$ to $\beta_3-((i/2)+1)n$ $+(i(i+2)/4)$	…	β_3-n-2 to β_3-2n+2	β_3-2 to β_3-n
Link vertex label	$\beta_3-(n(n-3)/2)$ $+((n-2)(n-5)/4)$ -4	$\beta_3-(n(n-3)/2)$ $+((n-2)(n-5)/4)$ -1	$\beta_3-(n(n-5)/2)$ $+((n-5)(n-7)/4)$ -1	…	$\beta_3-(in/2)-(i(i-2)/4)$ -1	…	β_3-n-1	β_3-1
Star edge weight range	$(n(n-3)/2)+((n-1)/2)$ $-((n-2)(n-5)/4)+4$	$((n-3)(n+1)/2)$ $-((n-2)(n-5)/4)+1$ to $((n-3)(n+1)/2)$ $-((n-2)(n-5)/4)+3$	$((n-5)(n+1)/2)$ $-((n-5)(n-7)/4)+1$ to $(n-5)/2+(n(n-3)/2)$ $-((n-3)(n-5)/4)$	…	$(i/2)+(in/2)$ $-(i(i-2)/4)+1$ to $(i/2)+((i/2)+1)n$ $-(i(i+2)/4)$	…	$n+2$ to $2n-1$	1 to n

Note: i) Link vertex label is the same as link edge weight; ii) $\beta_3=q-(n(n-1)/2)-2n+1$.

Here we give a different graceful (but not interlaced) labeling of $2-SB^n$ in which the apex gets the label 0. This new labeling depends on the parity of n. We consider $2-SB^n$ as two copies of SB^n with the same apex. Table 4.1 gives a labeling of one copy of SB^n, table 4.2 gives the other copy of SB^n when n is even and table 4.3 the other copy of SB^n when n is odd. Table 4.2 has, in fact, two parts. We have called them table 4.2 (even) and table 4.2 (odd). Table 4.2 (even) indicates the labeling for even sized stars involved in SB^n, that is, for $K_{1,2}, K_{1,4}, \ldots$. Table 4.2 (odd) indicates the labeling for odd sized stars involved in SB^n, that is, for $K_{1,1}, K_{1,3}, \ldots$. Similarly we have table 4.3 (even) and table 4.3 (odd). Figure 4.1 shows a graceful labeling of $2-SB^8$ while figure 4.2 shows a graceful labeling of $2-SB^7$. We have included a graceful labeling of SB^8 to be used for that of $2-SB^8$.

Now we give an algorithm to gracefully label $3-SB^n$ where $3-SB^n$ is obtained from SB^n and $2-SB^n$ by identifying their apex vertices.

Algorithm II

Step 1. Do the graceful labeling of SB^n part of $3-SB^n$ as per table 2.
Step 2. Do the graceful labeling of $2-SB^n$ part of $3-SB^n$ as per tables 4.1, 4.2 and 4.3.
Step 3. Identify the apex of SB^n with that of $2-SB^n$ and give label n to it.
Step 4. Add n to each label of $2-SB^n$ (except to the apex which is covered by step 3).

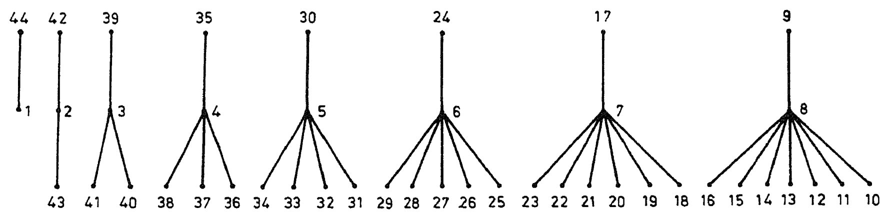

Figure 4.1. Graceful labeling of SB^8.

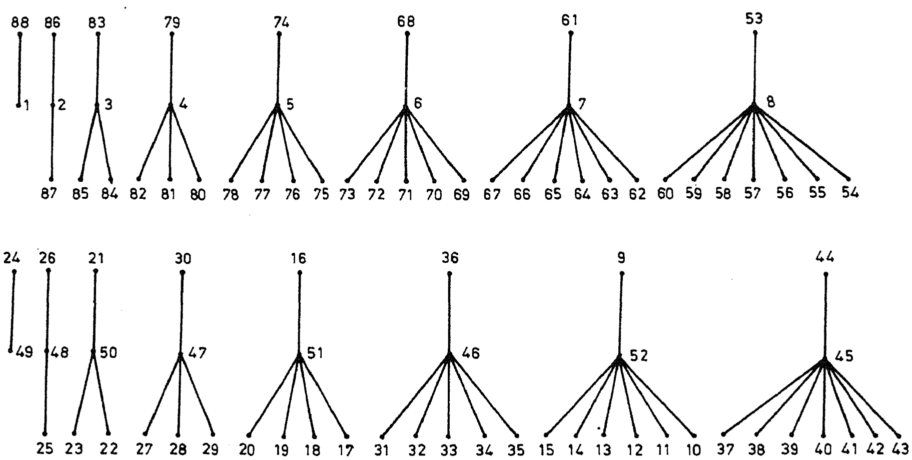

Figure 4.1. (continued). Graceful labeling of $2-SB^8$.

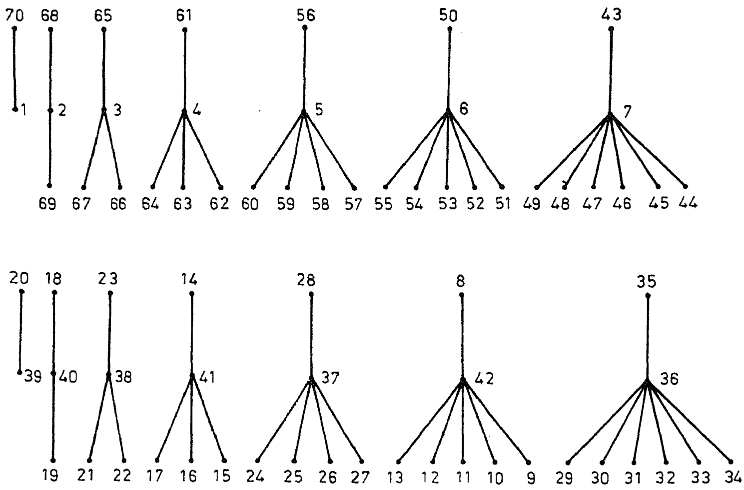

Figure 4.2. Graceful labeling of $2-\mathrm{SB}^7$.

Step 5. Change label x of SB^n to $n-x$ for all the central vertices of SB^n.
Step 6. For all other vertices of SB^n change label x to $3\,((n^2+3n)/2)+n+1-x$.

Remarks

1. Note that $3((n^2+3n)/2)=q(\mathrm{SB}^n)+q(2-\mathrm{SB}^n)$. Also note that $x \geqslant n+1$.
2. Steps 3, 4, 5 and 6 of Algorithm II are same as steps 1, 2, 3 and 4 of Algorithm I with $q(\mathrm{SB}^n(t,\alpha))$ replaced by $q(2-\mathrm{SB}^n)$. In other words, techniques of the algorithms are same.

Table 5 gives the edge weights of $3-\mathrm{SB}^n$ when this algorithm is executed. This table does not include the edge weights of $2-\mathrm{SB}^n$ part of $3-\mathrm{SB}^n$ since they are exactly the same as those of $2-\mathrm{SB}^n$ and are already listed in tables 4.1 and 4.2.

Figures 5(even) and 5(odd) give graceful labelings of $3-\mathrm{SB}^8$ and $3-\mathrm{SB}^7$ respectively. These figures are to be read along with figures 4.1, 2 and 4.2 respectively. As before, for convenience, the apex and the link edges are not drawn and the top-most vertex of each star is its link vertex.

5. Graceful labeling of $\mathrm{SB}^n(t,\alpha)$ with arbitrary t and α

In §2 we gave a graceful labeling of $\mathrm{SB}^n(t,\alpha)$ when $\alpha<t$. Here we give a graceful labeling of $\mathrm{SB}^n(t,\alpha)$ without any restriction on α and t. Let $\mathrm{B}(\alpha K_{1,t})$ denote the banana tree whose family of stars consists of α copies of $K_{1,t}$. One of the results of Chen, Lü and Yeh [1] gives a graceful labeling of $\mathrm{B}(\alpha, K_{1,t})$ when $\alpha \leqslant t$. We give here a graceful labeling of $\mathrm{B}(\alpha K_{1,t})$ with no restriction on α and t. Our graceful labeling gives label 0 to the apex and hence it is possible to combine $\mathrm{B}(\alpha K_{1,t})$ with SB^n to get a graceful labeling of $\mathrm{SB}^n(t,\alpha)$ without any restrictions on t and α.

Our graceful labeling of $\mathrm{B}(\alpha K_{1,t})$ with no restriction on α and t is given in table 6. Apex is labeled 0. In fact, after giving the table 6 we have also indicated the labeling function ϕ in a compact form.

Table 5. Edge weights of SB^n as a part of $3-SB^n$.

Stars	$K_{1,1}$	$K_{1,2}$	$\cdots$	$K_{1,i}$	$\cdots$	$K_{1,n-2}$	$K_{1,n-1}$	$K_{1,n}$
Link edge weight	$\beta-((n^2+3n)/2)$ $-n$	$\beta-((n^2+3n)/2)$ $-n+2$	$\cdots$	$\beta-n'-2n-1$	$\cdots$	$\beta-4n$	$\beta-3n-1$	$\beta-2n-1$
Range of star edge weights	$\beta-((n^2+3n)/2)$ $-n+1$	$\beta-((n^2+3n)/2)$ $-n+3$ to $\beta-((n^2+3n)/2)$ $-n+4$	$\cdots$	$\beta-n'-2n$ to $\beta-n'-n-i-1$	$\cdots$	$\beta-4n+1$ to $\beta-3n-2$	$\beta-3n$ to $\beta-2n-2$	$\beta-2n$ to $\beta-n-1$

Note: i) $p=3((n^2+3n)/2+(n+1)$; ii) $n'=n'(\mathrm{i})=n+(n-1)+(n-2)+\cdots+(n-(i-1))$.

Table 6. Edge weights of $B(\alpha K,t)$.

Stars	$K^{*_1}_{1,t}$	$K^{*_2}_{1,t}$	$K^{*_3}_{1,t}$	$K^{*_4}_{1,t}$	$\cdots$	$K^{*_{2i-1}}_{1,t}$	$K^{*_{2i}}_{1,t}$	$\cdots$	$K^{*_\alpha}_{1,t}$ α is odd	$K^{*_\alpha}_{1,t}$ α is even
Central vertex label	1	$q-t$	$2+t$	$q-2t+1$	$\cdots$	$((2i-2)t/2)$ $+((2i-2)/2)+1$	$q-it-(i-1)$	$\cdots$	$((\alpha-1)t/2)$ $+((\alpha-1)/2)+1$	$q-(\alpha t/2)$ $-((\alpha/2)-1$
Range of end vertex labels	$q-1$ to $q-(t-1)$	2 to t	$q-(t+2)$ to $q-2t$	$3+t$ to $2t+1$	$\cdots$	$q-((2i-2)t/2)$ $-((2i-2)/2)-1$ to $q-(2it/2)-(i-2)$	$(i-1)t+(i+1)$ to $it+(i-1)$	$\cdots$	$q-((\alpha-1)t/2)$ $-((\alpha-1)/2)-1$ to $q-((\alpha+1)t/2)$ $-((\alpha+1)/2)+2$	$((\alpha/2)-1)t$ $+((\alpha/2)+1$ to $(\alpha t/2)+(\alpha/2)-1$
Link vertex labels	q	$t+1$	$q-(t+1)$	$2+2t$	$\cdots$	$q-((2i-2)t/2)$ $-((2i-2)/2)$	$it+i$	$\cdots$	$q-((\alpha-1)t/2)$ $-((\alpha-1)/2)$	$(\alpha t/2)+(\alpha/2)$
Star edge weight range	$q-1$ to $q-t$	$q-(t+2)$ to $q-(2t+1)$	$q-(2t+3)$ to $q-(3t+2)$	$q-(3t+4)$ to $q-(4t+3)$	$\cdots$	$q-(2i-2)t-(2i-1)$ to $q-(2i-1)t-(2i-2)$	$q-(2i-1)t-(2i)$ to $q-2it-(2i-1)$	$\cdots$	$q-(\alpha-1)t-\alpha$ to $q-\alpha t-(\alpha-1)$	$q-(\alpha-1)t-\alpha$ to $q-\alpha t-(\alpha-1)$

Note: Link vertex label is the same as link edge weight.

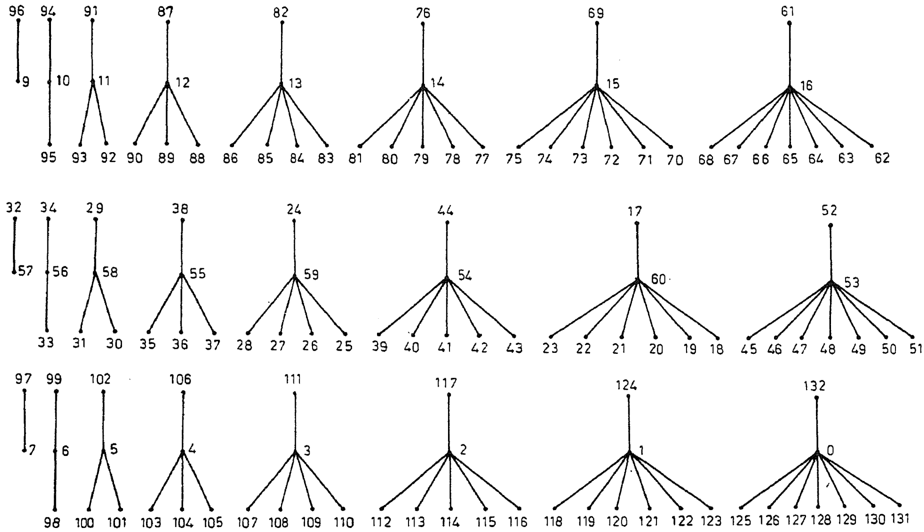

Figure 5 (even). Graceful labeling of $3-\mathrm{SB}^8$. The apex gets label 8.

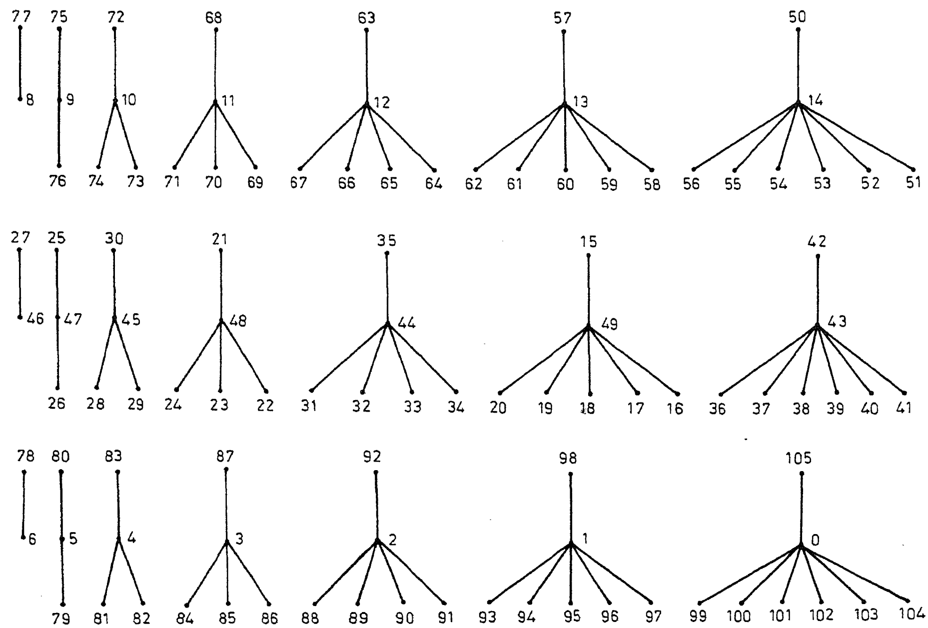

Figure 5 (odd). Graceful labeling of $3-\mathrm{SB}^7$. The apex gets label 7.

Refer to table 2 for the graceful labeling of SB^n as given by Chen, Lü and Yeh [1]. We have recast this graceful labeling in the form of a labeling function ψ after the description of the function ϕ.

We now give an algorithm to get a graceful labeling of $\mathrm{SB}^n(t,\alpha)$ where there are no restrictions on t and α.

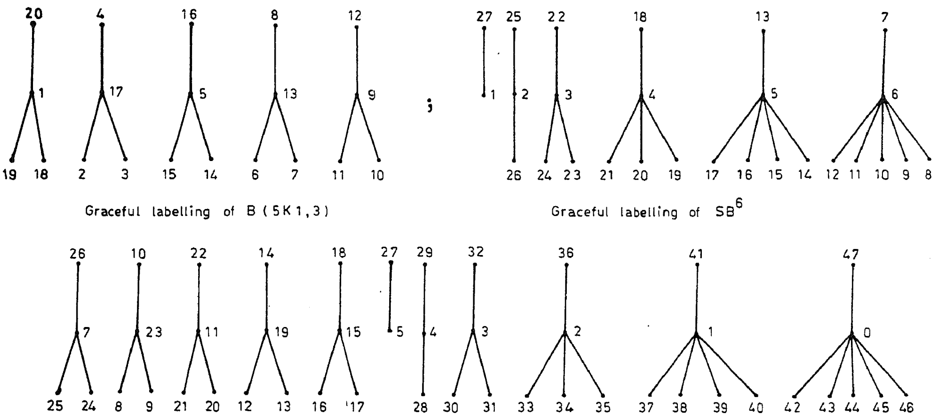

Figure 6. Graceful labeling of $\mathrm{SB}^6(3,5)$.

Algorithm III

Step 1. Identify the apex vertices of SB^n and $\mathrm{B}(\alpha K_{1,t})$ to get $\mathrm{SB}^n(t,\alpha)$. Give label n to the apex of $\mathrm{SB}^n(t,\alpha)$ thus obtained.
Step 2. Add n to each label of $\mathrm{B}(\alpha K_{1,t})$ part of $\mathrm{SB}^n(t,\alpha)$ except to the apex which is covered by step 1.
Step 3. For all the central vertices of SB^n change label x to $n-x$.
Step 4. For all other vertices of SB^n change label x to $((n^2+3n)/2)+\alpha(t+1)+(n+1)-x$.

Remark. We do not include a table for this case. The reason is, the edge weights of the edges of the $\mathrm{B}(\alpha K_{1,t})$ part of $\mathrm{SB}^n(t,\alpha)$ remain the same and they are already indicated in table 6. Though the edge weights of the SB^n part of $\mathrm{SB}^n(t,\alpha)$ change, one can refer to table 3 with $\beta_2=\alpha(t+1)$ instead of $((n^2+3n)/2)+\alpha(t+1)$.

Figure 6 gives a graceful labeling of $\mathrm{SB}^6(3,5)$ obtained as per Algorithm III.

The functions ϕ and ψ

Let v_j and u_j be the link vertex and central vertex respectively of the star $K_{1,t}^{*_j}$ of the banana tree $\mathrm{B}(\alpha K_{1,t})$, $1\leqslant j\leqslant\alpha$. Let w_j^r be the end vertex of the star $K_{1,t}^{*_r}$ of this banana tree for $1\leqslant j\leqslant t-1$, $1\leqslant r\leqslant\alpha$.

The compact form of the labeling function ϕ is as follows.

$$\phi(v_j)=q-\left(\frac{2i-2}{2}\right)t-\left(\frac{2i-2}{2}\right)\quad \text{for } j=2i-1$$

$$=it+i\quad \text{for } j=2i$$

$$\phi(u_j)=\left(\frac{2i-2}{2}\right)t+\left(\frac{2i-2}{2}\right)+1\quad \text{for } j=2i-1$$

$$=q-(it+(i-1))\quad \text{for } j=2i$$

$$\phi(w_j^r) = q - \left(\frac{2i-2)}{2}\right)t - \left(\frac{2i-2}{2}\right)$$

to for $1 \leqslant j \leqslant t-1$ and $r = 2i - 1$

$$= q - \left(\frac{2i}{2}\right)t - (i-2)$$

$$= (i-1)t + (i+1)$$

to for $1 \leqslant j \leqslant t-1$ and $r = 2i$

$$= it + (i-1).$$

Let v_j and u_j be the link vertex and central vertex of the star $K_{1,j}$ involved in SB^n for $1 \leqslant j \leqslant n$. Let w_j^r be the end vertex of the star $K_{1,r}$ of the banana tree SB^n, $1 \leqslant j \leqslant r-1$, $1 \leqslant r \leqslant n$.

$$\psi(v_j) = (n+1) + ni - \frac{i(i-1)}{2} \quad \text{for } j = n-i$$

$$\psi(u_j) = j$$

$$\psi(w_j^r) = (n+1) + ni - \frac{i(i-1)}{2} + 1$$

to for $r = n - i$, $1 \leqslant j \leqslant n-i$, $1 \leqslant i \leqslant n-i$.

$$= (n+1) + ni - \frac{i(i-1)}{2} + n - (i+1)$$

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Hardy's theorem for zeta-functions of quadratic forms*

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Abstract. Let $Q(u_1,\ldots,u_l)=\Sigma d_{ij}u_iu_j$ $(i,j=1$ to $l)$ be a positive definite quadratic form in $l(\geqslant 3)$ variables with integer coefficients $d_{ij}(=d_{ji})$. Put $s=\sigma+it$ and for $\sigma>(l/2)$ write

$$Z_Q(s)=\sum{}'(Q(u_1,\ldots,u_l))^{-s},$$

where the accent indicates that the sum is over all l-tuples of integers $(u_1,\ldots,u_l)$ with the exception of $(0,\ldots,0)$. It is well-known that this series converges for $\sigma>(l/2)$ and that $(s-(l/2))Z_Q(s)$ can be continued to an entire function of s. Let δ be any constant with $0<\delta<\frac{1}{100}$. Then it is proved that $Z_Q(s)$ has $\gg_\delta T\log T$ zeros in the rectangle $(|\sigma-\frac{1}{2}|\leqslant\delta, T\leqslant t\leqslant 2T)$.

Keywords. Quadratic forms; zeta-function; zeros near the line sigma equal to half.

1. Introduction

Let $Q(u_1,u_2,\ldots,u_l)$ be a positive definite quadratic form $\Sigma\, d_{ij}\,u_iu_j$, $(i,j=1$ to $l)$ in $l(\geqslant 3)$ variables and with integer coefficients d_{ij} $(=d_{ji}$ for $i,j)$. Put (with $s=\sigma+it$).

$$Z_Q(s)=\sum{}'(Q(u_1,u_2,\ldots,u_l))^{-s},$$

where the accent indicates that the summation is over all integer l-tuples $(u_1,u_2,\ldots,u_l)$ with the exception of $(0,0,\ldots,0)$. (It is known that $Z_Q(s)(s-(l/2))$ is an entire function.) Let $N(\alpha,T)$ denote the number of zeros of $Z_Q(s)$ in $\sigma\geqslant\alpha, T\leqslant t\leqslant 2T$. We prove the following theorem.

Main Theorem. *We have*

$$N(\alpha,T)\gg T\log T$$

if $\alpha=(l-1)/2-\delta$, $(\delta>0$ *any constant*) *provided* $l\geqslant 3$. *Also we have*

$$N(\beta,T)\ll T$$

if $\beta=(l-1)/2+\delta$.

For a neat consequence of this see Remark 2 below.

Remark 1. The proof of this theorem depends on the following two important results.

*Dedicated to Professor R P Bambah on his seventy-first birthday

First the lower bound

$$\frac{1}{T}\int_T^{2T} |Z_Q(\sigma+it)|\,\mathrm{d}t \gg T^{\delta}, \quad \left(\sigma=\frac{l-1}{2}-\delta\right),$$

where $\delta>0$ is a constant if $l\geqslant 3$. Next for $\frac{1}{2}+\varepsilon\leqslant(l-1)/2-\delta\leqslant(l-1)/2-\varepsilon$ (ε is a small positive constant), we have

$$\frac{1}{T}\int_T^{2T}\left|Z_Q\left(\frac{l-1}{2}-\delta+it\right)\right|^2 \mathrm{d}t \ll T^{2\delta}$$

for $l\geqslant 3$. Both the results follow from the ideas of R Balasubramanian and K Ramachandra (see $[\mathrm{RB},\mathrm{KR}]_1$, $[\mathrm{RB},\mathrm{KR}]_2$, $[\mathrm{KR}]_1$, $[\mathrm{KR}]_2$ and $[\mathrm{KR}]_3$). Also one has to use Theorem 3 of $[\mathrm{RB},\mathrm{KR}]_1$.

Remark 2. Using the functional equation of $Z_{\bar{Q}}(s)$ (with some associated quadratic form $\bar{Q}$) and applying the theorem we have the following corollary: $Z_Q(s)$ has $\gg T\log T$ zeros in $(|\sigma-\frac{1}{2}|\leqslant\delta,\ T\leqslant t\leqslant 2T)$. In a rough way we may say that the critical line (for $Z_Q(s)$) gets blown up into an inner critical strip $\frac{1}{2}\leqslant\sigma\leqslant(l-1)/2$ and that in the neighbourhood of the vertical borders there are plenty of zeros of $Z_Q(s)$. This is the justification for the title of the present paper.

2. Notation and preliminaries

1. $C_1, C_2, \ldots, A_1, A_2, \ldots$ denote effective positive constants, sometimes absolute.
2. $\varepsilon_1, \varepsilon_2, \ldots, \delta_1, \delta_2, \ldots$ denote small positive constants.
3. $f(x)\ll g(x)$ and $f(x)=O(g(x))$ will mean that $|f(x)|\leqslant C_1 g(x)$.
4. We write $s=\sigma+it$, $w=u+iv$.
5. $f(x)=o(g(x))$ means that $f(x)/g(x)$ as $x\to\infty$.

In any fixed strip $\alpha\leqslant\sigma\leqslant\beta$, as $t\to\infty$ we have

$$\Gamma(\sigma+it)=t^{\sigma+it-(1/2)}\mathrm{e}^{-(\pi/2)t-it+(i\pi/2)(\sigma-(1/2))}.\sqrt{2\pi}\left(1+O\left(\frac{1}{t}\right)\right). \tag{2.1}$$

$Z_Q(s)$ satisfies the functional equation (see [EH] or [HMS])

$$\left(\frac{\Delta^{1/l}}{2\pi}\right)^s\Gamma(s)Z_Q(s)=\left(\frac{\Delta^{1-(1/l)}}{2\pi}\right)^{(l/2)-s}\Gamma\left(\frac{l}{2}-s\right)Z_{\bar{Q}}\left(\frac{l}{2}-s\right), \tag{2.2}$$

where $\Delta=|\det((d_{ij}))|$. If we write

$$Z_Q(s)=\psi_Q(s)Z_{\bar{Q}}\left(\frac{l}{2}-s\right), \tag{2.3}$$

then, from (2.1) and (2.2), we obtain,

$$\psi_Q(s)=\left(\frac{\Delta^{1-(1/l)}}{2\pi}\right)^{l/2}\left(\frac{\Delta}{(2\pi)^2}\right)^{-\sigma}\mathrm{e}^{-i\pi(\sigma-(l/4))}t^{2((l/4)-\sigma)}\left(\frac{t\sqrt{\Delta}}{2\pi\mathrm{e}}\right)^{-2it}$$
$$\times\left(1+O\left(\frac{1}{t}\right)\right)=Ct^{2((l/4)-\sigma)}\left(\frac{t\sqrt{\Delta}}{2\pi\mathrm{e}}\right)^{-2it}\left(1+O\left(\frac{1}{t}\right)\right). \tag{2.4}$$

Hereafter, we write

$$Z_Q(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}\left(\text{in Re } s>\frac{l}{2}\right)$$

and its analytic continuations. The analytic continuation of $Z_Q(s)$ shows that in $|t|\geqslant 10$ we have in $-1\leqslant\sigma\leqslant l$ the bound $|Z_Q(s)|<t^A$ for some constant A depending on the quadratic form Q.

3. Some Lemmas

Lemma 3.1. *We have*

$$\sum_{n\leqslant x}a_n=C_2x^{(l/2)}+O(x^{(l/2-1/2)})$$

where C_2 depends on Δ and l.

Proof. See for example [EL] Hilfssatz 16.

Lemma 3.2. *Let Q be a primitive positive definite quadratic form in l-variables with integer coefficients. For $l\geqslant 3$, we have*

$$\sum_{n\leqslant x}a_n^2=C_3x^{l-1}+O(x^{(l-1)(4l-5)/(4l-3)}),$$

where C_3 is a positive real constant which depends on Q.

Proof. See Theorem 6.1 of [WM].

Lemma 3.3. *Let $\{b_n\}$ and $\{b'_n\}$, $n=1,2,\dots,M$ be any set of complex numbers. Then*

$$\int_0^T\left(\sum_{n=1}^{M}b_nn^{-it}\right)\left(\sum_{n=1}^{M}b'_nn^{it}\right)\mathrm{d}t=T\sum_{n=1}^{M}b_nb'_n+O\left(\left(\sum_{n=1}^{M}n|b_n|^2\right)^{1/2}\times\left(\sum_{n=1}^{M}n|b'_n|^2\right)^{1/2}\right).$$

Proof. See [HLM, RCV] or $[\mathrm{KR}]_4$.

Lemma 3.4. *For $T\geqslant 100$, we have*

$$\frac{1}{T}\int_T^{2T}|\zeta(\tfrac{1}{2}+\delta_1+it)|^4\,\mathrm{d}t\ll_{\delta_1}1,$$

where δ_1 is a fixed positive constant.

Proof. See for example [ECT].

Lemma 3.5. (*see* [KR, AS]). *Let I be any unit interval in $[T,2T]$ and define*

$$m(I)=\max_{\substack{t\in I\\(1/2)+\delta_2\leqslant\sigma\leqslant 2}}|\zeta(\sigma+it)|.$$

Then, we have

$$\sum_I (m(I))^4 \ll_{\delta_2} T.$$

COROLLARY

If $M(I) = \max_{t\in I}|\zeta(\frac{1}{4}+it)|$ *where* I *is any unit interval contained in* $[T, 2T]$, *then*

$$\sum_I (M(I))^4 \ll T^2.$$

Proof. Let $m(I) = |\zeta(\sigma_j + it_j)|$ and let $D(s_0) = D_{(\delta_2/2)}(s_0)$ denote the disc of radius $(\delta_2/2)$ with centre s_0. By Cauchy's theorem, we have $(s_j = \sigma_j + it_j)$,

$$|\zeta(s_j)|^4 \leqslant \frac{1}{A}\int\int_{D(s_j)} |\zeta(s)|^4 \, d\sigma \, dt,$$

where $A = \pi(\delta_2/2)^2$ is the area of $D(s_j)$. For any fixed j, $D(s_j)$ intersects $D(s_{j'})$ only for $O(1)$ values of j'. Now, summing over j, we obtain

$$\sum_I (m(I))^4 = \sum_j |\zeta(s_j)|^4$$

$$\ll \delta_2^{-2}\int_{(1+\delta_2)/2}^{100}\left(\int_{T-1}^{2T+1} |\zeta(s)|^4 \, dt\right) d\sigma$$

$$\ll_{\delta_2} T.$$

Now, the corollary follows on using the functional equation for $\zeta(s)$.

4. First power mean-lower bound

Theorem 4.1. *Let* $\delta > 0$ *be any fixed constant such that* $\frac{1}{2} + \varepsilon \leqslant (l-1)/2 - \delta \leqslant (l-1)/2 - \varepsilon$. *We make only the following hypothesis (which is satisfied by* a_n *in* $Z_Q(s)$ *from Lemmas* 3.1 *and* 3.2):

Hypothesis (*)$^+$. For each fixed l, we assume that for the corresponding a_n, the inequalities

$$\sum_{x\leqslant n\leqslant 2x} \frac{a_n}{n^{(l/2)-1}} \gg x$$

and

$$\sum_{x\leqslant n\leqslant 2x} \left(\frac{a_n}{n^{(l/2)-1}}\right)^2 \ll x$$

hold.

$^+$*Postscript.* Instead of Hypothesis (*) of Theorem 4.1 we can manage with the following hypothesis

$$\mathrm{Re}\sum_{x\leqslant n\leqslant 2x} a_n \gg x^{l/2} \quad \text{and} \quad \sum_{x\leqslant n\leqslant 2x} |a_n|^2 \ll x^{l-1}.$$

Then for $T \geqslant 10$, we have

$$\frac{1}{T}\int_T^{2T}\left|Z_Q\left(\frac{l-1}{2}-\delta+it\right)\right|\mathrm{d}t \gg T^{\delta}.$$

Note. We use the notation in the proof,

$$A \equiv A\left(\frac{l-1}{2}-\delta+it\right) \quad \text{and} \quad \zeta^* \equiv \zeta^*(\tfrac{1}{4}+it).$$

Proof. Let $M(I) = \max_{t\in I}|\zeta(\frac{1}{4}+it)|$ where t runs over all points in the unit interval I contained in $[T, 2T]$. From the corollary of Lemma 3.5, we have

$$\#\{I/M(I) \geqslant C_4 T^{1/4}\} \ll \frac{T}{C_4^4}, \tag{4.1.1}$$

where C_4 is a large positive constant. We define

$$A(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}(\mathrm{e}^{-(n/C_5T)} - \mathrm{e}^{-(n/C_6T)}), \tag{4.1.2}$$

where C_5 and C_6 satisfy $0 < C_6 < C_5 < 1$ (will be chosen suitably) and

$$\zeta^* \equiv \zeta^*\left(\frac{1}{4}+it\right) = \sum_{n\leqslant T} n^{-1/4-it}. \tag{4.1.3}$$

We divide the interval $[T, 2T]$ into disjoint unit intervals I. Now, consider

$$\int_T^{2T}|A|\mathrm{d}t \geqslant \sum_I\int_I\frac{|A\bar{\zeta}^*|\mathrm{d}t}{M(I)} \gg \frac{1}{C_4T^{1/4}}\sum_I{}'\int_I|A\bar{\zeta}^*|\mathrm{d}t, \tag{4.1.4}$$

where accent in the above sum indicates that the sum is over those I for which $M(I) \leqslant C_4T^{1/4}$. Hence from (4.1.4), we obtain

$$\begin{aligned}\int_T^{2T}|A|\mathrm{d}t &\gg \frac{1}{C_4T^{1/4}}\left\{\int_T^{2T-1}|A\bar{\zeta}^*|\mathrm{d}t - \int_T^{2T}\psi(t)|A\zeta^*|\mathrm{d}t\right\}\\ &\gg \frac{1}{C_4T^{1/4}}\left\{\left|\int_T^{2T-1}A\bar{\zeta}^*\mathrm{d}t\right| - \int_T^{2T}\psi(t)|A\zeta^*|\mathrm{d}t\right\},\end{aligned} \tag{4.1.5}$$

where $\psi(t)$ is the characteristic function of those I for which $M(I) \geqslant C_4T^{1/4}$. We note that from (4.1.1),

$$\int_T^{2T}\psi(t)\mathrm{d}t \ll \frac{T}{C_4^4}. \tag{4.1.6}$$

Now, from Lemma 3.3, we have

$$\begin{aligned}\int_T^{2T-1}A\bar{\zeta}^*\mathrm{d}t &= T\sum_{n\leqslant T}\frac{a_n(\mathrm{e}^{-(n/C_5T)} - \mathrm{e}^{-(n/C_6T)})}{n^{(l-1)/2-\delta+1/4}}\\ &\quad + O\left(\left(\sqrt{\sum_{n=1}^{\infty}\frac{a_n^2(\mathrm{e}^{-(n/C_5T)} - \mathrm{e}^{-(n/C_6T)})^2}{n^{l-1-2\delta}}\cdot n}\right)\left(\sqrt{\sum_{n\leqslant T}\frac{n}{n^{1/2}}}\right)\right)\\ &= J_1 + O(J_2). \text{ (say)}\end{aligned} \tag{4.1.7}$$

Now, for $\lambda < 1$, we have

$$
\begin{aligned}
J_1 &\geqslant T \sum_{n \leqslant T\lambda C_6} \frac{a_n}{n^{(l-1)/2-\delta+1/4}} \left\{1 - \frac{n}{C_5 T} - \left(1 - \frac{n}{C_6 T}\right) + O\left(\frac{n^2}{C_6^2 T^2}\right)\right\} \\
&\geqslant T \sum_{n \leqslant T\lambda C_6} \frac{a_n}{n^{(l-1)/2-\delta+1/4}} \left\{\frac{n}{2C_6 T} + O\left(\frac{n^2}{C_6^2 T^2}\right)\right\} \\
&\geqslant C_7 T \sum_{n \leqslant T\lambda C_6} \frac{1}{n^{(3/4-\delta)}} \left\{\frac{n}{2C_6 T} + O\left(\frac{n^2}{C_6^2 T^2}\right)\right\} \\
&\geqslant T\left\{C_8 \frac{(T\lambda C_6)^{(5/4+\delta)}}{C_6 T} - C_9 \frac{(T\lambda C_6)^{(9/4+\delta)}}{C_6^2 T^2}\right\},
\end{aligned}
\tag{4.1.8}
$$

provided $1 > C_5 \geqslant 20 C_6 > 0$. This implies that for sufficiently small λ, there exists an absolute constant C_{10} such that

$$
J_1 \geqslant C_{10} T^{(5/4+\delta)} C_6^{(1/4+\delta)}. \tag{4.1.9}
$$

Now,

$$
\begin{aligned}
J_2 &\ll T^{3/4} \sqrt{\sum_{n=1}^{\infty} \frac{a_n^2 \cdot n}{n^{l-1-2\delta}} e^{-(2n/C_5 T)}} \\
&\ll \sqrt{\sum_{n=1}^{\infty} n^{2\delta} e^{-(n/2C_5 T)}} \\
&\ll T^{(5/4+\delta)} C_5^{(1/2+\delta)}
\end{aligned}
\tag{4.1.10}
$$

since

$$
\sum_{n=1}^{\infty} n^{2\delta} e^{-(n/2C_5 T)} = \frac{1}{2\pi i} \int_{\mathrm{Re}w=2} \zeta(-2\delta + w)\Gamma(w)(2C_5 T)^w \mathrm{d}w \tag{4.1.11}
$$

and move the line of integration in (4.1.11) to $\mathrm{Re}w = 1 + 2\delta$ so that the residue of the pole at $w = 1 + 2\delta$ is $(2C_5 T)^{1+2\delta}\,\Gamma(1+2\delta)$. Note that, we have used the hypothesis $(*)$ in estimating J_1 and J_2. Therefore from (4.1.7), (4.1.9) and (4.1.10) we obtain

$$
\begin{aligned}
\int_T^{2T-1} A\bar{\zeta}^* \mathrm{d}t &> C_{10} T^{(5/4+\delta)} C_6^{(1/4+\delta)} - C_{11} T^{(5/4+\delta)} C_5^{(1/2+\delta)} \\
&= T^{(5/4+\delta)} C_{10} \cdot C_6^{(1/4+\delta)} \left(1 - \frac{C_{11}}{C_{10}} \cdot \frac{C_{12}^{(1/2+\delta)}}{C_6^{(1/4+\delta)}}\right).
\end{aligned}
\tag{4.1.12}
$$

We choose C_6 small and then C_5 such that

$$
C_5^{(1/2+\delta)} = \frac{C_{10}}{2C_{11}} C_6^{(1/4+\delta)}
$$

i.e.

$$
C_5 = \left(\frac{C_{10}}{2C_{11}}\right)^{1/(1/2+\delta)} \cdot C_6^{(1/4+\delta)/(1/2+\delta)} \geqslant 20 C_6.
$$

This is satisfied since C_6 is small and $(1/4+\delta)/(1/2+\delta) < 1$. Hence we have

$$
\int_T^{2T-1} A\zeta^* \mathrm{d}t > C_{12} T^{(5/4+\delta)}, \tag{4.1.13}
$$

where C_{12} depends only on δ. Now, from Hölder's inequality, it follows that

$$\int_T^{2T} \psi(t)|A\zeta^*|\,\mathrm{d}t \leqslant \left(\int_T^{2T} |A|^2\,\mathrm{d}t\right)^{1/2}\left(\int_T^{2T} \psi^4(t)\,\mathrm{d}t\right)^{1/4}\left(\int_T^{2T} |\zeta^*|^4\,\mathrm{d}t\right)^{1/4}. \tag{4.1.14}$$

From (4.1.1) and from Lemma 3.5, using the functional equation for $\zeta(s)$, we notice that

$$\left(\int_T^{2T} \psi^4(t)\,\mathrm{d}t\right)^{1/4}\left(\int_T^{2T} |\zeta^*|^4\,\mathrm{d}t\right)^{1/4} \ll \left(\frac{T}{C_4^4}\right)^{1/4}(T^2)^{1/4} \ll C_4^{-1}T^{3/4}. \tag{4.1.15}$$

Also from Lemma 3.3, it follows that

$$\begin{aligned}\int_T^{2T} |A|^2\,\mathrm{d}t &\ll \sum_{n=1}^{\infty} \frac{(T+n)|a_n|^2(\mathrm{e}^{-(n/C_5T)} - \mathrm{e}^{-(n/C_6T)})^2}{n^{l-1-2\delta}} \\ &\ll \sum_{n=1}^{\infty} \frac{(T+n)|a_n|^2\mathrm{e}^{-(2n/C_5T)}}{n^{l-1-2\delta}} \\ &\ll T\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-(2n/C_5T)}}{n^{1-2\delta}} + \sum_{n=1}^{\infty} n^{2\delta}\mathrm{e}^{-(2n/C_5T)} \\ &\ll T^{1+2\delta}\end{aligned} \tag{4.1.16}$$

on using the hypothesis and noticing the fact similar to (4.1.11). Therefore from (4.1.14), (4.1.15) and (4.1.16), we obtain

$$\int_T^{2T} \psi(t)|A\zeta^*|\,\mathrm{d}t \ll C_4^{-1}T^{(5/4+\delta)}. \tag{4.1.17}$$

Therefore from (4.1.5), (4.1.13) and (4.1.17), we get

$$\begin{aligned}\int_T^{2T} |A|\,\mathrm{d}t &> \frac{1}{C_4T^{1/4}}\{C_{12}T^{(5/4+\delta)} - C_4^{-1}C_{13}T^{(5/4+\delta)}\} \\ &\gg T^{1+\delta},\end{aligned} \tag{4.1.18}$$

since C_4 is large enough. Here C_{12} and C_{13} depend only on δ. Now let $\mathrm{Re}\,s = ((l-1)/2) - \delta$. By Mellin's transform, we have

$$\begin{aligned}A(s) &= \frac{1}{2\pi i}\int_{\mathrm{Re}\,w=100} Z_Q(s+w)((C_5T)^w - (C_6T)^w)\Gamma(w)\,\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_{\substack{\mathrm{Re}\,w=100\\ |v|\leqslant(\log T)^2}} Z_Q(s+w)((C_5T)^w - (C_6T)^w)\Gamma(w)\,\mathrm{d}w + O(T^{-1}).\end{aligned} \tag{4.1.19}$$

We move the line of integration in (4.1.19) to $\mathrm{Re}\,w = 0$ and we obtain

$$|A(s)| \ll \int_{|v|\leqslant(\log T)^2} \left|Z_Q(s+iv)\right|\left\|\frac{C_5^{iv} - C_6^{iv}}{v}\right\|\Gamma(1+iv)\Big|\,\mathrm{d}v + O(T^{-1})$$

and hence

$$\begin{aligned}
&\int_T^{2T} |A(s)| \\
&\ll \int_{|v|\leqslant(\log T)^2}\int_T^{2T}\left|Z_Q\left(\frac{l-1}{2}-\delta+it+iv\right)\right|\left|\frac{C_5^{iv}-C_6^{iv}}{v}\right|\left|\Gamma(1+iv)\right|\mathrm{d}v\mathrm{d}t \\
&\ll \int_{|v|\leqslant(\log T)^2}\int_{T-(\log T)^2}^{2T+(\log T)^2}\left|Z_Q\left(\frac{l-1}{2}-\delta+i\tau\right)\right|\left|\frac{C_5^{iv}-C_6^{iv}}{v}\right|\left|\Gamma(1+iv)\right|\mathrm{d}v\mathrm{d}\tau \\
&\ll \int_{T-(\log T)^2}^{2T+(\log T)^2}\left|Z_Q\left(\frac{l-1}{2}-\delta+i\tau\right)\right|\mathrm{d}\tau. \qquad (4.1.20)
\end{aligned}$$

From (4.1.18) and (4.1.20), the theorem follows, since we can define the integrand to be zero outside the interval $[T, 2T]$.

5. Mean-square upper bound

Theorem 5.1. *Let δ satisfy the condition as in Theorem* 4.1. *We make only the following hypothesis (which is satisfied by a_n in $Z_Q(s)$, from Lemma* 3.2).

Hypothesis $(*, *)$ For each l for the corresponding a_n, the inequality

$$\sum_{n\leqslant x}\left(\frac{a_n}{n^{(l/2-1)}}\right)^2 \ll x$$

hold.

Then for $T \geqslant 100$, we have

$$\frac{1}{T}\int_T^{2T}\left|Z_Q\left(\frac{l-1}{2}-\delta+it\right)\right|^2 \mathrm{d}t \ll T^{2\delta}.$$

Proof. It follows from the papers $[\mathrm{KR}]_2$ and $[\mathrm{KR}]_3$.

6. Balasubramanian–Ramachandra principle

Theorem 6.1. *For $T \geqslant T_0$, if*

$$\frac{1}{T}\int_T^{2T} |G(\sigma_1+it)|\mathrm{d}t > A_1\psi \qquad (6.1.1)$$

and

$$\frac{1}{T}\int_T^{2T} |G(\sigma_1+it)|^2\mathrm{d}t < A_2\psi^2 \qquad (6.1.2)$$

hold for a Dirichlet series $G(s)$ on a certain line σ_1 with positive constants A_1 and *A_2, then there exists at least $\geqslant [(A_1^2/2A_2)(T/H)] - 1$ intervals I of length H such that in each of the intervals I, the inequality*

$$\frac{1}{|I|}\int_I |G(\sigma_1+it)|\mathrm{d}t > \frac{A_1}{10}\psi \qquad (6.1.3)$$

holds where $H \leqslant T^{1-\varepsilon_1}$, and $\psi = \psi(T)$ tends to ∞.

Remark. This principle has been used in several occasions (for example see $[\mathrm{RB}, \mathrm{KR}]_1$, $[\mathrm{RB}, \mathrm{KR}]_2, \ldots$). For the sake of completeness, we sketch the proof.

Proof. We divide the interval $[T, 2T]$ into smaller disjoint (but abutting) intervals of length H(but with length $\leqslant H$ for an end interval). By defining G to be zero if $t \leqslant T$ or $t \geqslant 2T$, we get

$$A_1 \psi T < \int_T^{2T} |G(\sigma_1 + it)| \mathrm{d}t \leqslant \sum_I \int_I |G(\sigma_1 + it)| \mathrm{d}t. \tag{6.1.4}$$

Now, we omit these intervals I appearing in the sum of (6.1.4) for which

$$\int_I |G(\sigma_1 + it)| \mathrm{d}t \leqslant \frac{A_1}{2} H\psi. \tag{6.1.5}$$

Let N_1 be the number of those intervals I for each of which the inequality

$$\int_I |G(\sigma_1 + it)| \mathrm{d}t \geqslant \frac{A_1}{2} H\psi \tag{6.1.6}$$

holds. Therefore applying Hölder's inequality, we find that from (6.1.4), (6.1.5) and (6.1.6),

$$\begin{aligned} \frac{A_1}{2} \psi T &\leqslant \sum_I{}' \int_I |G(\sigma_1 + it)| \mathrm{d}t \\ &\leqslant \sqrt{N_1} \left\{ \sum_I{}' \left(\int_I |G(\sigma_1 + it)| \mathrm{d}t \right)^2 \right\}^{1/2} \\ &\leqslant \sqrt{N_1} \left\{ \sum_I \left(\int_I |G(\sigma_1 + it)| \mathrm{d}t \right)^2 \right\}^{1/2} \\ &\leqslant \sqrt{N_1} \left\{ \sum_I H \int_I |G(\sigma_1 + it)|^2 \mathrm{d}t \right\}^{1/2} \\ &\leqslant \sqrt{N_1 H} \left(\int_T^{2T} |G(\sigma_1 + it)|^2 \mathrm{d}t \right)^{1/2} \\ &\leqslant \sqrt{N_1 H} \psi T^{1/2} A_2^{1/2}, \end{aligned} \tag{6.1.7}$$

i.e. $N_1 \geqslant A_1^2 / A_2 \cdot T/H$, and the accent in the first two steps of the inequality (6.1.7) indicates that the sum runs over those intervals I for each of which the inequality (6.1.6) holds. This proves the theorem.

7. Proof of the main theorem

Taking $H = 1$, from Theorem 6.1, there are $\gg T$ well-spaced points t_r at which $|Z_Q(l-1)/2 - \delta + it_r)|$ is large. Now from Theorem 3 of $[\mathrm{RB}, \mathrm{KR}]_1$, each such point gives rise to $\gg \log T$ zeros of $Z_Q(s)$ in $\sigma \geqslant (l-1)/2 - \delta$. This completes the proof of the first part. Second part of the main theorem follows from the fact that

$$\frac{1}{T} \int_T^{2T} \left| Z_Q\left(\frac{l-1}{2} + it \right) \right|^2 \mathrm{d}t \ll T^{\varepsilon} \quad \forall \varepsilon > 0.$$

(For explanation see [ECT]).

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Degree of approximation of functions associated with Hardy–Littlewood series in the Hölder metric by Euler means

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Abstract. After establishing the Fourier character of the Hardy–Littlewood series the authors have studied the degree of approximation of functions associated with the same series in the Hölder metric using Euler means.

Keywords. Hardy–Littlewood series; Banach space; Hölder metric; Euler means.

1. Definition

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic continuous functions defined on $[-\pi,\pi]$ under sup-norm. For $0<\alpha\leqslant 1$ and some positive constant K, the function space H_α is given by

$$H_\alpha=\{f\in C_{2\pi}\colon |f(x)-f(y)|\leqslant K|x-y|^\alpha\}. \tag{1.1}$$

The space H_α is a Banach space (6) with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha=\|f\|_c+\sup_{x,y}\{\Delta^\alpha f(x,y)\} \tag{1.2}$$

where

$$\|f\|_c=\sup_{-\pi\leqslant x\leqslant\pi}|f(x)|$$

and

$$\Delta^\alpha f(x,y)=|f(x)-f(y)|\,|x-y|^{-\alpha},\ x\neq y. \tag{1.3}$$

We shall use the convention that $\Delta^0 f(x,y)=0$. The metric induced by norm (1.2) on H_α is called Hölder metric.

Let f be a periodic function of period 2π and integrable in the Lebesgue sense over $[-\pi,\pi]$. Let the Fourier series associated with f at x be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x). \tag{1.4}$$

Let us write

$$\Phi_x(u)=\tfrac{1}{2}\{f(x+u)+f(x-u)-2f(x)\}, \tag{1.5}$$

$$\chi_x(u)=\int_u^\pi \Phi_x(w)\tfrac{1}{2}\cot w/2\,dw. \tag{1.6}$$

Let $S_n(x)$ and $S_n^*(x)$ respectively denote the partial sum and the modified partial sum of (1.4), i.e.,

$$S_n(x) = \sum_{k=0}^{n} A_k(x),\ S_n^*(x) = \sum_{k=0}^{n-1} A_k(x) + \tfrac{1}{2}A_n(x).$$

It is known ([7], p. 50) that

$$S_n^*(x) - f(x) = \frac{2}{\pi}\int_0^\pi \frac{\Phi_x(u)\sin nu\,du}{2\tan u/2}. \tag{1.7}$$

Given any sequence $\{t_n\}$ its (E, q), $(q > 0)$ transformation is defined by ([4], p. 180)

$$E_n^q(t) = (q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} t_k. \tag{1.8}$$

2. Introduction

By writing Hardy–Littlewood series or in short HL-series, we mean the series

$$\sum_{n=1}^{\infty} \frac{S_n(x) - f(x)}{n}. \tag{2.1}$$

We take this opportunity to acknowledge the fact that this nomenclature for the series (2.1) is first due to Prof. R Mohanty (see [5]).

Hardy and Littlewood [2] have shown that (2.1) is summable $(C, 1)$ to the value

$$\frac{1}{\pi}\int_{0+}^{\pi}\left\{\left(\frac{\pi - u}{2}\right)\cot u/2 - \log(2\sin u/2)\right\}\Phi_x(u)\,du,$$

whenever the integral

$$\int_{0+}^{\pi} \Phi_x(u)\tfrac{1}{2}\cot u/2\,du \tag{2.2}$$

exists. Further ([2], see also [7], p. 122) if

$$\int_0^t |\Phi(u)|\,du = o(t) \text{ as } t \to 0+, \tag{2.3}$$

then (2.1) converges if and only if (2.2) exists. The interest of the HL-series lies in its relation to the integral (2.2), these relations being very similar to those between the conjugate series $\Sigma\, B_n(x)$ and the integral

$$\int_{0+}^{\pi} \frac{\psi_x(u)}{u}\,du, \tag{2.4}$$

where $\psi_x(u) = \frac{1}{2}\{f(x+u) - f(x-u)\}$. It is known [7] that if $f \in L$ then (2.4) exists almost everywhere. On the other hand there exists a continuous function f for which the integral (2.2) diverges for almost all x [2].

At this stage, we remark that the above results on HL-series remain unaltered if we replace the HL-series by

$$\tfrac{1}{2}c_0 + \sum_{n=1}^{\infty} \frac{S_n^*(x) - f(x)}{n}, \tag{2.5}$$

where

$$c_0 = \frac{2}{\pi}\int_0^{\pi} \Phi_x(u)\frac{u}{2}\cot u/2 du.$$

The series (2.5) is summable $(C, 1)$ to the value

$$\int_{0+}^{\pi} \Phi_x(u)\tfrac{1}{2}\cot u/2 du$$

whenever this integral exists. Thus the convergence or summability problem of (2.5) is same as that of (2.1) though their sums are different and hence we may term (2.5) as HL-series.

Let $T_n(x)$ denote the nth partial sum of the series (2.5), i.e.,

$$T_n(x) = \tfrac{1}{2}c_0 + \sum_{k=1}^{n} \frac{S_k^*(u) - f(x)}{k}. \tag{2.6}$$

3. Main results

Chandra [1] has studied the degree of approximation problem for Fourier series by Euler's means. The object of the present paper is to determine the degree of approximation of the series (2.5) by means of Euler's transformation in the Hölder metric. We prove the following theorem.

Theorem. *If* $0 \leqslant \beta < \alpha \leqslant 1$ *and* $f \in H_\alpha$ *then*

$$\| E_n^q(T,.) - \chi(N) \|_\beta = O(n^{\beta-\alpha}(\log n)^{1+\beta/\alpha}) \tag{3.1}$$

where

$$N = \frac{\pi(1+q)}{n}$$

and $E_n^q(T, x)$ *is the Euler's transformation of* $T_n(x)$.

Fourier character of HL-series (2.5)

Let

$$\chi(u) = \chi_x(u) = \int_u^{\pi} \Phi_x(w)\tfrac{1}{2}\cot w/2 dw. \tag{3.2}$$

It is known [3] that χ is even and Lebesgue integrable.

Let

$$\chi \sim \tfrac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos nt. \tag{3.3}$$

We have

$$\begin{aligned} c_0 &= \frac{2}{\pi}\int_0^{\pi} \chi_x(t)dt = \frac{2}{\pi}\int_0^{\pi}\left(\int_t^{\pi} \Phi(u)\tfrac{1}{2}\cot u/2 du\right)dt \\ &= \frac{2}{\pi}\int_0^{\pi} \Phi(u)\tfrac{1}{2}\cot u/2 du \int_0^{u} dt \\ &= \frac{2}{\pi}\int_0^{\pi} \Phi(u)\frac{u}{2}\cot u/2 du \end{aligned} \tag{3.4}$$

and for $n \geqslant 1$

$$\begin{aligned}
c_n &= \frac{2}{\pi}\int_0^{\pi} \chi_x(t)\cos nt\,dt \\
&= \frac{2}{\pi}\int_0^{\pi} \cos nt\left(\int_t^{\pi} \Phi(u)\tfrac{1}{2}\cot u/2\,du\right)dt \\
&= \frac{2}{\pi}\int_0^{\pi} \Phi(u)\tfrac{1}{2}\cot u/2\,du \int_0^{u} \cos nt\,dt \\
&= \frac{2}{\pi n}\int_0^{\pi} \frac{\Phi(u)\sin nu\,du}{2\tan u/2} \\
&= \frac{S_n^*(x)-f(x)}{n}.
\end{aligned} \tag{3.5}$$

Thus, we have the following.

PROPOSITION

The Hardy–Littlewood series (2.5) is the Fourier series of the even function $\chi(u)$ at $u=0$. In these circumstances

$$T_n(x) = \tfrac{1}{2}c_0 + \sum_{k=1}^{n} c_k = \frac{2}{\pi}\int_0^{\pi} \chi_x(u)D_n(u)\,du \tag{3.6}$$

where

$$D_n(u) = \frac{\sin(n+\frac{1}{2})u}{2\sin u/2}. \tag{3.7}$$

Throughout the paper we suppose $2\delta < \min[\pi/4,\ 1/q]$, $q>0$. We further use the following notations

$$F_x(u) = \chi_x(u) - \chi_x(N) \tag{3.8}$$

$$G(u) = F_x(u) - F_y(u) \tag{3.9}$$

$$p_q^n(u) = (q+1)^{-n}(1+q^2+2q\cos u)^{n/2} \tag{3.10}$$

$$g(u) = 1 + \frac{q\cos u}{(1-q^2\sin^2 u)^{1/2}},\ qu<1 \tag{3.11}$$

$$N = \frac{\pi(1+q)}{n} \tag{3.12}$$

$$b(y) = \tan^{-1}\left(\frac{\sin y}{q+\cos u}\right) \tag{3.13}$$

$$t_r = t_r(z) = z + \frac{r\pi}{n} + \sin^{-1}\left\{q\sin\left(z+\frac{r\pi}{n}\right)\right\} \text{ for } r=0,1,2, \tag{3.14}$$

$$t_0(z) = t(z) = t \tag{3.15}$$

$$R_n^q(x) = \int_{1/n}^{\log n/(n)^{1/2}} t^{-1}\{F_x(t) - F_x(t_1)\}\,p_q^n(t)\,dz \tag{3.16}$$

$$P(n,u)=(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\cos ku \tag{3.17}$$

$$Q(n,u)=(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin ku \tag{3.18}$$

$$K_n(F_x)=\int_{b(N)}^{b(\delta)}\left\{\left(\frac{1}{t}-\frac{1}{t_1}\right)F_x(t)p_q^n(t)g(z)\right.$$

$$\left.-\left(\frac{1}{t_1}-\frac{1}{t_2}\right)F_x(t_1)p_q^n(t_1)g(z+\pi/n)\right\}\sin nz\,dz \tag{3.19}$$

$$L_n(x)=E_n^q(T,x)-\chi_x(N). \tag{3.20}$$

4. Lemmas

To prove the theorem we will use the following lemmas.

Lemma 1. *If* $f\in H_\alpha$, $0<\alpha\leqslant 1$, *Then*

$$|\Phi_x(u)-\Phi_y(u)|=O(u^\alpha) \tag{4.1}$$

and also

$$|\Phi_x(u)-\Phi_y(u)|=O(|x-y|^\alpha). \tag{4.2}$$

The proof of the Lemma is an easy consequence of definition of $\Phi_x(u)$, $\Phi_y(u)$ and H_α.

Lemma 2. ([1], p. 101). *Let* $0\leqslant u\leqslant\pi$. *Then* $p_q^n(u)\leqslant e^{-Anu^2}$ (4.3)

where

$$A=2q(\pi(q+1))^{-2}$$

and

$$b(N)>n^{-1},\quad (n>4(q+1),\ q>0). \tag{4.4}$$

Lemma 3. ([1], p. 101). *Let* $0<z<\delta$. *Then*

$$t_r-t_{r-1}=O(n^{-1}),\quad (r=1,2) \tag{4.5}$$

and

$$2t_1-t-t_2=O(n^{-2})(z+\pi/n). \tag{4.6}$$

Lemma 4. ([1], p. 103). *Let* $0<z<\delta$. *Then*

$$p_q^n(t_1)g(z+\pi/n)-p_q^n(t)g(z)=O(n^{-1})\{z+\pi/n+n\sin t_1\}p_q^n(z). \tag{4.7}$$

Lemma 5. *Let* $\theta=\tan^{-1}\left(\dfrac{\sin u}{q+\cos u}\right)$ *and* $p_q^n(u)$ *be as in* (3.10); *then*

$$P(n,u)=p_q^n(u)\cos n\theta, \tag{4.8}$$

$$Q(n,u)=p_q^n(u)\sin n\theta, \tag{4.9}$$

$$P(n,u)=O(1), \tag{4.10}$$

$$Q(n,u)=O(nu). \tag{4.11}$$

Proof. By familiar computation

$$\begin{aligned}P(n,u)+iQ(n,u)&=(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}e^{iku}\\&=(q+1)^{-n}(q+e^{iu})^{n}\\&=p_q^n(u)(\cos n\theta+i\sin n\theta)\end{aligned}$$

which ensures (4.8) and (4.9). Since $|\cos ku|\leqslant 1$, $|\sin ku|\leqslant ku$ and $\Sigma_{k=0}^{n}\binom{n}{k}q^{n-k}=(q+1)^n$, estimates (4.10) and (4.11) follow at once.

Lemma 6. *If $f\in H_\alpha$ and $0<\alpha\leqslant 1$, then*

$$F_x(u)=O(1)\begin{cases}u^\alpha, & u>N\\ n^{-\alpha}, & u<N,\end{cases}\tag{4.12}$$

$$G(u)=O(1)\begin{cases}u^\alpha, & u>N\\ n^{-\alpha}, & u<N\end{cases}\tag{4.13}$$

and

$$G(u)=O(1)|x-y|^\alpha\begin{cases}\log n, & u>N\\ \log 1/u, & u<N.\end{cases}\tag{4.14}$$

Proof. As $f\in H_\alpha$, we have

$$\begin{aligned}F_x(u)&=\chi_x(u)-\chi_x(N)\\&=\int_u^N\Phi_x(w)\tfrac{1}{2}\cot w/2\,\mathrm{d}w\\&=O(1)\int_u^N w^{\alpha-1}\mathrm{d}w\\&=O(1)\begin{cases}u^\alpha, & u>N\\ n^{-\alpha}, & u<N.\end{cases}\end{aligned}$$

Again using Lemma 1, we get

$$\begin{aligned}G(u)&=\int_u^N\frac{\Phi_x(w)-\Phi_y(w)}{2\tan w/2}\mathrm{d}w=O(1)\int_u^N w^{\alpha-1}\mathrm{d}w\\&=O(1)\begin{cases}u^\alpha, & u>N\\ n^{-\alpha}, & u<N\end{cases}\end{aligned}$$

and

$$\begin{aligned}G(u)&=\int_u^N\frac{\{\Phi_x(w)-\Phi_y(w)\}}{2\tan w/2}\mathrm{d}w\\&=O(1)|x-y|^\alpha\int_u^N\frac{\mathrm{d}w}{w}\\&=O(1)|x-y|^\alpha\begin{cases}\log n, & u>N\\ \log 1/u, & u<N.\end{cases}\end{aligned}$$

This completes the proof.

Lemma 7. *If* $f \in \mathrm{Lip}\,\alpha$, $(0 < \alpha \leqslant 1)$ *then*

$$\| K_n(F) \| = O(n^{-\alpha}) + O(\| R_n^q \|), \tag{4.15}$$

where R_n^q *and* $K_n(F_x)$ *are defined in* (3.16) *and* (3.19) *respectively.*

Proof.

$$\begin{aligned}
K_n(F_x) &= \int_{b(N)}^{b(\delta)} \left\{ \left(\frac{1}{t} - \frac{1}{t_1}\right) F_x(t) p_q^n(t) g(z) \right. \\
&\quad \left. - \left(\frac{1}{t_1} - \frac{1}{t_2}\right) F_x(t_1) p_q^n(t_1) g(z + \pi/n) \right\} \sin nz \,\mathrm{d}z \\
&= \int_{b(N)}^{b(\delta)} \left[\left(\frac{t_2 - t_1}{t_1 t_2}\right) \{F_x(t) - F_x(t_1)\} p_q^n(t_1) g(z + \pi/n) \sin nz \right. \\
&\quad + \left(\frac{t_2 - t_1}{t t_1}\right) \{p_q^n(t) g(z) - p_q^n(t_1) g(z + \pi/n)\} F_x(t) \sin nz \\
&\quad + \frac{(t_2 - t_1)(t_2 - t)}{t t_1 t_2} \{p_q^n(t_1) g(z + \pi/n)\} F_x(t) \sin nz \\
&\quad \left. + \left(\frac{2t_1 - t - t_2}{t t_1}\right) p_q^n(t) g(z) F_x(t) \sin nz \right] \mathrm{d}z \\
&= K_n^1(x) + K_n^2(x) + K_n^3(x) + K_n^4(x), \quad \text{say.}
\end{aligned} \tag{4.16}$$

By using Lemmas 2 and 3 we have

$$\begin{aligned}
\| K_n^1 \|_c &= O(n^{-1}) \int_{b(N)}^{b(\delta)} t_1^{-1} t_2^{-1} |F_x(t) - F_x(t_1)| p_q^n(t_1) \mathrm{d}z \\
&= O(n^{-1}) n \int_{1/n}^{\delta} t_1^{-1} |(F_x(t) - F_x(t_1))| p_q^n(t_1) \mathrm{d}z \\
&= O(1) \left\{ \int_{1/n}^{\log n/(n)^{1/2}} t_1^{-1} |F_x(t) - F_x(t_1)| p_q^n(t_1) \mathrm{d}z \right. \\
&\quad \left. + \int_{\log n/(n)^{1/2}}^{\delta} t_1^{-1} |F_x(t) - F_x(t_1)| p_q^n(t_1) \mathrm{d}z \right\} \\
&= O(\| R_n^q \|) + O(1) \int_{\log n/(n)^{1/2}}^{\delta} t_1^{-1} |F_x(t) - F_x(t_1)| p_q^n(t_1) \mathrm{d}z \\
&= O(\| R_n^q \|) + O(1) \int_{\log n/(n)^{1/2}}^{\delta} t_1^{\alpha - 1} \mathrm{e}^{-Ant_1^2} \mathrm{d}z \\
&= O(\| R_n^q \|) + O(1) \int_{\log n/(n)^{1/2}}^{\delta} z^{\alpha - 1} \mathrm{e}^{-Anz^2} \mathrm{d}z \\
&= O(\| R_n^q \|) + O(n^{-1}) \int_{\log n/(n)^{1/2}}^{\delta} z^{\alpha - 2} \frac{\mathrm{d}}{\mathrm{d}z}(-\mathrm{e}^{-Anz^2}) \mathrm{d}z
\end{aligned}$$

$$= O(\|R_n^q\|) + O(n^{-1})\left\{|[z^{\alpha-2}(-\mathrm{e}^{-Anz^2})]_{\log n/(n)^{1/2}}^{\delta}| + \int_{\log n/(n)^{1/2}}^{\delta} z^{\alpha-3}\mathrm{e}^{-Anz^2}\,\mathrm{d}z\right\}$$

$$= O(\|R_n^q\|) + O(n^{-1})\frac{(\log n)^{\alpha-2}}{n^{\alpha/2-1}}\exp(-A(\log n)^2)$$

$$= O(\|R_n^q\|) + O(n^{-\alpha}). \tag{4.17}$$

Using Lemmas 2, 3, 4 and 6 we get

$$\|K_n^2\|_c = \left|\int_{b(N)}^{b(\delta)}\left(\frac{t_2-t_1}{tt_1}\right)\{p_q^n(t)g(z) - p_q^n(t_1)g(z+\pi/n)\}F_x(t)\sin nz\,\mathrm{d}z\right|$$

$$= O(n^{-1})\int_{b(N)}^{b(\delta)}\frac{n^{-1}(z+\pi/n+n\sin t_1)}{tt_1}p_q^n(z)|F_x(t)||\sin nz|\,\mathrm{d}z$$

$$= O(n^{-2})\int_{b(N)}^{b(\delta)}\frac{(nz)(nz)z^{\alpha}\mathrm{e}^{-Anz^2}\,\mathrm{d}z}{z^2} = O(1)\int_{1/n}^{\delta} z^{\alpha}\mathrm{e}^{-Anz^2}\,\mathrm{d}z$$

$$= O(n^{-1})\int_{1/n}^{\delta} z^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}z}(-\mathrm{e}^{-Anz^2})\mathrm{d}z$$

$$= O(n^{-1})\left\{|[z^{\alpha-1}(-\mathrm{e}^{-Anz^2})]_{1/n}^{\delta}| + (1-\alpha)\int_{1/n}^{\delta} z^{\alpha-2}(\mathrm{e}^{-Anz^2})\mathrm{d}z\right\}$$

$$= O(n^{-\alpha}). \tag{4.18}$$

Using Lemmas 2, 3 and 6 we get

$$\|K_n^3\|_c = \left\|\int_{b(N)}^{b(\delta)}\frac{(t_2-t_1)(t_2-t)}{tt_1t_2}p_q^n(t_1)g(z+\pi/n)F_x(t)\sin nz\,\mathrm{d}z\right\|$$

$$= O(n^{-2})\int_{1/n}^{\delta} t^{-3}\|F_x(t)\|\,\mathrm{d}z$$

$$= O(n^{-\alpha}). \tag{4.19}$$

Again by Lemmas 2, 3 and 6 we get

$$\|K_n^4\|_c = \left\|\int_{b(N)}^{b(\delta)}\left(\frac{2t_1-t-t_2}{tt_1}\right)p_q^n(t)F(t)g(z)\sin nz\,\mathrm{d}z\right\|$$

$$= O(n^{-2})\int_{1/n}^{\delta}\frac{(z+\pi/n)\|F(t)\|\,\mathrm{d}z}{tt_1}$$

$$= O(n^{-2})\int_{1/n}^{\delta} z^{\alpha-1}\,\mathrm{d}z$$

$$= O(n^{-2}). \tag{4.20}$$

Using (4.18), (4.19), (4.20) and (4.21) in (4.17) we get Lemma 7.

5. Proof of the theorem

From (3.6) and (1.8) we have

$$E_n^q(x) = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left[\frac{2}{\pi} \int_0^{\pi} \chi_x(u) D_k(u) \mathrm{d}u \right].$$

Hence

$$\begin{aligned}
L_n(x) &= E_n^q(x) - \chi_x(N) \\
&= E_n^q(x) - \frac{2}{\pi} \int_0^{\pi} \chi(N) D_k(u) \mathrm{d}u \\
&= \frac{2}{\pi} \int_0^{\pi} \{\chi(u) - \chi(N)\} \left\{ (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} D_k(u) \right\} \mathrm{d}u \\
&= \frac{2}{\pi} \int_0^{\pi} F_x(u) \left\{ (q+1)^{-n} \sum_{k=0}^{n} \binom{k}{n} q^{n-k} \frac{\sin(k+\frac{1}{2})u}{2\sin u/2} \right\} \mathrm{d}u \\
&= \frac{2}{\pi} \int_0^{\pi} F_x(u) \left\{ (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{\sin ku}{2\tan u/2} \right\} \mathrm{d}u \\
&\quad + \frac{2}{\pi} \int_0^{\pi} F_x(u) \left\{ (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{\cos ku}{2} \right\} \mathrm{d}u \\
&= \frac{2}{\pi} \int_0^{\pi} F_x(u) \frac{Q(n,u)}{2\tan u/2} \mathrm{d}u + \frac{1}{\pi} \int_0^{\pi} F_x(u) P(n,u) \mathrm{d}u. \qquad (5.1)
\end{aligned}$$

And hence

$$\begin{aligned}
L_n(x) - L_n(y) &= \frac{2}{\pi} \int_0^{\pi} \{F_x(u) - F_y(u)\} \frac{Q(n,u)}{2\tan u/2} \mathrm{d}u \\
&\quad + \frac{1}{\pi} \int_0^{\pi} \{F_x(u) - F_y(u)\} P(n,u) \mathrm{d}u \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{G(u) Q(n,u) \mathrm{d}u}{2\tan u/2} + \frac{1}{\pi} \int_0^{\pi} G(u) P(n,u) \mathrm{d}u \\
&= I + J, \quad \text{say.} \qquad (5.2)
\end{aligned}$$

Now

$$\begin{aligned}
J &= \frac{1}{\pi} \int_0^{\pi} G(u) P(n,u) \mathrm{d}u = \frac{1}{\pi} \left[\int_0^{N} + \int_N^{\log n/(n)^{1/2}} + \int_{\log n/(n)^{1/2}}^{\pi} \right] \\
&= J_1 + J_2 + J_3, \text{ say.} \qquad (5.3)
\end{aligned}$$

Using Lemmas 5 and 6, we obtain

$$J_1 = O(1) n^{-\alpha} \int_0^{N} \mathrm{d}u = O(n^{-1-\alpha}) \qquad (5.4)$$

and

$$\begin{aligned}
J_1 &= O(1) |x-y|^{\alpha} \int_0^{N} \log \frac{1}{u} \mathrm{d}u \\
&= O(1) |x-y|^{\alpha} \frac{\log n}{n}. \qquad (5.5)
\end{aligned}$$

Using Lemmas 2 and 6

$$\begin{aligned} J_2 &= \frac{1}{\pi}\int_N^{\log n/(n)^{1/2}} G(u)P(n,u)\mathrm{d}u \\ &= O(1)\int_N^{\log n/(n)^{1/2}} u^\alpha \mathrm{e}^{-Anu^2}\mathrm{d}u \\ &= O(n^{-1})\int_N^{\log n/(n)^{1/2}} u^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}u}(-\mathrm{e}^{-Anu^2})\mathrm{d}u \\ &= O(n^{-1})\left\{|[u^{\alpha-1}(-\mathrm{e}^{-Anu^2})]_N^{\log n/(n)^{1/2}}| + (1-\alpha)\int_N^{\log n/(n)^{1/2}} u^{\alpha-2}\mathrm{e}^{-Anu^2}\mathrm{d}u\right\} \\ &= O(n^{-\alpha}). \end{aligned} \tag{5.6}$$

Again by Lemmas 5 and 6

$$\begin{aligned} J_2 &= \frac{1}{\pi}\int_N^{\log n/(n)^{1/2}} G(u)P(n,u)\mathrm{d}u \\ &= O(1)|x-y|^\alpha \log n \int_N^{\log n/(n)^{1/2}} \mathrm{d}u \\ &= O(1)|x-y|^\alpha \frac{(\log n)^2}{\sqrt{n}}. \end{aligned} \tag{5.7}$$

By Lemmas 2 and 6

$$\begin{aligned} J_3 &= \frac{1}{\pi}\int_{\log n/(n)^{1/2}}^{\pi} G(u)p_q^n(u)\cos n\theta\,\mathrm{d}u \\ &= O(1)\int_{\log n/(n)^{1/2}}^{\pi} u^\alpha p_q^n(u)\mathrm{d}u \\ &= O(1)\int_{\log n/(n)^{1/2}}^{\pi} u^\alpha \mathrm{e}^{-Anu^2}\mathrm{d}u \\ &= O(1)\mathrm{e}^{-A(\log n)^2}\int_{\log n/(n)^{1/2}}^{\pi} u^\alpha \mathrm{d}u \\ &= O\left(\frac{1}{n^\Delta}\right), \quad \Delta > 0 \text{ however large} \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} J_3 &= \frac{1}{\pi}\int_{\log n/(n)^{1/2}}^{\pi} G(u)p_q^n(u)\cos n\theta\,\mathrm{d}u \\ &= O(1)|x-y|^\alpha \log n\int_{\log n/(n)^{1/2}}^{\pi} \mathrm{e}^{-Anu^2}\mathrm{d}u \\ &= O(1)|x-y|^\alpha \log n\,\mathrm{e}^{-A(\log n)^2}\int_{\log n/(n)^{1/2}}^{\pi} \mathrm{d}u \\ &= O(1)|x-y|^\alpha \frac{\log n}{n^\Delta}, \quad \Delta > 0, \text{ however large.} \end{aligned} \tag{5.9}$$

Collecting the estimates of $J_i (i = 1, 2, 3)$ from (5.4), (5.5), (5.6), (5.7), (5.8) and (5.9) and using them in (5.3) we get

$$J = O(1)\begin{cases} n^{-\alpha} \\ |x-y|^{\alpha} \end{cases} \frac{(\log n)^2}{\sqrt{n}}. \tag{5.10}$$

Now

$$\begin{aligned} I &= \frac{2}{\pi}\int_0^{\pi} \frac{G(u)Q(n,u)\mathrm{d}u}{2\tan u/2} \\ &= \frac{2}{\pi}\left\{\int_0^{N} + \int_N^{\delta} + \int_{\delta}^{\pi}\right\} \\ &= I_1 + I_2 + I_3, \text{ say.} \end{aligned} \tag{5.11}$$

By Lemmas 5 and 6

$$\begin{aligned} I_1 &= \frac{2}{\pi}\int_0^{N} \frac{G(u)Q(n,u)\mathrm{d}u}{2\tan u/2} \\ &= O(1)n^{-\alpha}\int_0^{N} \mathrm{d}u \\ &= O(n^{-\alpha}) \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} I_1 &= \frac{2}{\pi}\int_0^{N} \frac{G(u)Q(n,u)}{2\tan u/2}\mathrm{d}u \\ &= O(1)\,|x-y|^{\alpha} n\int_0^{N} \log\frac{1}{u}\mathrm{d}u \\ &= O(1)\,|x-y|^{\alpha}\log n. \end{aligned} \tag{5.13}$$

Combining (5.12) and (5.13) we get

$$I_1 = O(1)\begin{cases} n^{-\alpha} \\ |x-y|^{\alpha}\log n. \end{cases} \tag{5.14}$$

By Lemmas 2, 5 and 6, we get

$$\begin{aligned} I_3 &= \frac{2}{\pi}\int_{\delta}^{\pi} \frac{G(u)p_q^n(u)\sin n\theta\,\mathrm{d}u}{2\tan u/2} \\ &= O(1)\int_{\delta}^{\pi} u^{\alpha-1}\mathrm{e}^{-Anu^2}\mathrm{d}u \\ &= O(1)\mathrm{e}^{-An\delta^2}\int_{\delta}^{\pi} u^{\alpha-1}\mathrm{d}u \\ &= O(1)\mathrm{e}^{-An\delta^2} \\ &= O(n^{-\Delta}), \quad \Delta > 0 \text{ however large} \end{aligned} \tag{5.15}$$

and

$$I_3 = \frac{2}{\pi}\int_{\delta}^{\pi} \frac{G(u)p_q^n(u)\sin n\theta\,\mathrm{d}u}{2\tan u/2}$$

$$= O(1)|x-y|^\alpha \log n \int_\delta^\pi \frac{e^{-Anu^2}}{u} du$$

$$= O(1)|x-y|^\alpha e^{-An\delta^2} \log n \int_\delta^\pi u^{-1} du$$

$$= O(1)\frac{|x-y|^\alpha}{n^\Delta}, \quad \Delta > 0, \text{ however large.} \tag{5.16}$$

Combining (5.15) and (5.16) we get

$$I_3 = O(1)\begin{cases} n^{-\Delta}, & \Delta > 0 \text{ however large} \\ \dfrac{|x-y|^\alpha}{n^\Delta}. \end{cases} \tag{5.17}$$

By Lemma 5

$$I_2 = \frac{1}{\pi}\int_N^\delta G(u) \cot u/2 \, p_q^n(u) \sin n\theta \, du$$

$$= \frac{2}{\pi}\int_N^\delta \left[\frac{G(u)}{u} p_q^n(u) \sin n\theta + G(u)\left\{\frac{1}{2\tan u/2} - \frac{1}{u}\right\} p_q^n(u) \sin n\theta \right] du$$

$$= L + M, \quad \text{say.} \tag{5.18}$$

Using Lemmas 2 and 6 and the fact that $\{(\tan u/2)^{-1} - u^{-1}\} = O(u)$ we get

$$M = O(1)\int_N^\delta u^{\alpha+1} e^{-Anu^2} du$$

$$= O(n^{-1})\int_N^\delta u^\alpha \frac{d}{du}(-e^{-Anu^2}) du$$

$$= O(n^{-1})\left\{ |[u^\alpha(-e^{-Anu^2})]_N^\delta| + \int_N^\delta u^\alpha e^{-Anu^2} du \right\}$$

$$= O(n^{-1}). \tag{5.19}$$

Now for the estimation of L, we use the transformation

$$u = t(z) = z + \sin^{-1}(q \sin z),$$

which is same as $z = \tan^{-1}(\sin u/(q + \cos u))$.

By simple calculation, we have

$$du = g(z) dz, \quad \frac{\sin u}{q + \cos u} = \tan z$$

and

$$\sin n\theta = \sin n\left(\tan^{-1}\left(\frac{\sin u}{q + \cos u}\right)\right) = \sin nz$$

where $g(z)$ is defined in (3.11).

Throughout the present work, we write t for $t(z)$. Hence

$$L = \frac{2}{\pi}\int_N^\delta \frac{G(u)}{u} p_q^n(u) \sin n\theta \, du$$

$$= \frac{2}{\pi}\int_{b(N)}^{b(\delta)} \frac{G(u)}{t} p_q^n(t) \sin nz\, g(z)\, dz$$

$$= \frac{1}{\pi}\left[\int_{b(N)}^{b(\delta)} + \int_{b(N)+\pi/n}^{b(\delta)+\pi/n} + \int_{b(N)}^{b(N)+\pi/n} - \int_{b(\delta)}^{b(\delta)+\pi/n}\right]. \tag{5.20}$$

Now

$$\frac{1}{\pi}\int_{b(N)+\pi/n}^{\delta+\pi/n} \frac{G(t)}{t} p_q^n(t) g(z) \sin nz\, dz$$

$$= -\frac{1}{\pi}\int_{b(N)}^{b(\delta)} \frac{G(t_1)}{t_1} p_q^n(t_1) g(z+\pi/n) \sin nz\, dz. \tag{5.21}$$

Using (5.21) in (5.20) we get

$$L = \frac{1}{\pi}\int_{b(N)}^{b(\delta)} \{t^{-1}G(t)p_q^n(t)g(z) - t_1^{-1}G(t_1)p_q^n(t_1)g(z+\pi/n)\} \sin nz\, dz$$

$$+ \frac{1}{\pi}\left(\int_{b(N)}^{b(N)+\pi/n} - \int_{b(\delta)}^{b(\delta)+\pi/n}\right) \frac{G(t)}{t} p_q^n(t) g(z) \sin nz\, dz$$

$$= \frac{1}{\pi}T + \frac{1}{\pi}\left(\int_{b(N)}^{b(N)+\pi/n} - \int_{b(\delta)}^{b(\delta)+\pi/n}\right) \frac{G(t)}{t} p_q^n(t) g(z) \sin nz\, dz. \tag{5.22}$$

Using Lemma 6 and the fact that $g(z) = O(1)$, we get

$$\int_{b(N)}^{b(\delta)+\pi/n} \frac{G(t)}{t} p_q^n(t) g(z) \sin nz\, dz$$

$$= O(1)\int_{b(N)}^{b(N)+\pi/n} t^{\alpha-1}\, dz = O(1)\int_{b(N)}^{b(N)+\pi/n} z^{\alpha-1}\, dz$$

$$= O(n^{-\alpha}). \tag{5.23}$$

Again by Lemmas 2 and 6

$$\int_{b(\delta)}^{b(\delta)+\pi/n} \frac{G(t)}{t} p_q^n(t) g(z) \sin nz\, dz$$

$$= O(1)\int_{b(\delta)}^{b(\delta)+\pi/n} t^{\alpha-1} e^{-Ant^2}\, dz$$

$$= O(1) e^{-An\delta^2} = O(n^{-\Delta}), \quad \Delta > 0 \text{ however large.} \tag{5.24}$$

Using (5.23) and (5.24) in (5.22) we get

$$L = \frac{1}{\pi}T + O(n^{-\alpha}). \tag{5.25}$$

Using (5.19) and (5.25) in (5.18) we get

$$I_2 = \frac{1}{\pi}T + O(n^{-\alpha}). \tag{5.26}$$

Now

$$T = \int_{b(N)}^{b(\delta)} \left\{\frac{1}{t}G(t) - \frac{1}{t_1}G(t_1)\right\} p_q^n(t) g(z) \sin nz\, dz$$

$$+\int_{b(N)}^{b(\delta)}\{p_q^n(t)g(z)-p_q^n(t_1)g(z+\pi/n)\}\frac{G(t_1)}{t_1}\sin nz\,\mathrm{d}z$$
$$=T_1+T_2,\quad\text{say.}\tag{5.27}$$

By Lemmas 2 and 4

$$\begin{aligned}T_2&=O(1)\int_{b(N)}^{b(\delta)}n^{-1}\{z+\pi/n+n\sin t_1\}p_q^n(z)t_1^{\alpha-1}\mathrm{d}z\\&=O(1)\int_{1/n}^{\delta}p_q^n(z)t_1^{\alpha}\mathrm{d}z\\&=O\left(\frac{1}{n}\right)\int_{1/n}^{\delta}z^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}z}(-\mathrm{e}^{-Anz^2})\mathrm{d}z\\&=O(n^{-\alpha}),\end{aligned}\tag{5.28}$$

integrating by parts. Now

$$\begin{aligned}T_1&=\int_{b(N)}^{b(\delta)}\frac{\{G(t)-G(t_1)\}}{t_1}p_q^n(t)\mathrm{d}z\\&\quad+\left(\int_{b(N)}^{b(N)+(\pi/n)}+\int_{b(N)+\pi/n}^{b(\delta)}\right)\left(\frac{1}{t}-\frac{1}{t_1}\right)G(t)p_q^n(t)g(z)\sin nz\,\mathrm{d}z\\&=T_{1,1}+T_{1,2}+T_{1,3},\quad\text{say.}\end{aligned}\tag{5.29}$$

As

$$G(t)-G(t_1)=(F_x(t)-F_x(t_1))-(F_y(t)-F_y(t_1)),$$

we have

$$\begin{aligned}T_{1,1}&=O(1)\int_{1/n}^{\delta}\frac{|(G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\\&=O(1)\left[\int_{1/n}^{\log n/(n)^{1/2}}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right.\\&\qquad\left.+\int_{\log n/(n)^{1/2}}^{\delta}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right]\\&=O(1)\left[\int_{1/n}^{\log n/(n)^{1/2}}\frac{|F_x(t)-F_x(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right.\\&\qquad\left.+\int_{1/n}^{\log n/(n)^{1/2}}\frac{|F_y(t)-F_y(t_1)|}{t_1}p_q^n(t)\mathrm{d}z+\int_{\log n/(n)^{1/2}}^{\delta}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right]\\&=O(1)\left[R_n^q(x)+R_n^q(y)+\int_{\log n/(n)^{1/2}}^{\delta}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right].\end{aligned}\tag{5.30}$$

By Lemmas 4 and 6 we have

$$\int_{\log n/(n)^{1/2}}^{\delta}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z$$

$$= O(1)\int_{\log n/(n)^{1/2}}^{\delta} t_1^{\alpha-1}e^{-Anz^2}dz$$

$$= O\left(\frac{1}{n^{\Delta}}\right), \quad \Delta > 0 \text{ however large.} \tag{5.31}$$

Using (5.31) in (5.30), we get

$$T_{1,1} = O(1)(R_n^q(x) + R_n^q(y)) + O(n^{-\Delta}). \tag{5.32}$$

By Lemmas 3 and 6, we get

$$\begin{aligned} T_{1,2} &= O(1)\int_{b(N)}^{b(N)+\pi/n} \frac{n^{-1}t^{\alpha}nz\,dz}{t_1 t} \\ &= O(1)\int_{b(N)}^{b(N)+\pi/n} z^{\alpha-1}dz \\ &= O(n^{-\alpha}) \end{aligned} \tag{5.33}$$

$$\begin{aligned} 2T_{1,3} &= 2\int_{b(N)+\pi/n}^{\delta}\left(\frac{1}{t}-\frac{1}{t_1}\right)G(t)p_q^n(t)g(z)\sin nz\,dz \\ &= \left(\int_{b(N)}^{b(\delta)} + \int_{b(N)+\pi/n}^{b(\delta)+\pi/n}\right) + \int_{b(N)}^{b(N)+\pi/n} - \int_{b(\delta)}^{b(\delta)+\pi/n} \\ &= \int_{b(N)}^{b(\delta)} + \int_{b(N)+\pi/n}^{b(\delta)+\pi/n} + T_{1,2} + O(1)\int_{b(\delta)}^{b(\delta)+\pi/n}\left(\frac{1}{t}-\frac{1}{t_1}\right)|G(t)|\,dz. \end{aligned} \tag{5.34}$$

Since $t_1(z) = t(z+\pi/n)$ and $t_2(z) = t_1(z+\pi/n)$, replacing z by $z+\pi/n$, we get

$$\begin{aligned} &\int_{b(N)+\pi/n}^{b(\delta)+\pi/n}\left(\frac{1}{t}-\frac{1}{t_1}\right)G(t)p_q^n(t)g(z)\sin nz\,dz \\ &= -\int_{b(N)}^{b(\delta)}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)G(t_1)p_q^n(t_1)g(z+\pi/n)\sin nz\,dz. \end{aligned} \tag{5.35}$$

By Lemmas 6

$$\begin{aligned} &\int_{b(\delta)}^{b(\delta)+\pi/n}\left(\frac{1}{t}-\frac{1}{t_1}\right)|G(t)|\,dz \\ &= O(n^{-1})\int_{b(\delta)}^{b(\delta)+\pi/n}\frac{t^{\alpha-1}}{t_1}dz \\ &= O(n^{-1}). \end{aligned} \tag{5.36}$$

Using (5.33), (5.35) and (5.36) in (5.34), we have

$$\begin{aligned} 2T_{1,3} &= \int_{b(N)}^{b(\delta)}\left\{\left(\frac{1}{t}-\frac{1}{t_1}\right)G(t)p_q^n(t)g(z)\right. \\ &\quad \left. - \left(\frac{1}{t_1}-\frac{1}{t_2}\right)G(t_1)p_q^n(t_1)g(z+\pi/n)\right\}\sin nz\,dz + O(n^{-\alpha}) \\ &= K_n(F_x) - K_n(F_y) + O(n^{-\alpha}). \end{aligned}$$

Hence by Lemma 7 we get

$$2T_{1,3} = O(\|R_n^q\|) + O(n^{-\alpha}). \tag{5.37}$$

Using (5.32), (5.33) and (5.37) in (5.29), we get

$$T_1 = R_n^q(x) + R_n^q(y) + O(n^{-\alpha}) + O(\|R_n^q\|). \tag{5.38}$$

Using (5.28) and (5.38) in (5.27) we have

$$\begin{aligned} T &= R_n^q(x) + R_n^q(y) + O(\|R_n^q\|) + O(n^{-\alpha}) \\ &= O(\|R_n^q\|) + O(n^{-\alpha}). \end{aligned} \tag{5.39}$$

Using (5.39) in (5.26), we can write

$$I_2 = O(\|R_n^q\|) + O(n^{-\alpha}). \tag{5.40}$$

By Lemmas 5 and 6

$$\begin{aligned} I_2 &= \frac{2}{\pi}\int_N^\delta \frac{G(u)Q(n,u)\mathrm{d}u}{2\tan u/2} \\ &= O(1)|x-y|^\alpha \log n \int_N^\delta \frac{\mathrm{d}u}{u} \\ &= O(1)|x-y|^\alpha (\log n)^2. \end{aligned} \tag{5.41}$$

Since $f \in H_\alpha$, we can write

$$\begin{aligned} 2\{F_x(t) - F_x(t_1)\} &= 2\int_t^{t_1} \frac{\Phi_x(u)\mathrm{d}u}{2\tan u/2} \\ &= O(1)\int_t^{t_1} u^{\alpha-1}\mathrm{d}u = O(n^{-\alpha}). \end{aligned} \tag{5.42}$$

Hence

$$\begin{aligned} \|R_n^q\| &= \int_{1/n}^{\log n/(n)^{1/2}} t^{-1}\|F(t) - F(t_1)\| p_q^n(t)\mathrm{d}z \\ &= O(n^{-\alpha})\int_{1/n}^{\log n/(n)^{1/2}} \frac{\mathrm{d}z}{z} \\ &= O(1)n^{-\alpha}\log n. \end{aligned} \tag{5.43}$$

Using (5.43) in (5.40) and combining with (5.41) we get

$$I_2 = O(1)\begin{cases} n^{-\alpha}\log n \\ |x-y|^\alpha (\log n)^2. \end{cases} \tag{5.44}$$

Using (5.10), (5.11), (5.14), (5.17), (5.44) in (5.2), we get

$$|L_n(x) - L_n(y)| = O(1)\begin{cases} n^{-\alpha}\log n \\ |x-y|^\alpha (\log n)^2. \end{cases} \tag{5.45}$$

Using (5.45), we get

$$|L_n(x) - L_n(y)| = |L_n(x) - L_n(y)|^{\beta/\alpha}|L_n(x) - L_n(y)|^{1-\beta/\alpha}$$

$$= O(1)(|x-y|^{\alpha}(\log n)^2)^{\beta/\alpha}(n^{-\alpha}\log n)^{1-\beta/\alpha}$$
$$= O(1)|x-y|^{\beta} n^{\beta-\alpha}(\log n)^{1+\beta/\alpha}$$

which further ensures that

$$\sup_{\substack{x,y\\ x\neq y}} |\Delta^{\beta} L_n(x,y)| = \sup_{\substack{x,y\\ x\neq y}} \frac{|L_n(x) - L_n(y)|}{|x-y|^{\beta}}$$
$$= O(1)n^{\beta-\alpha}(\log n)^{1+\beta/\alpha}. \tag{5.46}$$

Again $f \in H_{\alpha} \Rightarrow \Phi_x(\upsilon) = O(|\upsilon|^{\alpha})$ and so proceeding as above, we obtain

$$\|L_n(\cdot)\|_c = \sup_{-\pi \leqslant x < \pi} |L_n(x)| = O(1)n^{-\alpha}\log n. \tag{5.47}$$

Now the theorem is completely proved by combining (5.46) and (5.47).

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On the first cohomology of cocompact arithmetic groups

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Abstract. Results of Matsushima and Raghunathan imply that the first cohomology of a cocompact irreducible lattice in a semisimple Lie group G, with coefficients in an irreducible finite dimensional representation of G, vanishes unless the Lie group is $SO(n,1)$ or $SU(n,1)$ and the highest weight of the representation is an integral multiple of that of the standard representation.

We show here that every cocompact arithmetic lattice in $SO(n,1)$ contains a subgroup of finite index whose first cohomology is non-zero when the representation is one of the exceptional types mentioned above.

Keywords. Cohomology groups; arithmetic groups.

1. Introduction

In this paper we prove some results on the non-vanishing of certain first cohomology groups. We recall that if G is a real semisimple group without compact factors such that G is not locally isomorphic to $SU(n,1)$ or $SO(n,1)$ and $\Gamma \subset G$ is any irreducible *cocompact* lattice and ρ is a finite dimensional irreducible representation of G, then $H^1(\Gamma,\rho)$ is zero. This vanishing theorem is proved in [R1] when ρ is non-trivial, and when ρ is trivial, the vanishing theorem is proved as a consequence of "property T" (in [K1] when G is not locally isomorphic to $Sp(n,1)$ or the real rank one form of F_4 and [Kos] in the remaining cases). See also [Mat] where a large number of groups are covered.

In the remaining cases of $G = SO(n,1)$ or $SU(n,1)$ suppose V is the standard representation of G on $\mathbf{C}^{n+1}$, V^* the dual, $\mathrm{sym}^l(V)$ (respectively $\mathrm{sym}^l(V^*)$) the lth-symmetric power of V (resp. of V^*), Q the quadratic form on V which is preserved by $SO(n,1)$, $Q\mathrm{sym}^{l-2}(V^*)$ the space of elements of $\mathrm{sym}^l(V^*)$ (i.e. polynomials) which are divisible by Q. Let H_l be the quotient space $\mathrm{sym}^l(V^*)/Q\mathrm{sym}^{l-2}(V^*)$. If $G = SU(n,1)$, $\rho \neq \mathrm{sym}^l(V)$ and $\rho \neq \mathrm{sym}^l(V^*)$ for any l, then $H^1(\Gamma,\rho) = 0$ for any cocompact lattice $\Gamma \subset SU(n,1)$. If $G = SO(n,1)$ and $\rho \neq H_l$ for any l, then $H^1(\Gamma,\rho) = 0$ for any cocompact lattice $\Gamma \subset SO(n,1)$. These two vanishing theorems are proved in [R1].

Thus only the cases (1) $G = SU(n,1)$, $\rho = \mathrm{sym}^l(V)$ or $\mathrm{sym}^l(V^*)$ and (2) $G = SO(n,1)$, $\rho = H_l$, remain to be considered. We note that in these two cases, the representation ρ may be described as the irreducible representation of G whose highest weight is l-times the highest weight of the standard representation.

In this paper we prove the following:

Theorem 1. *Let $n \geqslant 4$ be an integer and assume that $n \neq 7$. Let $\Delta \subset SO(n,1)$ be a cocompact arithmetic lattice. Let $l \geqslant 0$ be an integer, V_{n+1} the standard representation of*

$SO(n,1)$ and Q the quadratic form preserved by $SO(n,1)$ on V_{n+1}. Let H_l be the quotient $\mathrm{sym}^l(V^*_{n+1})/Q\mathrm{sym}^{l-2}(V^*_{n+1})$. *Then there exists a subgroup Δ' of finite index in Δ such that*

$$H^1(\Delta', H_l) \neq 0.$$

Remarks. (1) Theorem 1 holds even when $n=7$, provided Δ comes from an arithmetic structure which is not of the type (in the notation of [T]) ${}^3D_{40}$ or ${}^6D_{40}$.

(2) Theorem 1 is proved for all $n \geqslant 6$ in [L]. The case $n=5$ and $l=0$ is handled in [L-M] and [R-V]. The case of n even and $l=0$ is proved in [M1]. The case n even and l is non-zero is proved in [M2]. Thus, Theorem 1 is new only for $n=5$ and $l \neq 0$. However, our proof, which is a continuation of [R-V], works uniformly for all $n \geqslant 4$ and $l \geqslant 0$ and yields additional information which we describe below in Theorem 2. The main point of interest in our proof is that the non-vanishing of cohomology is obtained as a consequence of a relative congruence subgroup property which was proved in [R-V] and is therefore completely different from the usual proofs involving representation theory.

1.1. *Notation.* In this paper we only consider arithmetic lattices of $SU(n,1)$ which are of the following kind. Assume that K is a totally real number field, L is a totally imaginary quadratic extension of K, V is an $(n+1)$-dimensional vector space over L, $h_0: V \times V \to L$ a bi-additive map which is hermitian with respect to the action of the nontrivial element of the Galois group of L over K. Let G be the K-algebraic group $SU(h_0)$. We assume that $G(K \otimes \mathbf{R})$ is isomorphic to the product of $SU(n,1)$ with a compact group. The groups Γ that we consider are the ones coming from these arithmetic structures on $SU(n,1)$.

1.2. *Notation.* We now describe all the arithmetic lattices of $SO(n,1)$ $(n \geqslant 4)$ except those of $SO(7,1)$ which arise from K-forms of the type ${}^6D_{40}$ or ${}^3D_{40}$. They arise as follows.

Let K be a totally real number-field, D a central simple algebra over K of degree $d \leqslant 2$, V an m-dimensional D-vector space, and ι an involution on D given by $(tr(x)-x)$ if d is 2 and (x) if d is 1, for all x in D. Let $h: V \times V \to D$ be a biadditive map such that for all $\lambda, \mu \in D$ and $v, w \in V$ we have $h(\lambda v, \mu w) = \iota(\lambda) h(v,w)\mu$. Let $H = SU(V,h)$ be the special unitary group of this form h. We assume that h is so chosen that

$$H(K \otimes \mathbf{R}) = SO(n,1) \times \text{a compact group},$$

where $n+1 = md$.

By [T], the only arithmetic lattices in $SO(n,1)$ $(n \geqslant 4,\ n \neq 7)$ arise as $SO(n,1)$-conjugates of arithmetic subgroups of $H(K)$, for some H as above.

1.3. *Notation.* Let $\Delta \subset SO(n,1)$ be an arithmetic lattice as in (1.2). Let L/K be a quadratic extension which is totally imaginary and which splits D (it is easy to see, using weak approximation on K, that such an L exists). Let, for x in L, $x \to \bar{x}$ be the action of the non-trivial element of the Galois group of L/K. Let $D_L = D \otimes_K L$. On D_L, define the involution ι by $\iota(\lambda \otimes a) = \iota(\lambda) \otimes a$. Let $V_L = V \otimes_K L$ and define $h_L: V_L \times V_L \to D_L$ by $h_L(v_1 \otimes a_1, v_2 \otimes a_2) = h(v_1, v_2) \otimes a_1 a_2$ for all $v_1, v_2 \in V$ and $a_1, a_2 \in L$. Consider the K-algebraic group $G = SU(V_L, h_L)$. Then it is easy to see that

$$G(K \otimes_Q \mathbf{R}) = SU(n,1) \times \text{a compact group}.$$

Moreover, the arithmetic structure on $SU(n,1)$ is the same as the one defined in (1.1) for a suitable h_0.

Thus, given an arithmetic lattice Δ of the type (1.2) in $SO(n,1)$, there are natural ways of extending Δ to an arithmetic lattice Γ in $SU(n,1)$, of the kind described in (1.1). We will fix one such.

We have defined, in Theorem 1, the spaces V_{n+1}^* and the spaces H_l. We have the quotient map $\mathrm{sym}^l(V_{n+1}^*)\to H_l$. If G is as in (1.1), and $\sigma\in(\mathrm{Aut}G)(K)$, we have the homomorphism $\sigma(\Gamma)\to\Gamma$, given by $x\to\sigma^{-1}x$. This induces the map

$$H^1(\Gamma,\mathrm{sym}^l(V_{n+1}^*))\to H^1(\sigma(\Gamma),\mathrm{sym}^l(V_{n+1}^*)). \tag{1}$$

We also have the inclusion map $\sigma(\Gamma)\cap H\subset\sigma(\Gamma)$ and the quotient map above which induce the map

$$H^1(\sigma(\Gamma),\mathrm{sym}^l(V_{n+1}^*))\to H^1(\sigma(\Gamma)\cap H,H_l)). \tag{2}$$

The composite of (1) and (2) yields a map, which we denote by

$$\mathrm{Res}_\sigma: H^1(\Gamma,\mathrm{sym}^l(V_{n+1}^*))\to H^1(\sigma(\Gamma)\cap H,H_l).$$

Denote by Res the product map $\Pi_{\sigma\in(\mathrm{Aut}G)(K)}\mathrm{Res}_\sigma$. Then we have

Theorem 2. *If $n\geqslant 6$, and $n+1$ is even, then the map*

$$\mathrm{Res}: H^1(\Gamma,\mathrm{sym}^l(V_{n+1}^*))\to \prod_{\sigma\in(\mathrm{Aut}G)(K)} H^1(\sigma(\Gamma)\cap H,H_l)$$

is injective.

This paper is organized as follows. In § 2, we prove a proposition (see (2.4)) which relates the congruence subgroup kernels of two groups G and H with the injectivity of certain restriction maps closely related to those that occur in Theorems 2 and 3. We also show that the assumptions of (2.4) on G and H are satisfied for a large class of groups G and H (see 2.7). In § 3, we prove Theorem 2 using (2.7) and (2.4). In § 4, we prove Theorem 2. A more involved version of (2.4), namely Proposition (2.5) is used. We also prove an analogue of Theorem (2.7) in (4.4). In § 5, we deduce Theorem 1 from Theorem (4.6), and a Theorem in [B-W] for certain cocompact arithmetic lattices in $SU(n,1)$.

2. The congruence subgroup kernel and H^1

2.1. *Notation.* Let G be a linear algebraic simply connected semisimple group defined over a number field K. Assume that G is absolutely almost simple. Let H be a simply connected semi-simple group over K, $i:H\to G$ a morphism of algebraic groups over K with finite kernel. If A is an algebra over K, we denote by $G(A)$ the group of A-rational points of G. We assume that $\Pi_{v\in\infty}G(K_v)$ and $\Pi_{v\in\infty}H(K_v)$ are both noncompact. The group $G(K)$ may be given the structure of a topological group whose topology (called *arithmetic* (resp. *congruence*) *topology*) is obtained by designating *arithmetic* (resp. *congruence*) subgroups of $G(K)$ as open. The completion $G(a)$ (resp. $G(c)$) of $G(K)$, with respect to the arithmetic (resp. congruence) topology is called the *arithmetic* (resp. *congruence*) *completion* of G (note that by assumption $\Pi_{v\in\infty}G(K_v)$ is

not compact and therefore, by strong approximation, $G(c) = G(\mathbf{A}_f)$, where $\mathbf{A}_f$ denotes the ring of finite adeles over K). The identity map $G(K) \to G(K)$ induces a natural continuous homomorphism $G(a) \to G(c)$ whose kernel, as may be easily checked, is a compact profinite group. We have thus an exact sequence of groups given by $1 \to C(G) \to G(a) \to G(c) \to 1$, where $C(G)$ is called the *congruence subgroup kernel of* G. The map $i: H \to G$ induces natural continuous homomorphisms $i(a): H(a) \to G(a)$ and $i(c): H(c) \to G(c)$ such that $i(a)(C(H)) \subset C(G)$. Moreover the rectangles in the following diagram commute:

$$\begin{array}{ccccccccc} 1 & \to & C(G) & \to & G(a) & \to & G(c) & \to & 1 \\ & & \uparrow i(a) & & \uparrow i(a) & & \uparrow i(c) & & . \\ 1 & \to & C(H) & \to & H(a) & \to & H(c) & \to & 1 \end{array}$$

By taking $H = G$ and $f: G \to G$ an automorphism of G over K, we see that the group $(\mathrm{Aut}G)(K)$ acts on $C(G)$. Moreover $G(a)$ normalizes $C(G)$. We denote by $C(H, G, i)$ the closed subgroup of $C(G)$ generated by the collection $\{\sigma(i(a)(C(H))); \sigma \in (\mathrm{Aut}G)(K)\}$ of subgroups of $C(G)$. This group is normalized by $G(K) \subset G(a)$ and since $G(K)$ is dense in $G(a)$, we see that $C(H, G, i)$ is normalized by $G(a)$. We denote by $C_{H,G,i}$ (or by C when there is no ambiguity about H and i) the quotient group $C(G)/C(H, G, i)$. Let $\hat{G}$ be the quotient group $G(a)/C$. Then we have surjections $G(a) \to \hat{G} \to G(c)$ and $G(K)$ is a subgroup of $\hat{G}$. We write $G(K) \cap \hat{G}$ for the topological space obtained by the relative topology on $G(K)$ in $\hat{G}$. It is then easy to see that an arithmetic group Δ is open in $G(K) \cap \hat{G}$ if and only if $\Delta \cap \sigma(H)$ is a congruence subgroup of $\sigma(H)$ for all $\sigma \in (\mathrm{Aut}G)(K)$.

We now consider (mainly for handling the case of $SO(4,1)$ and $SO(5,1)$), a quotient of $\hat{G}$. Let Δ be an arithmetic group in $G(K)$ which satisfies the condition $(*)$ below:

$(*)$ there exists a congruence subgroup Γ of $G(K)$ such that for all $\sigma \in H(K)$ we have: $\sigma(H) \cap \Delta \supset \sigma(H) \cap \Gamma$.

In particular, $\sigma(H) \cap \Delta$ is a congruence subgroup of $G(K)$, i.e. Δ is open in $G(K) \cap \hat{G}$, by the remark made above. The completion of $G(K)$ with respect to the topology generated by designating Δ which satisfy $(*)$ to be open, is denoted by G^*. We clearly have surjections $\hat{G} \to G^* \to G(c)$. Write C^* for the image of C in G^*. Note that $C^* = C^*_{H,G,i}$ depends on i and H.

It is immediate from the definitions that $C = \lim(\bar{\Delta}/\Delta)$ where Δ runs through arithmetic subgroups which are open in $G(K) \cap \hat{G}$ and $\bar{\Delta}$ is the smallest congruence subgroup of $G(K)$ which contains Δ. It is also immediate that $C^* = \lim(\bar{\Delta}/\Delta)$ where Δ now runs through arithmetic subgroups which are open in $G^* \cap G(K)$ and Δ is as before (here, lim denotes the inverse limit). In particular, we observe that C^* is contained in the closure of Δ in G^* for every Δ, i.e. the closure Γ^* (of any *congruence* subgroup Γ of $G(K)$) contains C^*.

2.2. *Notation.* Let $\rho: (\mathrm{Aut}G)(K) \to GL(E)$ be a rational representation on a finite dimensional complex vector space E. Suppose that $\Gamma \subset G(K)$ (resp. $\Delta \subset H(K)$) is a congruence subgroup of G (resp. H). We have homomorphisms $\pi: G(K) \to (\mathrm{Aut}G)(K)$ and $\pi \circ i: H(K) \to (\mathrm{Aut}G)(K)$. Consider the cohomology group $H^1(\Gamma, \rho \circ \pi)$ (resp. $H^1(\Delta, \rho \circ \pi \circ i)$). We have the homomorphism $\sigma: \sigma^{-1}(\Gamma) \to \Gamma$ for all $\sigma \in (\mathrm{Aut}G)(K)$ which induces a map $H^1(\sigma): H^1(\Gamma, \rho) \to H^1(\sigma^{-1}(\Gamma), \rho \circ \sigma)$ which is given on a one-cocycle $Z(\gamma)$

by $H^1(\sigma)(Z(\gamma))=\rho(\sigma)^{-1}(Z(\sigma(\gamma))$. Thus we have a map $H^1(\Gamma,\rho)\xrightarrow{H^1(\sigma)}H^1(\sigma^{-1}(\Gamma),\rho)$. We also have the "restriction" map $H^1(\sigma^{-1}(\Gamma),\rho)\xrightarrow{H^1(i)}H^1(i^{-1}\sigma^{-1}(\Gamma),\rho)$. The composition of these two maps will be denoted $\mathrm{Res}_\sigma\colon H^1(\Gamma,\rho)\to H^1(i^{-1}\sigma^{-1}(\Gamma),\rho)$. We denote by Res the product map $\Pi_{\sigma\in(\mathrm{Aut}G)(K)}\mathrm{Res}_\sigma$ where

$$\mathrm{Res}\colon H^1(\Gamma,\rho)\to\prod_{\sigma\in(\mathrm{Aut}G)(K)}H^1(i^{-1}\sigma^{-1}(\Gamma),\rho).$$

The representation ρ is defined over the algebraic closure $\bar{K}$ of K (and hence on a finite extension K' of K), since $(\mathrm{Aut}G)$ is a reductive algebraic group over K. Therefore $E=E_{K'}\otimes_{K'}\mathbf{C}$ where $E_{K'}$ is a K'-vector space, and $\rho(G(K))\subset\rho(G(K'))\subset GL(E_{K'})$.

Let l be a non-archimedian local field containing K'. Let K_v and $\mathbf{Q}_p$ be the closures of K and $\mathbf{Q}$ in l. Write $E_l=E_{K'}\otimes_{K'}l$. Let $\Omega\subset GL(E_l)$ be a compact subgroup. Then it is easily checked that there exists a compact open subgroup $\mathscr{E}(\Omega)$ of E_l such that $\mathscr{E}(\Omega)$ is stable under multiplication by the integers O_l in l, and under the action of the group Ω.

If $\Gamma\subset G(K)$ is an arithmetic group, we have $\rho(\Gamma)\subset\rho(G(l))\subset GL(E_l)$ and since Γ is contained in a compact subgroup of $G(l)$, it is clear that $\rho(\Gamma)$ lies in a compact subgroup Ω of $GL(E_l)$. Let $\mathscr{E}=\mathscr{E}(\Omega)$ as above. Then $\mathscr{E}$ is stable under the action of Γ on E_l. We have

$$H^1(\Gamma,\rho)=H^1(\Gamma,E)=H^1(\Gamma,E_{K'})\otimes_{K'}\mathbf{C},$$

$$H^1(\Gamma,E_{K'})\otimes_{K'}l=H^1(\Gamma,E_l)=H^1(\Gamma,\mathscr{E})\otimes_{O_l}l.$$

Given $\xi\in H^1(\Gamma,\rho)$, we may restrict the class of ξ to the subgroup $\Gamma\cap\sigma(H)$ for every $\sigma\in(\mathrm{Aut}G)(K)$. We thus obtain a map

$$(**)\ H^1(\Gamma,\rho)\to\prod_\sigma H^1(\Gamma\cap\sigma(H),\rho)$$

which we again denote by Res.

2.3. *Lemma. Suppose that C^* is finite. Then, the kernel of the map*

$$\mathrm{Res}\colon H^1(\Gamma,\mathscr{E})\to\prod_\sigma H^1(\Gamma\cap\sigma(H),\mathscr{E})$$

is a torsion group. Here Γ is a congruence subgroup of $G(K)$ and Res *is the restriction map got by replacing ρ by $\mathscr{E}$ in* $(**)$.

Proof. Suppose $\xi\in H^1(\Gamma,\mathscr{E})$ is such that $\mathrm{Res}(\xi)=0$. Let Z be the one-cocycle representing $\xi, Z\colon\Gamma\to\mathscr{E}$. We may view Z as a homomorphism τ_Z of Γ into the semi-direct product $GL(\mathscr{E})\ltimes\mathscr{E}$:

$$\tau_Z(\gamma)=\begin{pmatrix}\rho(\gamma) & Z(\gamma)\\ 0 & 1\end{pmatrix}.$$

Since $\mathrm{Res}(\xi)=0$, we have, for each σ, and $\gamma\in\sigma(H)\cap\Gamma$, a vector $v\in\mathscr{E}$ such that

$$\tau_Z(\gamma)=\begin{pmatrix}1 & -v\\ 0 & 1\end{pmatrix}\begin{pmatrix}\rho(\gamma) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}.$$

Let $\mathcal{U}$ be an open subgroup of $GL(\mathscr{E})\ltimes \mathscr{E}$. It is easy to see that $\mathcal{U}$ contains an open subgroup $\mathcal{U}'$ of the form $\mathcal{U}_1\ltimes \mathcal{U}_2$ where $\mathcal{U}_1 \subset GL(\mathscr{E})$ is an open torsion free subgroup and

$$\mathcal{U}_2 \supset \{(\rho(g)-1)(w); g\in\mathcal{U}_1, w\in\mathscr{E}\}.$$

It is easy to see that

$$\tau_Z^{-1}(\mathcal{U})\cap\sigma(H)\cap\Gamma\supset\rho^{-1}(\mathcal{U}_1)\cap\sigma(H)\cap\Gamma.$$

The group $\rho^{-1}(\mathcal{U}_1\cap\Gamma)$ is a congruence subgroup since the map $\rho: G(K)\to GL(E_l)$ extends to a map

$$\rho: G(K_v)\to G(l)\to GL(E_l).$$

Therefore, $\tau_Z^{-1}(\mathcal{U})$ is open in $G^*\cap G(K)$. This shows that τ_Z extends to a continuous homomorphism $\tau_Z^*:\Gamma^*\to GL(\mathscr{E})\ltimes\mathscr{E}$. Then $G(K)\cap\rho^{-1}(\mathcal{U}_1)=\Gamma_1$ is a congruence subgroup of $G(K)$ and $\tau_Z^*(\Gamma_1^*)\subset\mathcal{U}_1\ltimes\mathscr{E}$. Since $\Gamma_1^*\supset C^*$, and by assumption, C^* is finite, the group $\tau_Z^*(C^*)$ is a finite subgroup of the torsion-free group $\mathcal{U}_1$ and is therefore trivial. Thus, we get a homomorphism

$$\tau_Z^*:\Gamma(c)\to GL(\mathscr{E})\ltimes\mathscr{E}\subset GL(E_l)\ltimes E_l,$$

where $\Gamma(c)$ is the (congruence) closure of Γ in $G(c)$. It is easily shown that $H^1(\Gamma(c),E_l)=0$. Therefore, $Z(\gamma)=\rho(\gamma)w-w$ for some w in E_l for all $\gamma\in\Gamma$. Since $\mathscr{E}$ is an open subgroup of E_l, there exists an integer $M\geqslant 0$ such that $p^M w\in\mathscr{E}$ (recall that $l\supset\mathbf{Q}_p$). Hence $p^M\xi=0$ in $H^1(\Gamma,\mathscr{E})$ i.e. ξ is a torsion.

2.4. PROPOSITION

If C^* is finite, then

$$\mathrm{Res}: H^1(\Gamma,\rho)\to\prod_\sigma H^1(\Gamma\cap\sigma(H),\rho)$$

is injective.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H^1(\Gamma,\mathscr{E}) & \xrightarrow{a} & \prod_\sigma H^1(\Gamma,\mathscr{E}) \\ \downarrow c & & \downarrow b \\ H^1(\Gamma,E_l) & \xrightarrow{d} & \prod_\sigma H^1(\Gamma,E_l), \end{array}$$

where a, d are restriction maps; b, c are induced by the inclusion of $\mathscr{E}$ in E_l and σ runs through all the elements of $(\mathrm{Aut}G)(K)$.

Let F be the kernel of d. Then there exists a finite subset Σ of $(\mathrm{Aut}G)(K)$ such that if we let σ run through only the elements of Σ in the commutative diagram above but denote the maps by the same letters, then kernel of d is again F. This is a consequence of the well-known fact that for the arithmetic group Γ, the space $H^1(\Gamma,E_l)$ is finite dimensional over l. We will therefore assume that in the above diagram the σ's lie in Σ.

Clearly, $\ker(b)$ and $\ker(c)$ are torsion groups and by Lemma (1.3), $\ker(a)$ is also torsion. Then,

$$\ker(d)\cap\mathrm{Im}(c)=c(c^{-1}(\ker(d)))=c(\ker(dc))=c(\ker(ba))$$

is a torsion group. Since $H^1(\Gamma, E_l)$ is torsion-free it follows that $\ker(d)\cap \mathrm{Im}(c)=0$. But $\mathrm{Im}(c)\otimes_{O_l} l$ is $H^1(\Gamma, E_l)$ and therefore $\mathrm{Im}(c)$ is an open subgroup of $H^1(\Gamma, E_l)$. Hence $0=\ker(d)\cap \mathrm{Im}(c)$ is an open subgroup of the l-vector space $\ker(d)$ i.e. $\ker(d)=0$. This proves the proposition.

We now reformulate Proposition (2.4) slightly differently, replacing $\Gamma\cap\sigma(H)$ by the group $\sigma(\Gamma)\cap H$ and considering the restriction map defined at the beginning of (2.2). This is because, we prefer to work with a fixed H and varying groups $\sigma(\Gamma)$. Thus, we have

2.4. PROPOSITION (reformulated)

Let (G, H, i) *be as in* (2.1). *Assume that* C *is finite.* ρ, Γ *are as in* (2.2), *then the map*

$$\mathrm{Res}: H^1(\Gamma,\rho)\to \prod_{\sigma\in(\mathrm{Aut}G)(K)} H^1(i^{-1}\sigma^{-1}(\Gamma),\rho)$$

is injective. Consequently if $H^1(\Gamma,\rho)\neq 0$ *for some congruence subgroup* Γ *of* $G(K)$ *then* $H^1(\Delta,\rho)\neq 0$ *for some congruence subgroup* Δ *of* $H(K)$.

2.5. *Notation.* In place of H, we may even take infinitely many K-groups H_m, *with* K-homomorphisms $i_m: H_m\to G$ with finite kernel. Instead of $C(G,H,i)$ consider the closed subgroup B of $C(G)$ generated by the collection $\{C(G,H_m,i_m)\}$ of subgroups of $C(G)$. The quotient group $C(G)/B$ is still denoted C. We define a new topology on $G(K)$ by designating an arithmetic subgroup $\Delta\subset G(K)$ to be open if there exists a *congruence* subgroup Γ of $G(K)$ such that for all $\sigma\in(\mathrm{Aut}G)(K)$, and all ϕ, we have

$$\sigma(H_\phi)\cap\Delta\supset\sigma(H_\phi)\cap\Gamma.$$

Then, the completion of $G(K)$ with respect to this topology (we must check that a left Cauchy sequence is right Cauchy; this can be readily checked; then, one can form the completion; cf. [S1]) is again denoted by G^*. We have, as before, a surjection $\hat{G}\to G^*$ and the image of C under this map is again denoted by C^*. As in (2.2) we may consider the restriction maps

$$H^1(\Gamma,\rho)\to \prod_{\sigma\in(\mathrm{Aut}G)(K)} H^1(i_m^{-1}\sigma^{-1}(\Gamma),\rho)$$

for each m. We may then take the product of all these maps and obtain the big restriction map

$$H^1(\Gamma,\rho)\to \prod_m \prod_{\sigma\in(\mathrm{Aut}G)(K)} H^1(i_m^{-1}\sigma^{-1}(\Gamma),\rho).$$

We then obtain the following.

2.6. PROPOSITION

Let (G, H_m, i_m) *be as in* (2.4). *Assume that* C^* *is finite.* ρ,Γ *are as in* (2.2), *then the map*

$$H^1(\Gamma,\rho)\to \prod_m \prod_{\sigma\in(\mathrm{Aut}G)(K)} H^1(i_m^{-1}\sigma^{-1}(\Gamma),\rho)$$

is injective. Consequently if $H^1(\Gamma,\rho)\neq 0$ *for some congruence subgroup* Γ *of* $G(K)$, *then* $H^1(\Delta,\rho)\neq 0$ *for some congruence subgroup* Δ *of* $H_m(K)$ *for some* m.

Proof. The proof is exactly the same as that of Proposition (2.4), except that instead of one group H, we have to consider infinitely many groups H_m. We will need this proposition to handle the case of $SO(5,1)$ and $SO(4,1)$.

2.7. *Notation.* Suppose K is a totally real number field, D a central simple algebra over K such that every Archimedian completion K_v of K splits D. Let W be a finite dimensional right D-module equipped with a D-valued form h on $W \times W$ such that h is Hermitian with respect to the standard involution $\imath(\imath(x) = \mathrm{tr}(x) - x$ on D if the degree of D over K is 2 and $\imath(x) = x$ if $D = K$). Let $H = \mathrm{Spin}(h)$ be the spin group of the Hermitian form h. We assume that h is so chosen, that for a fixed Archimedean completion ∞ of K, the group $H(K_\infty)$ is isomorphic to $\mathrm{Spin}(2m\text{-}1, 1)$ and for all other archimedean completions v of K, the group $H(K_v)$ is isomorphic to $\mathrm{Spin}(2m)$ (here, m is the dimension of W over D). We now choose a totally imaginary quadratic extension L over K such that L splits D. Let $a \to \bar{a}$ be the nontrivial Galois automorphism of (L/K). Write $W_L = W \otimes_K L$, $D_L = D \otimes_K L$. Given $\lambda = d \otimes a \in D_L$ with $d \in D$ and $a \in L$, let $\bar{\lambda} = \imath(d) \otimes \bar{a} \in D_L$. Then we get an involution $x \to \bar{x}$ of the second kind on D_L. On the free D_L module $W_L \times W_L$ define a K-biinvariant map by writing, for $w_1, w_2 \in W$ and $\lambda, \mu \in D_L, h_L(w_1 \otimes \lambda, w_2 \otimes \mu) = \bar{\lambda} h(w_1, w_2)\mu$. Let $G = SU(h_L)$ be the special unitary group of the form h_L on W_L which is Hermitian with respect to the involution on D_L defined above. We have then a K-homomorphism $\imath : H \to G$ with finite kernel.

2.8. *Theorem. In the notation of* (1.1), *if the degree of D over K is* 2, $K \neq \mathbf{Q}$ *and* $\dim_D(W) \geqslant 3$, *then* $C(G, H, \imath)$ *is finite.*

We refer to [R-V] for the proof. We also note that if $K = \mathbf{Q}$, but the other assumptions of (2.8) hold, then, by the Hasse–Minkowski theorem for quadratic forms, it follows that the $\mathbf{Q}$-ranks of H and G are both 1. Thus, the arithmetic lattice is not cocompact.

3. Proof of Theorem 2

3.1. *Notation.* Let Ω be an abstract group. We consider the category $\mathscr{C} = \mathscr{C}(\Omega)$ of finite dimensional completely reducible complex representations of Ω. If $E_1 \subset E_2 \in \mathscr{C}$, then $H^1(\Omega, E_1)$ is a direct summand of $H^1(\Omega, E_2)$, since $E_2 = E_1 \oplus E_1'$ for some $E_1' \in \mathscr{C}$ and (1) $H^1(\Omega, E_2) = H^1(\Omega, E_1) \oplus H^1(\Omega, E_1')$. Suppose that $f: \Omega \to \Omega'$ is a homomorphism of groups such that if ρ is a semisimple representation of Ω', then the composite of ρ and f is a semisimple representation of Ω. Let E' be a semisimple representation of Ω'. Let E'' be an Ω invariant subspace of E'. We may write $E' = E'' \oplus E$ as Ω modules. There is a canonical restriction map (2) $\mathrm{Res}: H^1(\Omega', E') \to H^1(\Omega, E')$. In view of (1), we get a projection map from $H^1(\Omega, E')$ into $H^1(\Omega, E'')$. We denote again by Res, the composite of this projection with the restriction map.

3.2. *Notation.* Suppose m, n are integers with $1 \leqslant m < n$. Let V_n be an n-dimensional vector space, and $\mathrm{sym}^l(V_n^*)$ the space of polynomials of degree l on V_n. If $0 \leqslant i$, we have, $\mathrm{sym}^i(V_n/V_m)^* \subset \mathrm{sym}^l(V_n^*)$ and

$$\mathrm{sym}^l(V_n^*) = \oplus_{0 \leqslant i \leqslant l} \mathrm{sym}^i(V_m^*) \otimes \mathrm{sym}^{l-i}(V_n/V_m)^*.$$

Let Q be a non-degenerate quadratic form on V_n. In particular, we have $Q \in \mathrm{sym}^2(V_n^*)$. We have

$$Q\mathrm{sym}^{l-2}(V_n^*) = \{P \in \mathrm{sym}^l(V_n^*);\ Q \text{ divides } P\}.$$

With these notations, we have

3.3. *Lemma. For all* $i \geqslant 0$, *and* $m \leqslant n$, *we have*

$$Q\mathrm{sym}^{l-2}(V_n^*) \cap [\mathrm{sym}^i(V_m^*) \otimes \mathrm{sym}^{l-i}(V_n/V_m)^*] = 0.$$

Proof. Suppose that $Qf = \Sigma A_r B_r$, where f, A_r, B_r belong respectively to the spaces $\mathrm{sym}^{l-2}(V_n^*)$, $\mathrm{sym}^i(V_m^*)$, and $\mathrm{sym}^{l-i}((V_n/V_m)^*)$. We may write $V_n = V_m \oplus W_m$ for some subspace W_m of V_n and for g in $\mathrm{sym}^k(V_n^*)$, we may write $g = g(x, y)$ for x in V_m and y in W_m. Then, for all complex numbers a, we have

$$Q(ax, y)f(ax, y) = \sum A_r(ax)B_r(y) = a^i \sum A_r B_r = a^i Qf.$$

This shows that Qf is divisible by $Q(ax, y)$ for all $a \neq 0$. This is impossible by the unique factorization of polynomials, unless $Qf = 0$.

3.4. *Notation.* We continue the notation of (3.2). By Lemma (3.3) we may assume that $E_i = \mathrm{sym}^i(V_m^*) \otimes \mathrm{sym}^{l-i}(V_n/V_m)^*$ is a subspace of

$$H_l = \mathrm{sym}^l V_n^* / Q\mathrm{sym}^{l-2} V_n^*.$$

Now let Ω be a reductive subgroup of $SO(Q)$ which maps V_m into itself. Then the spaces E_i, H_l are all semisimple Ω-modules, and therefore, $H_l = E_i \oplus E_i'$ for an Ω-stable submodule E_i' of H_l. We thus get the following inclusion of Ω-modules

$$\mathrm{sym}^l V_n^* = \oplus E_i \subset \oplus H_l \tag{1}$$

where the last sum is taken $(l+1)$-times

3.5. COROLLARY of (2.8)

Let $n+1$ *be even and suppose that* $n \geqslant 7$. *Then, the map*

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l V_{n+1}^*) \to \prod_{\sigma \in (\mathrm{Aut}G)(K)} H^1(\sigma(\Gamma) \cap H, \mathrm{sym}^l(V_{n+1}^*))$$

is injective.

Proof. Observe that $K \neq \mathbf{Q}$ (cf. the remark after the statement of Theorem (2.8). The corollary follows from Theorem (2.8) and Proposition (2.4).

3.6. *Notation.* Assume that $n+1$ is even and that $n \geqslant 7$. Let H be as in (2.2). By the Morita theory (see [Sch]) we may assume that the degree of D is always 2 (since $(n+1)$-dimensional quadratic spaces may be thought of as $((n+1)/2)$-dimensional Hermitian spaces over $D = M_2(K)$). We write, as in (2.2), $H = \mathrm{Spin}(W, h)$. As is easily seen, we can find a basis $e_1, \ldots, e_k$ of W over D such that if $W' = e_1 D \oplus \cdots \oplus e_{k-1} D$, then $\mathrm{Spin}(W', h) = \Omega$ is a K-algebraic group with $\Omega(K \otimes \mathbf{R}) = \mathrm{Spin}(n-2, 1) \times K'$ where K' is compact. Furthermore, $W = W' \oplus e_k D$ is an orthogonal direct sum.

3.7. *Lemma. Let* $(n+1)$ *be even with* $n \geqslant 7$. *Let* H *and* Ω *be as in* (3.6). *Then the restriction map*

$$\mathrm{Res}: H^1(\Delta, \mathrm{sym}^l V_{n+1}^*) \to \prod_{\tau \in (\mathrm{Aut} H)(K)} H^1(\tau(\Delta) \cap \Omega, \mathrm{sym}^l V_{n+1}^*)$$

in injective.

Proof. If $K = \mathbf{Q}$, then $\mathbf{Q}$-rank$(H) = \mathbf{Q}$-rank$(\Omega) = 1$ and therefore the arithmetic group is not cocompact. Therefore, $K \neq \mathbf{Q}$ and, by the main theorem of [R-V], the group $C_H/C(H,\Omega)$ is finite (this is where we need $n \geqslant 7$) and by Proposition (2.4), the lemma follows.

Theorem 3. *Let* H, G *be as in* (2.2) *with* $n+1$ *even and* $n \geqslant 7$. *Then, for every congruence subgroup* Γ *of* $G(K)$, *the map*

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l(V_{n+1}^*)) \to \prod_{\sigma \in (\mathrm{Aut} G)(K)} H^1(\sigma(\Gamma) \cap H, H_l)$$

is injective.

Proof. Let $\xi \in H^1(\Gamma, \mathrm{sym}^l(V_{n+1}^*))$ be such that $\mathrm{Res}(\xi)$ is zero. Then, for every $\tau \in (\mathrm{Aut} H)(K)$, we see that the restriction of ξ, as an element of $H^1(\tau(\sigma(\Gamma) \cap H) \cap \Omega, H_l)$ is zero. In the notation of (3.4) this means that the restriction of ξ, as an element of $H^1(\tau(\sigma(\Gamma) \cap H) \cap \Omega, E_i))$ is zero. But, by (1) of (3.4), it follows that $\mathrm{Res}(\xi)$ as an element of $H^1(\tau(\sigma(\Gamma) \cap H) \cap \Omega, \mathrm{sym}^l(V_{n+1}^*)$ is zero. By Lemma (3.7), $\mathrm{Res}(\xi)$ as an element of $H^1(\sigma(\Gamma) \cap H, \mathrm{sym}^l(V_{n+1}^*))$ is zero. Now (3.5) shows that ξ is zero.

3.8. *Notation.* We will now assume that $n+1$ is odd, with $n+1 \geqslant 9$. In the notation of (2.2), we have $D = K$ and Q is a quadratic form on an $n+1$-dimensional K-vector space W; set $H_1 = \mathrm{Spin}(W, Q)$. Let $V = W \oplus e_{n+1} K$ be an $(n+2)$-dimensional K-vector space. We may define a new quadratic form h on V by writing, for w in W and $\lambda \in K$

$$h(w+\lambda, w+\lambda) = Q(w,w) + \lambda^2 \theta$$

for some scalar $\theta \in K$ such that the group $H = \mathrm{Spin}(V, h)(K \otimes_{\mathbf{Q}} \mathbf{R})$ is isomorphic to the product of $\mathrm{Spin}(n+1, 1)$ with a compact group. We may also find a subspace W' of codimension 1 in W such that (a) $W = W' \oplus e_{n+1} K$ is an orthogonal direct sum for some vector e_{n+1} and (b) if h' denotes the restriction of h to W', then the group $H' = \mathrm{Spin}(W', h')(K \otimes_{\mathbf{Q}} \mathbf{R})$ is isomorphic to the product of $\mathrm{Spin}(n-1, 1)$ with a compact group. Let $\Delta \subset H(K)$ be a congruence subgroup. Then, by Lemma (3.7) (notice that $n+2$ is even!), we have

$$\mathrm{Res}\, H^1(\Delta, \mathrm{sym}^l V_{n+2}^*) \to \prod_{\sigma \in (\mathrm{Aut} H)(K)} H^1(\sigma(\Delta) \cap H_1, \mathrm{sym}^l V_{n+2}^*) \qquad (1)$$

is injective.

Since $V = W \oplus e_{n+1} K$, we see that $V_{n+2} = V_{n+1} \oplus e_{n+2} \mathbf{C}$. By (1) of (3.3), we get the following relations of H_1-modules

$$\mathrm{sym}^l V_{n+1}^* = \oplus E_i \subset \oplus H_l \qquad (2)$$

where $(*)$ $H_l = \mathrm{sym}^l V_{n+1}^* / Q \mathrm{sym}^{l-2} V_{n+1}^*$. We also note that $h = Q$ on W.

3.9. Theorem. *Suppose that $n+1$ is odd. We use the notation of* (2.2) *and* (3.9). *The map*

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*) \to \prod_{\sigma \in (\mathrm{Aut}G)(K)} H^1(\sigma(\Gamma) \cap H_1, H_l)$$

is an injection, provided $n \geqslant 8$.

Proof. Let $W'' = e_{n+1}K \oplus e_{n+2}K$. For every $\tau \in \mathrm{Spin}(W'', h)(K)$ we consider the map

$$\mathrm{Res}\, H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*) \to \prod_{\sigma \in (\mathrm{Aut}G)(K)} H^1(\sigma(\Gamma) \cap \tau(H_1), \tau(H_l))$$

where $\tau(H_l)$ is the quotient space defined in $(*)$ with V_{n+1} replaced by $\tau(V_{n+1})$.

Consider now the map Res of the Theorem. Suppose that for some $\xi \in H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*)$, we have $\mathrm{Res}(\xi) = 0$. Considering the restriction to the smaller group H', $\mathrm{Res}_\sigma : H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*) \to H^1(\sigma(\Gamma) \cap H', H_l)$ we obtain: $\mathrm{Res}_\sigma = 0$ in $H^1(\sigma(\Gamma) \cap H', H_l)$, for all $\sigma \in \mathrm{Aut}G(K)$. Now (2) shows that $\mathrm{sym}^l V_{n+1}^*$ is an H' submodule of $\oplus H_l$. Hence, by the remarks in (3.1), $\mathrm{Res}_{\sigma^*} = 0$ in $H^1(\sigma(\Gamma) \cap H', \mathrm{sym}^l V_{n+1}^*)$. By the naturality of the restriction map, we may replace V_{n+1} by the space $\tau(V_{n+1})$ and obtain for all σ and τ: $\mathrm{Res}(\xi) = 0 \in H^1(\sigma(\Gamma) \cap \tau(H'), \mathrm{sym}^l \tau(V_{n+1})^*)$. But, as an H'-module, the sum $\Sigma_\tau \mathrm{sym}^l \tau V_{n+1})^*$ is the same as the sum $\Sigma_i \mathrm{sym}^l V_n^* \otimes \mathrm{sym}^{l-i}(\tau(V_{n+1})/V_n)^*$. This, in turn, is $\mathrm{sym}^l V_{n+2}^*$. We now apply the remarks in (3.1) to conclude that ξ lies in the kernel of the map

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*) \to \prod_{\sigma \in (\mathrm{Aut}G)(K)} H^1(\sigma(\Gamma) \cap H_1, \mathrm{sym}^l V_{n+2}^*)$$

and from (1), we get $\mathrm{Res}(\xi) = 0$ in $\Pi_{\sigma \in (\mathrm{Aut}G)(K)} H^1(\sigma(\Gamma) \cap H, \mathrm{sym}^l V_{n+2}^*)$. Now Theorem 2 (replace n by $n+1$ there) shows that $\xi = 0$.

4. The cases $n = 4$ and 5

4.1. *Notation* We now assume that $n = 4$ or 5. We use the same notation in these two different cases in order to treat them simultaneously.

If $n = 4$, then we start with an arithmetic lattice in $SO(4, 1)$, which, as we remarked in (2.2), comes from a quadratic form Q_5 over K. We now choose $\theta \in K^*$ as in (3.1) so that if (W_5, Q_5) is the quadratic space on which $SO(Q_5)$ operates, then $W_5 \oplus e_6 K = W_6$ has the quadratic form $Q_6 = Q_5 \oplus \theta\lambda^2$ on it. Write $E = W_5$. Moreover, θ is so chosen that if $H_6 = \mathrm{Spin}(W_6, Q_6)$, then $H_6(K \otimes_{\mathbf{Q}} \mathbf{R})$ is the product of Spin(5, 1) with a compact group. Clearly, H_6 contains $H_5 = H$ as a K-subgroup. We may write W_5 as an orthogonal direct sum $W_4 \oplus e_5 k$ with respect to Q_5 such that, if Q_4 denotes the restriction of Q_5 to W_4 and $H_4 = \mathrm{Spin}(W_4, Q_4)$, then $H_4(K_\infty) = \mathrm{Spin}(3, 1)$. We may also choose a two-dimensional space E' over K with a quadratic form $\mathbf{Q}'$ so that if $W_8 = W_6 \oplus E'$, $Q_8 = Q_6 \oplus Q'$, and $H_8 = \mathrm{Spin}(W_8, Q_8)$, then $H_8(K \otimes_{\mathbf{Q}} \mathbf{R})$ is isomorphic to the product of Spin (7, 1) with a compact group. Choose a quadratic extension (L/K) as in (2.7) and let $G_8 = SU(W_8 \otimes L, Q_8 \otimes L)$ as in (2.7).

If $n = 5$, then an arithmetic lattice in $SO(n, 1)$ comes from a Hermitian form over a central simple algebra D (of degree 2 over K) as in (1.2). Let $E = E_3$ be a free module over D of rank 3 and h_3 a D-valued Hermitian form on E_3 with respect to the standard involution on D described in (2.7). Set $H_6 = H = \mathrm{Spin}(E_3, h_3)$. Let h_2 denote the restriction of H_3 to E_2. Write $H_4 = \mathrm{Spin}(E_2, h_2)$. Furthermore, we may choose E_2 so that $H_4(K_\infty) = \mathrm{Spin}(3, 1)$. We also choose a 1-dimensional space E_1 over D, and a Hermitian form h' on it such that if $h_4 = h_3 \oplus h'$ is the Hermitian form on $E_4 = E_3 \oplus E_1$ and $H_8 = \mathrm{Spin}(E_4, h_4)$, then $H_4(K \otimes_{\mathbf{Q}} \mathbf{R})$ is the product of Spin(7, 1) with

a compact group. Let (L/K) denote a quadratic extension as in (3.1) and write, as in (2.2), $G_8 = SU(W_4 \otimes L, h_4 \otimes L)$.

We note once again that H_4, H_6, H, E, G_8 stands for two different things depending on the cases $n = 4$ or 5. From now on n is 4 or 5. Note that $G_8(K \otimes_{\mathbf{Q}} \mathbf{R})$ is the product of $SU(7,1)$ and a compact group. Consider the standard action of $SU(7,1)$ on $\mathbf{C}^8 = V_8$. If $k = 3, 4$ or 5, then for the action of $H_{k+1}(K)$ on V_8 via $H_{k+1}(K_\infty) = \mathrm{Spin}(k,1)$, we have $V_8 = V_{k+1} \oplus \mathbf{C}^{8-k-1}$ where $\mathbf{C}^{8-k-1}$ is the trivial $\mathrm{Spin}(k,1)$ module of dimension $(8-k-1)$. We have thus surjective homomorphisms $V_8^* \to V_{k+1}^*$ and the restriction maps of (3.1) may be defined. We also denote by Q_{n+1} the quadratic form preserved by H_{n+1} on V_{n+1} and write $H_l = \mathrm{sym}^l V_{n+1}^* / Q_{n+1} \mathrm{sym}^{l-2} V_{n+1}^*$.

4.2. *Lemma. The restriction map*

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(V_8^*)) \to \prod_{\sigma \in (\mathrm{Aut} G_8)(K)} H^1(\sigma(\Gamma_8) \cap H_6, \mathrm{sym}^l(V_8^*))$$

is injective.

This is just a restatement of Lemma (3.7).

4.3. *Notation.* Let $\{H_\phi\}$ denote the collection of simply connected K-subgroups of H_6 such that for all ϕ, (1) $H_\phi(K_\infty) = \mathrm{Spin}(3,1)$ and (2) $H_\phi = \mathrm{Spin}(E_\phi, h_\phi)$ where E_ϕ is a D-submodule of E such that $E = E_\phi \oplus E'_\phi$ as an orthogonal direct sum, H_ϕ is the restriction of h to E_ϕ. Let $C(H_6\{H_\phi\})$ denote the group defined in (2.5). We also write $H_\phi = H(E_\phi)$.

4.4. PROPOSITION

The group $(C(H_6)/C(H_6, \{H_\phi\}))^*$ *is finite.*

Proof. Let Δ be an open subgroup of $H_6(K) \cap H^*$. Let Ω be a congruence subgroup such that for all σ and all ϕ we have

$$\Delta \cap \sigma(H_\phi) \supset \sigma(H_\phi \cap \Omega).$$

Let Δ^* denote the smallest congruence subgroup of $H_6(K)$ containing Δ. Let v be a vector in $W(K)$ such that vD is isotropic over K_∞. Let r_v be the reflection with respect to v in $\mathrm{Aut}H(K)$. Let Ω' be a congruence subgroup such that $r_v \Omega' r_v^{-1} \subset \Omega$. For $\delta \in \Delta^* \cap \Omega'$ we have, $r_v \delta r_v^{-1} \delta^{-1}$ lies in the group $H(vD + \delta(v)D) \cap \Omega$ and the latter lies in Δ. But any element of Δ^* may be represented by an element of $\Delta^* \cap \Omega'$ modulo Δ since $\Delta\Omega' \supset \Delta^*$. This shows that on the quotient group Δ^*/Δ, the element r_v acts trivially. Since the group $(C(H_6)/C(H_6, \{H_\phi\}))^*$ is the inverse limit of the groups Δ^*/Δ (see (2.1)), it follows that r_v acts trivially, and hence elements of $H(K)$ of the form $r_v x r_v^{-1} x^{-1}$ act trivially on the group $(C(H_6)/C(H_6, \{H_\phi\}))^*$. The proposition now follows by the projective simplicity of $H_6(K)$ (see [To]).

4.5. COROLLARY

Let Δ_6 *be a congruence subgroup of* $H_6(K)$. *Then the restriction map* (*for every finite dimensional representation* ρ *of* $SO(5,1)$)

$$\mathrm{Res}: H^1(\Delta_6, \rho) \to \prod_\phi \prod_{\tau \in (\mathrm{Aut} H_6)} H^1(\tau(\Delta_6) \cap H_\phi, \rho)$$

is injective.

This is immediate from Proposition (4.4) and Proposition (2.6).

4.6. Theorem. *Let $\Gamma_8 \subset G_8(K)$ be a congruence subgroup. Then, in the notation of* (4.1), *the restriction map*

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(V_8^*)) \to \prod_{\sigma \in (\mathrm{Aut}\, G_8)(K)} H^1(\sigma(\Gamma_8) \cap H, H_l)$$

is injective.

Proof. Suppose that $\xi \in H^1(\Gamma_8, \mathrm{sym}^l(V_8^*))$ is in the kernel of Res. Restricting further to $\sigma(\Gamma_8 \cap H_\phi)$, we obtain, for each σ and ϕ, $\mathrm{Res}(\xi) = 0$ in $H^1(\sigma(\Gamma_8 \cap H_\phi, H_l)$. Corresponding to the inclusion $E_\phi \subset E$, we get a subspace V_ϕ of V_{n+1} and surjections $V_{n+1}^* \to V_\phi^*$. Hence, as an H_ϕ module, $\mathrm{sym}^l(V_{n+1}^*)$ decomposes as $\oplus E_\phi(i)$ where $E_\phi(i) = \mathrm{sym}^i(V_\phi^*) \otimes \mathrm{sym}^{l-i}(V_{n+1}/V_\phi)^*$. By Lemma (3.3), each $E_\phi(i)$ is an H_ϕ stable subspace of H_l and therefore, $\mathrm{Res}(\xi) = 0$ in $H^1(\sigma(\Gamma_8 \cap H_\phi, E_\phi(i))$ for all i.

We now replace σ by $\tau\sigma$ where $V_8 = V_\phi \oplus \mathbf{C}^4$ for the action of H_ϕ (and $\mathbf{C}^4$ is the trivial H_ϕ module), and $\tau \in G_8$ leaves the space V_ϕ pointwise fixed. By using the naturality of the restriction map, it follows that $\mathrm{Res}(\xi) = 0$ in $H^1(\sigma(\Gamma_8 \cap H_\phi, \tau^{-1}(E_\phi(i))$ for all i, τ, σ. But, τ is so chosen that $\tau(H_\phi) = H_\phi$ and $\tau(V_4) = V_4$. Hence, for all τ, σ, i we obtain, $\mathrm{Res}(\xi) = 0$ in

$$H^1(\sigma(\Gamma_8 \cap H_\phi, \mathrm{sym}^i V_4^* \otimes \mathrm{sym}^{l-i}(\tau(V_6)/V_4)^*)).$$

But, as is easily checked, the τ's act irreducibly on $\mathrm{sym}^l(\mathbf{C}^4) = \mathrm{sym}^l(V_8/V_4)$ and hence the sum over all i of $\Sigma_\tau \mathrm{sym}^i V_4^* \otimes \mathrm{sym}^{l-i}(\tau(V_6)/V_4)^*)$ is the sum over all i of $\mathrm{sym}^i V_4^* \otimes \mathrm{sym}^{l-i}(V_8/V_4)^*)$ which is $\mathrm{sym}^l V_8^*$. Hence, $\mathrm{Res}(\xi) = 0$ in the group $H^1(\sigma(\Gamma_8 \cap H_\phi, \mathrm{sym}^l V_8^*)$ for all σ, ϕ.

Therefore, if Δ_8 denotes any one of the groups $\sigma(\Gamma_8) \cap H$ and $\xi_\Delta = \mathrm{Res}(\xi) \in H^1(\Delta_8, \mathrm{sym}^l V_8^*)$, then $\mathrm{Res}(\xi) \in H^1(\Delta_8 \cap H_\phi, \mathrm{sym}^l V_8^*)$ for every ϕ. By corollary (4.5), this means that $\xi_\Delta = 0$. Now (4.2) shows that $\xi = 0$ and the Theorem is proved.

5. Proof of Theorem 1

5.1. $n+1$ *is even and greater than* 8. By [T], every arithmetic lattice Δ in $SO(n,1)$ arises as one of those in (1.2) with the degree of D over K being 2. We may extend Δ (by replacing it with a subgroup of finite index, if necessary) to an arithmetic lattice Γ of $SU(n,1)$ as in (1.3). By replacing Δ and Γ by subgroups of finite index if necessary, we may assume, by [B-W], Ch. [8], Theorem (5.9), that

$$H^1(\Gamma, \mathrm{sym}^l(V^*)) \neq 0. \tag{1}$$

Now, Theorem 2 shows that if $n \geq 9$, then, the map

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l(V^*)) \to \prod_\sigma H^1(\sigma(\Gamma) \cap \mathrm{Spin}(n,1), H_l) \tag{2}$$

is injective. In particular, we see that the map

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l(V^*)) \to \prod_\sigma H^1(\sigma(\Gamma) \cap SO(n,1), H_l) \tag{3}$$

is injective. Now (1) and (3) imply that for one of the groups $\Delta' = \sigma(\Gamma) \cap SO(n,1)$ we have

$$H^1(\Delta', H_l) \neq 0.$$

This proves Theorem 1 for odd $n \geqslant 9$.

5.2. *$n+1$ is odd and greater than* 7. By [T], every arithmetic lattice Δ in $SO(n,1)$ is one of the form described in (3.8). Let Γ be an arithmetic lattice in $G = SU(n+1,1)$ as in (3.8). As in (5.1) we may assume that

$$H^1(\Gamma, \mathrm{sym}^l(V^*)) \neq 0. \tag{1}$$

Now, Theorem (3.9), and arguments similar to those in (5.1) to replace $\mathrm{Spin}(n,1)$ by $SO(n,1)$ show that

$$H^1(\Delta', H_l) \neq 0,$$

where $\Delta' = \sigma(\Gamma) \cap SO(n,1)$ for some $\sigma \in (\mathrm{Aut}G)(K)$. This proves Theorem 1 for even $n \geqslant 8$.

5.3. *$n = 4$ or* 5. We have already described the arithmetic lattices of $SO(n,1)$ in (4.1). Let Γ_8 be an arithmetic subgroup of $SU(7,1)$ as in (4.6). We may assume that

$$H^1(\Gamma_8, \mathrm{sym}^l(V^*)) \neq 0 \tag{1}$$

by replacing it by a subgroup of finite index if necessary. Now Theorem (4.6) implies that

$$H^1(\Delta', H_l) \neq 0,$$

where $\Delta' = \sigma(\Gamma_8) \cap SO(n,1)$ for some $\sigma \in (\mathrm{Aut}G_8)(K)$. This proves Theorem 1 for $n = 5$ or 4.

5.4. $n = 6$. As observed before, arithmetic lattices in $SO(6,1)$ arise as unit groups of quadratic forms Q_7 in 7 variables over K. Let W_7, Q_7 be the quadratic space over K. We may write $W_7 = W_6 \oplus e_7 K$, and denote by Q_6 the restriction of Q_7 to W_6, so that

$$SO(W_6, Q_6)(K \otimes_{\mathbf{Q}} \mathbf{R}) = SO(5,1) \times K_6,$$

where K_6 is a compact group. We may also find a quadratic space $W_8 = W_7 \oplus e_8 K$ with a quadratic form Q_8 whose restriction to W_7 is Q_7 and

$$SO(W_8, Q_8)(K \otimes_{\mathbf{Q}} \mathbf{R}) = SO(7,1) \times K_8,$$

where K_8 is a compact group. Write

$$H_l = \mathrm{sym}^l(W_7^*)/Q_7 \mathrm{sym}^{l-2}(W_7^*)$$

and

$$H_l(6) = \mathrm{sym}^l(W_6^*)/Q_6 \mathrm{sym}^{l-2}(W_6^*).$$

Then, as a representation of $H_6 = \mathrm{Spin}(5,1)$, $H_l(6)$ is a direct summand of H_l. By Theorem (4.4), the map

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(W_8^*)) \to \prod_\sigma H^1(\sigma(\Gamma_8) \cap \mathrm{Spin}(5,1), H_l(6))$$

is injective. Therefore

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(W_8^*)) \to \prod_\sigma H^1(\sigma(\Gamma_8) \cap \mathrm{Spin}(5,1), H_l)$$

is injective. In particular

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(W_8^*)) \to \prod_{\sigma} H^1(\sigma(\Gamma_8) \cap \mathrm{Spin}(6,1), H_l) \tag{1}$$

is injective. We may choose, by Theorem 2 (and by [B-W], Theorem (5.9), Ch. (8)), Γ_8 such that

$$H^1(\Gamma_8, \mathrm{sym}^l(W_8^*) \neq 0. \tag{2}$$

Thus, Theorem 1 for $n = 6$, follows from (1) and (2).

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Solution of singular integral equations with logarithmic and Cauchy kernels

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Abstract. A direct method of solution is presented for singular integral equations of the first kind, involving the combination of a logarithmic and a Cauchy type singularity. Two typical cases are considered, in one of which the range of integration is a single finite interval and, in the other, the range of integration is a union of disjoint finite intervals. More such general equations associated with a finite number (greater than two) of finite, disjoint, intervals can also be handled by the technique employed here.

Keywords. Water waves; scattering; Cauchy kernel; logarithmic kernel.

1. Introduction

The singular integral equation

$$\int_L f(t)\left[K\ln\left|\frac{x-t}{x+t}\right| + \frac{1}{x+t} + \frac{1}{x-t}\right]\mathrm{d}t = g(x), \quad x\in L,$$

$$(K>0, \text{ a known constant}), \tag{1.1}$$

with L representing either (i) a single finite interval (a,b), or (ii) a union of two disjoint finite intervals of the type $(a,b)\cup(c,d)$, where $0<a<b<c<d$, has been considered for solution by employing a direct method which reduces such integral equations to an equation having only a Cauchy type singularity.

Singular integral equations of the form (1.1), where the unknown function $f(x)$ possesses square-root singularities at the end points of L (see Mandal [3]) occur in the study of scattering and radiation of surface water waves, in the linearized theory, by vertical barriers possessing either (i) a single gap, in the case when $L=(a,b)$ or (ii) two gaps, in the case when $L=(a,b)\cup(c,d)$. Recently, Banerjea and Mandal [6] have presented a method of solving eq. (1.1) when $L=(a,b)$, by reducing it to a singular integral equation over two disjoint intervals, with a Cauchy type singularity, and employing a function theoretic approach to the resulting equation. Banerjea and Mandal [6] have also mentioned that eq. (1.1) cannot be reduced to an equation with a Cauchy-type singularity, of a type which has been possible to equations of this form, (see Ursell [2]) where L represents a single interval of the form $(0,a)$ or (b,∞), in which case, the end behaviours of the unknown function $f(x)$ have a different structure as compared to the ones required for (1.1).

In the present paper, we have shown that a method, similar to Ursell's [2] is applicable to the integral equation (1.1), which reduces the equation to the one which involves only a Cauchy type singularity and which can be solved by using well-known results available in the books of Muskhelishvili [1] and Gakhov [4], as well as in a recently published paper of Chakrabarti and George [5].

The all-important identity that has helped in arriving at the closed form solution of the integral equation (1.1) by our approach is the one as given by the relation

$$\ln\left|\frac{x-\alpha}{x+\alpha}\right| = 2x\int_0^{\alpha}\frac{dt}{t^2-x^2}, \quad 0<x<\alpha. \tag{1.2}$$

In the next section, we take up the solution of (1.1) in the two cases (i) and (ii) described above, and present the solutions in closed forms.

2. Methods of solutions

Case (i). $L=(a,b)$

In this case, we look for the solution of (1.1) along with the end point requirements

$$f(x)\approx\begin{cases}O(|x-a|^{-1/2}) & \text{as} \quad x\to a,\\ O(|x-b|^{-1/2}) & \text{as} \quad x\to b.\end{cases} \tag{2.1}$$

We set, as in the work of Ursell [2],

$$F(x)=\int_x^b f(t)\mathrm{d}t \quad \text{for} \quad a<x<b \tag{2.2}$$

so that

$$F(b)=0 \quad \text{and} \quad F'(x)=-f(x) \quad \text{for} \quad a<x<b. \tag{2.3}$$

Substituting for $f(x)$ from (2.3) in (1.1), we obtain the singular integral equation

$$\int_a^b\frac{\lambda(t)\mathrm{d}t}{t^2-x^2}=h(x) \quad \text{for} \quad a<x<b, \tag{2.4}$$

where

$$\lambda(x)=F'(x)+KF(x) \quad \text{for} \quad a<x<b, \text{ so that}$$

$$f(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-Kx}\int_x^b\lambda(u)\mathrm{e}^{Ku}\mathrm{d}u\right] \quad \text{for} \quad a<x<b \tag{2.5}$$

and

$$h(x)=\frac{1}{2x}\left\{g(x)-KF(a)\ln\left|\frac{x-a}{x+a}\right|\right\} \quad \text{for} \quad a<x<b \tag{2.6}$$

with

$$\lambda(x)\approx\begin{cases}O(|x-a|^{-1/2}) & \text{as} \quad x\to a,\\ O(|x-b|^{-1/2}) & \text{as} \quad x\to b.\end{cases} \tag{2.7}$$

The solution of the integral equation (2.4) is well-known and is given by (see Gakhov [4])

$$\lambda(x)=\frac{2Cx}{R(x)}-\frac{4x}{\pi^2}\int_a^b\frac{tR(t)h(t)\mathrm{d}t}{R(x)(t^2-x^2)} \quad \text{for} \quad a<x<b, \tag{2.8}$$

where

$$R(t)=\sqrt{(t^2-a^2)(b^2-t^2)} \quad \text{for} \quad a<t<b, \tag{2.9}$$

and C is an arbitrary constant.

We next mention the following results to be obtained by standard methods (see Gakhov [4] problem 18, pp. 81)

$$\int_a^b\frac{R(t)g(t)\mathrm{d}t}{R(x)(t^2-x^2)}=\frac{C_2}{R(x)}+\Delta(x)\int_a^b\frac{1}{\Delta(t)}\frac{g(t)\mathrm{d}t}{t^2-x^2} \quad \text{for} \quad a<x<b, \tag{R1}$$

where

$$C_2 = \int_a^b \frac{g(t)}{\Delta(t)}\mathrm{d}t, \quad \Delta(x) = \sqrt{\frac{x^2-a^2}{b^2-x^2}} \quad \text{for} \quad a<x<b,$$

$$\ln\left|\frac{t-a}{t+a}\right| = 2t\int_0^a \frac{\mathrm{d}\xi}{\xi^2-t^2} \quad \text{for} \quad 0<t<a, \tag{R2}$$

$$\int_a^b \frac{tR(t)\mathrm{d}t}{t^2-u^2} = \frac{\pi}{2}\left[\frac{a^2+b^2}{2}-u^2\right] \quad \text{for} \quad (a<u<b), \tag{R3}$$

$$\int_a^b \frac{tR(t)\mathrm{d}t}{t^2-u^2} = \pi\left\{-\sqrt{(a^2-u^2)(b^2-u^2)}-u^2+\frac{a^2+b^2}{2}\right\} \quad \text{for} \quad (0<u<a), \tag{R4}$$

$$\int_a^b R(t)\ln\left|\frac{t-a}{t+a}\right|\frac{\mathrm{d}t}{t^2-u^2} = \pi a + \pi\int_0^a \frac{\sqrt{(a^2-t^2)(b^2-t^2)}\mathrm{d}t}{t^2-u^2} \quad \text{for } a<u<b. \tag{R5}$$

Using the above mentioned results (R1) to (R5), from the relation (2.8) we obtain

$$\lambda(x) = \frac{x}{R(x)}\left[Q+\frac{2K}{\pi}F(a)G(a,b,x)\right] - \frac{2x}{\pi^2}\Delta(x)\int_a^b \frac{1}{\Delta(t)}\frac{g(t)\mathrm{d}t}{t^2-x^2} \quad \text{for} \quad a<x<b, \tag{2.10}$$

where

$$Q = 2\left[C-\frac{C_2}{\pi^2}+\frac{KaF(a)}{\pi}\right] \quad \text{and}$$

$$G(a,b,x) = \int_0^a \frac{\sqrt{(a^2-\xi^2)(b^2-\xi^2)}\mathrm{d}\xi}{\xi^2-x^2}, \quad a<x<b. \tag{2.11}$$

From the relations (2.2), (2.5) and the expression (2.10), we obtain the relation connecting the constants Q and $F(a)$ as given by

$$\mathrm{e}^{Ka}F(a) = \int_a^b u\mathrm{e}^{Ku}\left(\frac{[Q+(2K/\pi)F(a)G(a,b,u)]}{R(u)} - \frac{2}{\pi^2}\Delta(u)\int_a^b \frac{1}{\Delta(t)}\frac{g(t)\mathrm{d}t}{t^2-u^2}\right)\mathrm{d}u. \tag{2.12}$$

Also, from relations (2.5) and (2.10) we finally obtain the complete solution of (1.1) in the case when $L=(a,b)$, as given by

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\mathrm{e}^{-Kx}\int_b^x u\mathrm{e}^{Ku}\left(\frac{[\hat{Q}-(2K/\pi)F(a)G(a,b,u)]}{R(u)} + \frac{2}{\pi^2}\Delta(u)\int_a^b \frac{1}{\Delta(t)}\frac{g(t)\mathrm{d}t}{t^2-u^2}\right)\mathrm{d}u\right\} \quad \text{for} \quad a<x<b, \tag{2.13}$$

where

$$\hat{Q} = -Q.$$

The solution (2.13) agrees, in principle, with the one obtained by Banerjea and Mandal [6] recently.

The solution of (1.1) can also be obtained, in this case, by setting

$$\tilde{F}(x)=\int_a^x f(t)\mathrm{d}t \quad \text{for} \quad a<x<b. \tag{2.14}$$

Proceeding in a similar manner as discussed above, we find the following alternative form of the solution $f(x)$ as given by

$$f(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left\{\mathrm{e}^{-Kx}\int_a^x u\mathrm{e}^{Ku}\left(\frac{[\tilde{Q}-(2K/\pi)\tilde{F}(b)G(a,b,u)]}{R(u)}\right.\right.$$
$$\left.\left.+\frac{2}{\pi^2}\Delta(u)\int_a^b\frac{1}{\Delta(t)}\frac{g(t)\mathrm{d}t}{t^2-u^2}\right)\mathrm{d}u\right\} \quad \text{for} \quad a<x<b, \tag{2.15}$$

where the constants $\tilde{Q}$ and $\tilde{F}(b)$ are related as

$$\mathrm{e}^{Ka}\tilde{F}(b)=\int_a^b u\mathrm{e}^{Ku}\left(\frac{[\tilde{Q}-(2K/\pi)\tilde{F}(b)G(a,b,u)]}{R(u)}+\frac{2}{\pi^2}\Delta(u)\int_a^b\frac{1}{\Delta(t)}\frac{g(t)\mathrm{d}t}{t^2-u^2}\right)\mathrm{d}u. \tag{2.16}$$

Next, we will discuss the method of solution of (1.1) when L is the union of two disjoint finite intervals.

Case (ii). $L=(a,b)\cup(c,d)$

In this case, we desire to solve (1.1) where

$$f(x)\approx \begin{array}{ll} O(|x-a|^{-1/2}) & \text{as} \quad x\to a,\\ O(|x-b|^{-1/2}) & \text{as} \quad x\to b,\\ O(|x-c|^{-1/2}) & \text{as} \quad x\to c,\\ O(|x-d|^{-1/2}) & \text{as} \quad x\to d. \end{array} \tag{2.17}$$

We set

$$f(x)=\begin{cases} f_1(x) & \text{for} \quad x\in(a,b),\\ f_2(x) & \text{for} \quad x\in(c,d) \end{cases} \tag{2.18}$$

and

$$g(x)=\begin{cases} g_1(x) & \text{for} \quad x\in(a,b),\\ g_2(x) & \text{for} \quad x\in(c,d). \end{cases} \tag{2.19}$$

Then, defining

$$F_1(x)=\int_x^b f_1(t)\mathrm{d}t \quad \text{for} \quad a<x<b, \tag{2.20}$$

so that

$$F_1'(x)=-f_1(x) \quad \text{for} \quad a<x<b,$$
$$F_1(b)=0 \tag{2.21}$$

and

$$F_2(x)=\int_x^d f_2(t)\mathrm{d}t \quad \text{for} \quad c<x<d, \tag{2.22}$$

so that

$$F_2'(x)=-f_2(x) \quad \text{for} \quad c<x<d \quad \text{and}$$
$$F_2(d)=0. \tag{2.23}$$

We reduce (1.1) into the following new equation

$$\int_a^b \frac{\lambda_1(t)\mathrm{d}t}{t^2-x^2} + \int_c^d \frac{\lambda_2(t)\mathrm{d}t}{t^2-x^2} = \frac{\tilde{g}(x)}{2x} \quad \text{for} \quad x \in (a,b) \cup (c,d), \tag{2.24}$$

where

$$\begin{aligned} \lambda_1(x) &= F_1'(x) + KF_1(x) \quad \text{for} \quad a<x<b \quad \text{and} \\ \lambda_2(x) &= F_2'(x) + KF_2(x) \quad \text{for} \quad c<x<d, \end{aligned} \tag{2.25}$$

and

$$\tilde{g}(x) = \left\{ g(x) - KF_1(a)\ln\left|\frac{x-a}{x+a}\right| - KF_2(c)\ln\left|\frac{x-c}{x+c}\right| \right\} \quad \text{for} \quad x \in (a,b) \cup (c,d), \tag{2.26}$$

along with the end conditions that

$$\lambda_1(x) \approx \begin{cases} O(|x-a|^{-1/2}) & \text{as} \quad x \to a, \\ O(|x-b|^{-1/2}) & \text{as} \quad x \to b \end{cases} \tag{2.27}$$

and

$$\lambda_2(x) \approx \begin{cases} O(|x-c|^{-1/2}) & \text{as} \quad x \to c, \\ O(|x-d|^{-1/2}) & \text{as} \quad x \to d. \end{cases} \tag{2.28}$$

The solution of the integral equation (2.24) along with the end conditions (2.27) and (2.28) can be derived by using an analysis similar to the one available in the paper of Chakrabarti and George [5], with appropriate modifications (see Appendix) and we find that

$$\begin{aligned} \lambda_1(x) = \frac{-2(A_1x^2+B_1)x}{R_1(x)\hat{R}_2(x)} - \frac{2x}{\pi^2}\Bigg[& \int_a^b \frac{\tilde{g}_1(t)R_1(t)\hat{R}_2(t)\mathrm{d}t}{R_1(x)\hat{R}_2(x)(t^2-x^2)} \\ & - \int_c^d \frac{\hat{R}_1(t)R_2(t)\tilde{g}_2(t)\mathrm{d}t}{R_1(x)\hat{R}_2(x)(t^2-x^2)} \Bigg] \quad \text{for } a<x<b \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} \lambda_2(x) = \frac{2(A_1x^2 \mp B_1)x}{\hat{R}_1(x)R_2(x)} + \frac{2x}{\pi^2}\Bigg[& \int_a^b \frac{\tilde{g}_1(t)\hat{R}_1(t)R_2(t)\mathrm{d}t}{\hat{R}_1(x)R_2(x)(t^2-x^2)} \\ & - \int_c^d \frac{\hat{R}_1(t)R_2(t)\tilde{g}_2(t)\mathrm{d}t}{\hat{R}_1(x)R_2(x)(t^2-x^2)} \Bigg] \quad \text{for} \quad c<x<d, \end{aligned} \tag{2.30}$$

where A_1, B_1 are arbitrary constants,

$$\begin{aligned} \tilde{g}_1(x) &= \tilde{g}(x) \quad \text{for} \quad x \in (a,b) \quad \text{and} \\ \tilde{g}_2(x) &= \tilde{g}(x) \quad \text{for} \quad x \in (c,d), \end{aligned} \tag{2.31}$$

$$R_1(x) = \sqrt{(x^2-a^2)(b^2-x^2)} \quad \text{for} \quad a<x<b, \tag{2.32}$$

$$\hat{R}_1(x) = \begin{cases} \sqrt{(x^2-a^2)(x^2-b^2)} & \text{for} \quad x>b, \\ \sqrt{(a^2-x^2)(b^2-x^2)} & \text{for} \quad 0<x<a, \end{cases} \tag{2.33}$$

$$R_2(x) = \sqrt{(x^2-c^2)(d^2-x^2)} \quad \text{for} \quad c<x<d, \tag{2.34}$$

$$\hat{R}_2(x) = \begin{cases} \sqrt{(d^2-x^2)(c^2-x^2)} & \text{for} \quad x<c \\ \sqrt{(x^2-c^2)(x^2-d^2)} & \text{for} \quad x>d. \end{cases} \tag{2.35}$$

We then use the following results to simplify the expressions for $\lambda_1(x)$ and $\lambda_2(x)$:

$$\int_a^b R_1(t)\hat{R}_2(t)\ln\left|\frac{t-p}{t+p}\right|\frac{dt}{t^2-x^2}=\int_0^p \frac{G_1(x,\xi)d\xi}{\xi^2-x^2}, \tag{R6}$$

where $p=a$ or c, and

$$G_1(x,\xi)=2\left[\int_a^b tR_1(t)\hat{R}_2(t)\left\{\frac{1}{t^2-x^2}+\frac{1}{\xi^2-t^2}\right\}dt\right],$$

for $x\in(a,b)$ or $x\in(c,d)$

$$\int_a^b \frac{R_1(t)\hat{R}_2(t)g_1(t)dt}{R_1(x)\hat{R}_2(x)(t^2-x^2)}=\frac{C_3}{R_1(x)\hat{R}_2(x)}+\frac{\Delta_1(x)}{\hat{R}_2(x)}\int_a^b \frac{\hat{R}_2(t)\ g_1(t)dt}{\Delta_1(t)(t^2-x^2)}$$

for $a<x<b$, (R7)

where

$$C_3=\int_a^b \frac{\hat{R}_2(t)}{\Delta_1(t)}g_1(t)dt$$

and

$$\Delta_1(x)=\sqrt{\frac{x^2-a^2}{b^2-x^2}}\quad \text{for}\quad a<x<b,$$

$$\int_c^d \frac{\hat{R}_1(t)R_2(t)g_2(t)dt}{R_1(x)\hat{R}_2(x)(t^2-x^2)}=\frac{C_4}{R_1(x)\hat{R}_2(x)}-\frac{\Delta_2(x)}{\hat{R}_1(x)}\int_c^d \frac{\hat{R}_1(t)\ g_2(t)dt}{\Delta_2(t)(t^2-x^2)},\quad a<x<b, \tag{R8}$$

where

$$C_4=\int_c^d \frac{\hat{R}_1(t)}{\Delta_2(t)}g_2(t)dt$$

and

$$\Delta_2(x)=\sqrt{\frac{c^2-x^2}{d^2-x^2}},\quad a<x<b,$$

$$\int_c^d R_2(t)\hat{R}_1(t)\ln\left|\frac{t-q}{t+q}\right|\frac{dt}{t^2-x^2}=\int_0^q \frac{G_2(x,\xi)d\xi}{\xi^2-x^2}, \tag{R9}$$

where $q=a$ or $q=c$ and

$$G_2(x,\xi)=2\left[\int_c^d t\hat{R}_1(t)R_2(t)\left\{\frac{1}{t^2-x^2}+\frac{1}{\xi^2-t^2}\right\}dt\right]$$

for $x\in(a,b)$ or for $x\in(c,d)$.

$$\int_a^b \frac{R_1(t)\hat{R}_2(t)g_1(t)dt}{R_2(x)\hat{R}_1(x)(t^2-x^2)}=\frac{C_3}{R_2(x)\hat{R}_1(x)}+\frac{\Delta_3(x)}{R_2(x)}\int_a^b \frac{\hat{R}_2(t)\ g_1(t)dt}{\Delta_3(t)(t^2-x^2)}$$

for $c<x<d$, (R10)

where C_3 is the same as defined in (R7) and

$$\Delta_3(x)=\sqrt{\frac{x^2-a^2}{x^2-b^2}} \quad \text{for} \quad c<x<d,$$

$$\int_c^d \frac{\hat{R}_1(t)R_2(t)g_2(t)\mathrm{d}t}{R_2(x)\hat{R}_1(x)(t^2-x^2)}=\frac{C_4}{R_2(x)\hat{R}_1(x)}-\frac{\Delta_4(x)}{\hat{R}_1(x)}\int_c^d \frac{\hat{R}_1(t)\ g_2(t)\mathrm{d}t}{\Delta_4(t)(t^2-x^2)} \quad \text{for} \quad c<x<d, \tag{R11}$$

where C_4 is the same as defined in (R8) and

$$\Delta_4(x)=\sqrt{\frac{x^2-c^2}{d^2-x^2}} \quad \text{for} \quad c<x<d.$$

Using the results (R6) to (R11), we obtain

$$\lambda_1(x)=\frac{-2[A_1x^2+Q_1+H_1(x)]x}{R_1(x)\hat{R}_2(x)}-\frac{2x}{\pi^2}P_1(x) \quad \text{for} \quad a<x<b \tag{2.36}$$

and

$$\lambda_2(x)=\frac{2[A_1x^2+Q_1+H_1(x)]x}{\hat{R}_1(x)R_2(x)}+\frac{2x}{\pi^2}P_2(x) \quad \text{for} \quad c<x<d, \tag{2.37}$$

where

$$Q_1=B_1+\frac{C_3-C_4}{\pi^2},$$

$$H_1(x)=\frac{K}{\pi^2}\left\{F_1(a)\int_0^a \frac{[G_2(x,\xi)-G_1(x,\xi)]\mathrm{d}\xi}{\xi^2-x^2} + F_2(c)\int_0^c \frac{[G_2(x,\xi)-G_1(x,\xi)]\mathrm{d}\xi}{\xi^2-x^2}\right\}, \tag{2.38}$$

$$P_1(x)=\frac{\Delta_1(x)}{\hat{R}_2(x)}\int_a^b \frac{\hat{R}_2(t)g_1(t)\mathrm{d}t}{\Delta_1(t)(t^2-x^2)}+\frac{\Delta_2(x)}{R_1(x)}\int_c^d \frac{\hat{R}_1(t)g_2(t)\mathrm{d}t}{\Delta_2(t)(t^2-x^2)} \quad \text{for} \quad a<x<b \tag{2.39}$$

and

$$P_2(x)=\frac{\Delta_3(x)}{R_2(x)}\int_a^b \frac{\hat{R}_2(t)g_1(t)\mathrm{d}t}{\Delta_3(t)(t^2-x^2)}-\frac{\Delta_4(x)}{\hat{R}_1(x)}\int_c^d \frac{\hat{R}_1(t)g_2(t)\mathrm{d}t}{\Delta_4(t)(t^2-x^2)} \quad \text{for} \quad c<x<d. \tag{2.40}$$

Now using the relations (2.25), we find that

$$\mathrm{e}^{Ka}F_1(a)=\int_a^b \mathrm{e}^{Ku}\lambda_1(u)\mathrm{d}u \quad \text{and}$$

$$\mathrm{e}^{Kc}F_2(c)=\int_c^d \mathrm{e}^{Ku}\lambda_2(u)\mathrm{d}u, \tag{2.41}$$

and

$$f_1(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-Kx}\int_x^b \mathrm{e}^{Ku}\lambda_1(u)\mathrm{d}u\right], \quad \text{for} \quad a<x<b \quad \text{and}$$

$$f_2(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-Kx}\int_x^d \mathrm{e}^{Ku}\lambda_2(u)\mathrm{d}u\right] \quad \text{for} \quad c<x<d, \tag{2.42}$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are given by the expressions (2.36) and (2.37) respectively.

The relations (2.42) completely solve (1.1), in this case, if the two relations in (2.41) are utilized to determine the constants $F_1(a)$ and $F_2(c)$ in terms of A_1 and Q_1, which remain arbitrary. Alternative forms of the functions f_1 and f_2 can also be derived, as has been done in the case (i), where the solution has been expressed through the relations (2.15) and (2.16).

3. Conclusion

A unified approach has been developed to solve the singular integral equations of water wave problems, which involve a union of disjoint finite intervals, at the end points of which the unknown function is required to satisfy square-root type integrable singularities. The cases of single as well as double intervals have been analyzed here in detail. For more number of intervals (greater than two), the method is similar, even though a little more involved.

Appendix

Solution of (2.24)

Equation (2.24) can be cast into the form

$$T_1\lambda_1^*(\eta)+\tilde{T}_2\lambda_2^*(\eta)=g_1^*(\eta) \quad \text{for} \quad \eta\in(a_1,b_1) \tag{A1}$$

and

$$\tilde{T}_1\lambda_1^*(\eta)+T_2\lambda_2^*(\eta)=g_2^*(\eta) \quad \text{for} \quad \eta\in(c_1,d_1) \tag{A2}$$

after using the transformations

$$x^2=\eta,\ t^2=\xi,\ \lambda_1^*(\xi)=\frac{\lambda_1(\sqrt{\xi})}{2\sqrt{\xi}},\ \lambda_2^*(\xi)=\frac{\lambda_2(\sqrt{\xi})}{2\sqrt{\xi}},\ a^2=a_1,\ b^2=b_1,\ c^2=c_1,$$

$$d^2=d_1,\ g_1^*(\eta)=\frac{\tilde{g}_1(\sqrt{\eta})}{2\sqrt{\eta}},\ \eta\in(a_1,b_1),\ g_2^*(\eta)=\frac{\tilde{g}_2(\sqrt{\eta})}{2\sqrt{\eta}},\ \eta\in(c_1,d_1) \tag{A3}$$

and employing the operators T_1, T_2, $\tilde{T}_1$ and $\tilde{T}_2$ as defined by the relations

$$T_1\lambda_1^*(\eta)=\int_{a_1}^{b_1}\frac{\lambda_1^*(\xi)\mathrm{d}\xi}{\xi-\eta} \quad \text{for} \quad \eta\in(a_1,b_1),$$

$$T_2\lambda_2^*(\eta)=\int_{c_1}^{d_1}\frac{\lambda_2^*(\xi)\mathrm{d}\xi}{\xi-\eta} \quad \text{for} \quad \eta\in(c_1,d_1),$$

$$\tilde{T}_1\lambda_1^*(\eta)=\int_{a_1}^{b_1}\frac{\lambda_1^*(\xi)\mathrm{d}\xi}{\xi-\eta} \quad \text{for} \quad \eta\notin(a_1,b_1),$$

$$\tilde{T}_2\lambda_2^*(\eta)=\int_{c_1}^{d_1}\frac{\lambda_2^*(\xi)\mathrm{d}\xi}{\xi-\eta} \quad \text{for} \quad \eta\notin(c_1,d_1).$$

We then define the inverse operators T_1^{-1} and T_2^{-1} as given by

$$T_1^{-1}\lambda_1^*(\eta)=\frac{D_1}{\Delta_3(\eta)}-\frac{T_1(\Delta_3(\eta)\lambda_1^*(\eta))}{\pi^2\Delta_3(\eta)} \quad \text{for} \quad \eta\in(a_1,b_1) \quad \text{and}$$

$$T_2^{-1}\lambda_2^*(\eta)=\frac{D_2}{\Delta_4(\eta)}-\frac{T_2(\Delta_4(\eta)\lambda_2^*(\eta))}{\pi^2\Delta_4(\eta)} \quad \text{for} \quad \eta\in(c_1,d_1), \tag{A4}$$

where D_1 and D_2 are two arbitrary constants and

$$\Delta_3(\eta)=\sqrt{(\eta-a_1)(b_1-\eta)} \quad \text{for} \quad \eta\in(a_1,b_1) \quad \text{and}$$
$$\Delta_4(\eta)=\sqrt{(\eta-c_1)(d_1-\eta)} \quad \text{for} \quad \eta\in(c_1,d_1). \tag{A5}$$

We easily derive the following results ($\tilde{R}1$–$\tilde{R}8$), by using standard methods.

$$T_1(\Delta_3(\eta))=\pi\left[-\eta+\frac{a_1+b_1}{2}\right] \quad \text{for} \quad \eta\in(a_1,b_1), \tag{$\tilde{R}1$}$$

$$T_1(\Delta_3(\eta)\,\tilde{T}_2\lambda_2^*(\eta))=\pi\int_{c_1}^{d_1}\lambda_2^*(\eta)\mathrm{d}\eta-\pi\tilde{T}_2(\hat{\Delta}_3(\eta)\lambda_2^*(\eta)) \quad \text{for} \quad a_1<\eta<b_1, \tag{$\tilde{R}2$}$$

where $\hat{\Delta}_3(\eta)=\sqrt{(\eta-a_1)(\eta-b_1)}$ for $\eta\notin(a_1,b_1)$,

$$\tilde{T}_1\left(\frac{1}{\Delta_3(\eta)}\tilde{T}_2(\hat{\Delta}_3(\eta)\lambda_2^*(\eta))\right)=\pi T_2(\lambda_2^*(\eta))-\frac{\pi}{\hat{\Delta}_3(\eta)}T_2(\hat{\Delta}_3(\eta)\lambda_2^*(\eta)) \quad \text{for} \quad \eta\in(c_1,d_1), \tag{$\tilde{R}3$}$$

$$\tilde{T}_1\left(\frac{1}{\Delta_3(\eta)}T_1(\Delta_1(\eta)g_1^*(\eta))\right)=-\frac{\pi}{\hat{\Delta}_3(\eta)}\tilde{T}_1(\Delta_3(\eta)g_1^*(\eta)) \quad \text{for} \quad \eta\in(c_1,d_1), \tag{$\tilde{R}4$}$$

$$T_2(\Delta_4(\eta)\,\tilde{T}_1(\Delta_3(\eta)g_1^*(\eta)))=\pi\int_{a_1}^{b_1}\Delta_3(\eta)g_1^*(\eta)\mathrm{d}\eta+\pi\tilde{T}_1(\Delta_3(\eta)\hat{\Delta}_4(\eta)g_1^*(\eta)) \quad \text{for } \eta\in(c_1,d_1), \tag{$\tilde{R}5$}$$

where $\hat{\Delta}_4(\eta)=\sqrt{(c_1-\eta)(d_1-\eta)}$,

$$\tilde{T}_2\left(\frac{1}{\Delta_4(\eta)}\tilde{T}_1(\Delta_3(\eta)\,\hat{\Delta}_4(\eta)g_1(\eta))\right)=\frac{\pi}{\hat{\Delta}_4(\eta)}T_1(\Delta_3(\eta)\,\hat{\Delta}_4(\eta)g_1(\eta))-\pi T_1(\Delta_3(\eta)g_1(\eta)), \tag{$\tilde{R}6$}$$

$$\tilde{T}_2\left(\frac{1}{\Delta_4(\eta)}T_2(\hat{\Delta}_3(\eta)\Delta_4(\eta)g_2^*(\eta))\right)=\frac{\pi}{\hat{\Delta}_4(\eta)}\tilde{T}_2(\Delta_4(\eta)\,\hat{\Delta}_3(\eta)g_2^*(\eta)), \tag{$\tilde{R}7$}$$

$$\tilde{T}_2\left(\frac{\eta}{\hat{\Delta}_4(\eta)}\right)=\pi\left(1+\frac{\eta}{\hat{\Delta}_4(\eta)}\right) \quad \text{for } \eta\in(a_1,b_1). \tag{$\tilde{R}8$}$$

Applying the operator T_1^{-1} to eq. (A1) and using the result ($\tilde{R}2$), we obtain

$$\lambda_1^*(\eta)=\frac{A_1}{\Delta_3(\eta)}-\frac{\pi\tilde{T}_2(\Delta_3(\eta))+T_1(\Delta_3(\eta)g_1^*(\eta))}{\pi^2\Delta_3(\eta)}, \tag{A6}$$

where $A_1=D_1+(1/\pi)\int_{c_1}^{d_1}\lambda_2^*(s)\mathrm{d}s$ is an arbitrary constant.

Using the expression (A6) along with the results ($\tilde{R}3$) and ($\tilde{R}4$), we can rewrite (A2) in the form

$$T_2(\hat{\Delta}_3(\eta)\lambda_2^*(\eta))=A_1\pi-\frac{1}{\pi}\tilde{T}_1(\Delta_3(\eta)g_1^*(\eta))+\hat{\Delta}_3(\eta)g_2^*(\eta) \quad \text{for} \quad \eta\in(c_1,d_1). \tag{A7}$$

Applying the inverse operator T_2^{-1} to eq. (A7), we obtain, after using the results ($\tilde{R}1$), ($\tilde{R}5$), that

$$\lambda_2^*(\eta) = \frac{A_1\eta + B_1}{\Delta_4(\eta)\hat{\Delta}_3(\eta)} - \frac{T_2(\Delta_4(\eta)\hat{\Delta}_3(\eta)g_2^*(\eta)) - \tilde{T}_1(\Delta_3(\eta)\hat{\Delta}_4(\eta)g_1^*(\eta))}{\pi^2\hat{\Delta}_3(\eta)\Delta_4(\eta)}$$

$$\text{for } \eta \in (c_1, d_1), \qquad \text{(A8)}$$

where $B_1 (= D_2 - (A_1/2)(c_1 + d_1) + \pi^{-2}\int_{a_1}^{b_1}\Delta_3(\eta)g_1^*(\eta)\mathrm{d}\eta)$ is an arbitrary constant.

The relations (A6) and (A8), after using the results ($\tilde{R}5$) to ($\tilde{R}8$) ultimately give

$$\lambda_1^*(\eta) = \frac{-A_1\eta - B_1}{\Delta_3(\eta)\hat{\Delta}_4(\eta)} - \frac{T_1(\Delta_3(\eta)\hat{\Delta}_4(\eta)g_1^*(\eta)) - \tilde{T}_2(\Delta_4(\eta)\hat{\Delta}_3(\eta)g_2^*(\eta))}{\pi^2\hat{\Delta}_4(\eta)\Delta_3(\eta)}$$

$$\text{for } a_1 < \eta < b_1. \qquad \text{(A9)}$$

Going back to the original variables x, t, λ_1 and λ_2, we get the results (2.29) and (2.30) used in the paper.

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Finite part representations of hyper singular integral equation of acoustic scattering and radiation by open smooth surfaces

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Abstract. The Green's function solution of the Helmholtz's equation for acoustic scattering by hard surfaces and radiation by vibrating surfaces, lead in both the cases, to a hyper singular surface boundary integral equation. Considering a general open surface, a simple proof has been given to show that the integral is to be interpreted like the Hadmard finite part of a divergent integral in one variable. The equation is reformulated as a Cauchy principal value integral equation, but also containing the potential at the control point. It is amenable to numerical treatment by conventional methods. An alternative formulation in the better known form, containing the tangential derivative of the potential is also given. The two dimensional problem for an open arc is separately treated for its simpler feature.

Keywords. Finite part; hyper singular integral; integral equation; Cauchy principal value; acoustic scattering; acoustic radiation; open smooth surfaces.

1. Introduction

It has been recognized in recent years that the Green's function formulation of scattering and radiation of acoustic or elastic waves by surfaces on which Neumann boundary condition holds, leads to hyper singular integral equations. In the acoustic case, the surfaces are non-soft, that is, hard or partially absorbing, open or closed. References to and regularization of the hyper singular nature by conversion to Cauchy principal value (CPV) integral equations involving (tangential) derivatives of the unknown potential or integro-differential equations have been made in diverse literature (Burton and Miller [1], Meyer *et al* [9], Terai [11], Martin and Rizzo [7], Krishnasamy *et al* [5]). In the context of vector elastic waves, the surfaces are usually stress-free crack surfaces in the solid and considerable literature exists generally without direct reference to hyper singularity. References to the few which treat hyper singularity may be found in Martin and Rizzo [7] and Krishnasamy *et al* [5].

The hyper singularity in the boundary integral equation arises when the normal derivative implied by the boundary conditions is carried into the Cauchy principal value integral containing normal derivative of Green's function. Validation of the hyper singular integrals in different contexts as Hadamard finite part (HFP) [2] is generally obtainable (Krishnasamy *et al* [5]). In view of the straightforward formulation, there have been several suggestions following Iokimidis [3], to treat these numerically, using Gaussian quadrature formula developed by Kutt [6] for integrals in one dimension.

Herein, we treat the hyper singular integral equation that arises in the context of acoustic scattering or radiation by a hard open surface (three as well as two dimensional, that is, an arc). By adopting a definition of finite part of the hyper singular surface integral as a generalization of the usual HFP of a curvilinear integral stated in

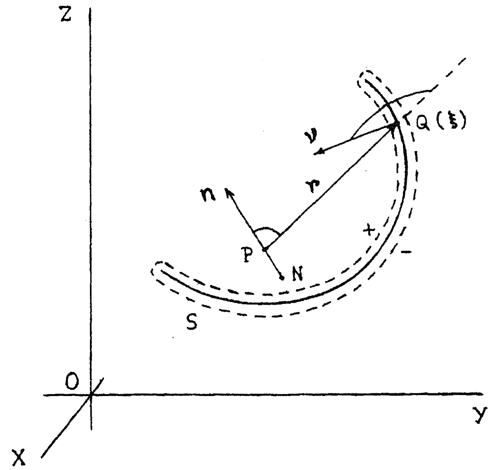

Figure 1. The geometry of the problem. N is the foot of the perpendicular from P on the surface S.

Martin and Rizzo [7], we present a short validation of the equation. The proof of Krishnasamy *et al* [5] based on the original definition of Hadamard for one dimensional integral is somewhat longer. Next the equation is transformed in a straightforward way to derive CPV linear integral equations, which can be numerically treated in conventional ways. Familiar type, containing the (tangential) derivative of the potential is also derived. The case of two dimensional arc is separately treated for its simplicity and elucidation.

2. The hyper singular integral equation

The acoustic field potential ϕ satisfies Helmholtz's equation

$$\nabla^2\phi + k^2\phi = 0, \tag{1}$$

where k is the wave number ω/c, $\omega/2\pi$ = frequency and c = wave velocity. By Green's identity (cf. figure 1), the solution of (1) can be written as

$$\phi = \frac{1}{\kappa}\int_S \left[\phi\frac{\partial G}{\partial \nu} - G\frac{\partial \phi}{\partial \nu}\right]_-^+ \mathrm{d}S, \tag{2}$$

where $\kappa = -2\pi$ or 4π with regard to the (open) surface S being two or three dimensional, and G is the free space Green's function $(i\pi/2)H_0^{(2)}(kr)$ or e^{-ikr}/r, $H_0^2(\cdot)$ being the Hankel function of second kind and zero order. For an acoustically hard surface scattering problem, the normal component of velocity at the variable point Q, namely $\partial\phi/\partial\nu$ is zero while it is prescribed $(-i\rho\omega v(Q))$ and is continuous across S for the radiation problem. Hence (2) simplifies to

$$\phi = \frac{1}{\kappa}\int_S \Phi\frac{\partial G}{\partial \nu}\mathrm{d}S, \quad \Phi(Q) = \phi^+ - \phi^-. \tag{3}$$

Boundary integral equation for Φ, completing the solution of (3), is obtained by again invoking the boundary condition on S. For the radiation problem $\partial\phi/\partial n = v(N)$ as the control point P tends to N where $v(N)$ is the amplitude of the normal component of

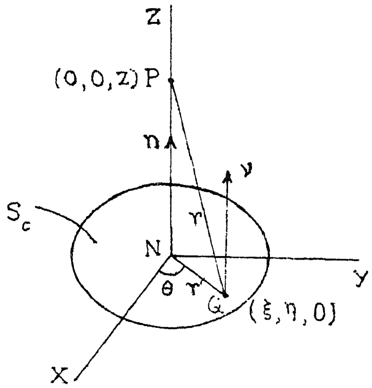

Figure 2. Local coordinates for circular region S_c.

displacement. We thus get

$$\frac{\partial}{\partial n} \fint_S \Phi(Q) \frac{\partial G}{\partial \nu} \mathrm{d}S = \Psi(N), \quad \text{as } P \to N, \tag{4}$$

where $\Psi = \kappa v(N)$. In the scattering problem, ϕ can be considered as the scattered field in a total field ϕ^{tot} and incident field ϕ^{inc}. For acoustically hard surface $\partial\phi^{\text{tot}}/\partial n = 0$ as P tends to N and we again have eq. (4), where $\Psi(N) = -\kappa \partial \phi^{\text{inc}}(N)/\partial n$. In the surface integral of eq. (4) as P tends normally to the point N on $S, \partial G/\partial \nu \, \mathrm{d}S$ becomes $O(1/r)$ singular. Hence the CPV of the integral denoted by $\fint$ has to be taken. If further, the derivative $\partial/\partial n$ is taken within the integral sign, there results

$$\fint_S \Phi(Q) \frac{\partial^2 G}{\partial n \partial \nu} \mathrm{d}S = \Psi(P), \quad \text{as } P \to N. \tag{5}$$

Now $\partial^2 G/\partial n \partial \nu \, \mathrm{d}S$ becomes $O(1/r^2)$ singular as $P \to N$ and the integral is divergent. Justification and meaning in the sense of Hadamard finite part (HFP) denoted by $\fint$ has been provided by Krishnasamy *et al* [5]. In the following we give a shorter justification.

Lemma 1. *For infinitesimal region S_0 with centroid N*

$$\int_{S_0} \frac{\partial^2 G}{\partial n \partial \nu} \mathrm{d}S \to 0, \quad \textit{as } S_0 \textit{ shrinks to } N.$$

Let 2ε be the diameter of the smallest circular region with centre N containing S_0. Then the integral in magnitude, is less than or equal to that over S_c. Now introducing local coordinates with centre N as origin (figure 2).

$$\frac{\partial^2 G}{\partial n \partial \nu} = \frac{\partial^2}{\partial z^2} \left(\frac{\mathrm{e}^{-ikr}}{r} \right).$$

Hence

$$\int_{S_c} \frac{\partial G}{\partial n \partial \nu} \mathrm{d}S = \int_{\theta=0}^{2\pi} \mathrm{d}\theta \int_{r'=0}^{\varepsilon} \mathrm{e}^{-ikr} \left[\left(\frac{3}{r^3} + \frac{3ik}{r^2} - \frac{k^2}{r} \right) \frac{z^2}{r^2} - \frac{1}{r^3} - \frac{ik}{r^2} \right] r' \mathrm{d}r'$$

$$= - \int_{\theta=0}^{2\pi} \int_{z}^{(\varepsilon^2+z^2)^{1/2}} \frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{\mathrm{e}^{-ikr}}{r} \left\{ (1+ikr)\frac{z^2}{r^2} - 1 \right\} \right] \mathrm{d}r$$

$$= -2\pi\left[\frac{e^{-ik(\varepsilon^2+z^2)^{1/2}}}{\sqrt{\varepsilon^2+z^2}}\left\{\frac{ikz^2}{\sqrt{\varepsilon^2+z^2}}-\frac{\varepsilon^2}{\varepsilon^2+z^2}\right\}-ike^{-ikz}\right]$$

$$\to 0 \text{ as } \varepsilon \to 0.$$

(Terai [11]).

Reverting to the hyper singular integral, we define its finite part in the manner of the integral in one variable (eq. (22))

$$\oint_S \Phi(Q)\frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S \overset{\text{def}}{=} \int_{S-S_0}\Phi(Q)\frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S + \Phi(N)\int_{S_0}\frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S, \quad \text{as } S_0 \to N$$

$$= \int_{S-S_0}\Phi(Q)\frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S, \quad \text{as } S_0 \to N$$

by lemma 1. Hence,

$$\frac{\partial}{\partial n}\oint_S \Phi(Q)\frac{\partial G}{\partial \nu}\mathrm{d}S = \frac{\partial}{\partial n}\int_{S-S_0}\Phi(Q)\frac{\partial G}{\partial \nu}\mathrm{d}S, \quad \text{as } S_0 \to N$$

$$= \int_{S-S_0}\Phi(Q)\frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S, \quad \text{as } S_0 \to N$$

$$= \oint_S \Phi(Q)\frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S. \tag{6}$$

In the case of an open arc in two dimensions, the above does not hold. It can however be proved more easily as indicated in Remark of § 6.

Referring to figure 1, the hyper singular kernel can be written in view of

$$G = G(r) \quad \text{and} \quad \frac{\partial r}{\partial \nu} = \cos(\mathbf{r}, \boldsymbol{\nu}),$$

$$\frac{\partial^2 r}{\partial n \partial \nu} = -\frac{\cos(\mathbf{n}, \boldsymbol{\nu})}{r} + \frac{1}{r}\cos(\mathbf{r}, \mathbf{n})\cos(\mathbf{r}, \boldsymbol{\nu})$$

as

$$\frac{\partial^2 G}{\partial n \partial \nu} = -\left(\frac{\partial^2 G}{\partial r^2} - \frac{1}{r}\frac{\partial G}{\partial r}\right)\cos(\mathbf{r}, \mathbf{n})\cos(\mathbf{r}, \boldsymbol{\nu}) - \frac{1}{r}\frac{\partial G}{\partial r}\cos(\mathbf{n}, \boldsymbol{\nu}). \tag{7}$$

In the following we consider reduction of the finite part integral of eq. (5) to CPV and ordinary integrals. We need separate treatment for three- and two-dimensional cases.

3. Three dimensional case

In this case, $G = e^{-ikr}/r$ and eq. (7) becomes

$$\frac{\partial^2 G}{\partial n \partial \nu} = -e^{-ikr}\left(\frac{3}{r^3} + \frac{3ik}{r^2} - \frac{k^2}{r}\right)\cos(\mathbf{r}, \mathbf{n})\cos(\mathbf{r}, \boldsymbol{\nu}) + e^{-ikr}\left(\frac{1}{r^3} + \frac{ik}{r^2}\right)\cos(\mathbf{n}, \boldsymbol{\nu}). \tag{8}$$

The $O(r^{-3})$ terms give rise to finite part hyper singular integrals in eq. (5), while $O(r^{-2})$ and $O(r^{-1})$ terms yield CPV singular and regular integrals respectively. For the left

hand side of the equation we can write

$$\neq\!\!\!\!\!\!\int_S \Phi(Q)\frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S = \fint_S [\Phi(Q)-\Phi(N)]\frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S + \Phi(N)\neq\!\!\!\!\!\!\int_S \frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S. \tag{9}$$

Significantly, by Lemma 1, the singularity in the last integral can in fact be ignored and it is possible to transform the integral by Stokes theorem. Noting that $G = G(r)$ with $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$, $\nabla_x G = -\nabla G$, where ∇_x is the divergence operator in the space of $P(x, y, z)$ and ∇ is the same operator in the space of the variable point $Q(\xi, \eta, \zeta)$ of S. Hence

$$\begin{aligned}\frac{\partial^2 G}{\partial n \partial \nu} &= \mathbf{n}\cdot\nabla_x(\mathbf{v}\cdot\nabla G)\\ &= \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)\left(\lambda\frac{\partial G}{\partial \xi} + \mu\frac{\partial G}{\partial \eta} + \nu\frac{\partial G}{\partial \zeta}\right),\end{aligned}$$

where (l, m, n) and (λ, μ, ν) are respectively the projections of $\mathbf{n}$ and ν on the coordinate axes. Since $\nabla_x(\partial G/\partial\xi) = -\nabla(\partial G/\partial\xi)\ldots$, we can rearrange to obtain

$$\frac{\partial^2 G}{\partial n \partial \nu} = -(\mathbf{n}\cdot\mathbf{v})\nabla^2 G + \mathbf{v}\cdot\mathrm{curl}[\mathbf{n}\times\nabla G]. \tag{10}$$

Thus, noting that $\nabla^2 G = -k^2 G$ and applying Stokes theorem to the second term

$$\neq\!\!\!\!\!\!\int_S \frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S = k^2\int_S (\mathbf{n}\cdot\mathbf{v})G\mathrm{d}S + \int_C [\mathbf{n}\times\nabla G]\cdot\mathrm{d}\boldsymbol{\xi}, \tag{11}$$

the right hand side of which consists of regular integrals.

In the first integral on the right hand side of eq. (10), there is a large part in the integrand which yields regular integrals. To separate the Cauchy principal value integral in it, we introduce the potential $G_0 = 1/r$ for which

$$\frac{\partial^2 G_0}{\partial n \partial \nu} = -\frac{3}{r^3}\cos(\mathbf{r},\mathbf{n})\cos(\mathbf{r},\mathbf{v}) + \frac{1}{r^3}\cos(\mathbf{n},\mathbf{v}). \tag{12}$$

We can then write

$$\begin{aligned}\fint_S [\Phi(Q)-\Phi(N)]\frac{\partial^2 G}{\partial n \partial \nu}\mathrm{d}S &= \int_S [\Phi(Q)-\Phi(N)]\frac{\partial^2 (G-G_0)}{\partial n \partial \nu}\mathrm{d}S\\ &\quad + \fint_S [\Phi(Q)-\Phi(N)]\frac{\partial^2 G_0}{\partial n \partial \nu}\mathrm{d}S.\end{aligned} \tag{13}$$

The full integrand of the first integral on the right hand side of eq. (12) can be explicitly written down from eqs (8) and (13). For the second we use identity (10) with $\nabla^2 G_0 = 0$ and ∇G_0 calculable explicitly, we get

$$\begin{aligned}\frac{\partial^2 G_0}{\partial n \partial \nu} = -2\frac{\mathbf{n}\cdot\mathbf{v}}{r^3} + \frac{3}{r^3}\Bigg[&\left(l\frac{\eta-y}{r} - m\frac{\xi-x}{r}\right)\left(\lambda\frac{\eta-y}{r} - \mu\frac{\xi-x}{r}\right)\\ &+ \left(m\frac{\zeta-z}{r} - n\frac{\eta-y}{r}\right)\left(\mu\frac{\zeta-z}{r} - \nu\frac{\eta-y}{r}\right)\\ &+ \left(n\frac{\xi-x}{r} - l\frac{\zeta-z}{r}\right)\left(\nu\frac{\xi-x}{r} - \lambda\frac{\zeta-z}{r}\right)\Bigg].\end{aligned} \tag{14}$$

The first term on the right hand side leads to CPV integral while the second leads to a regular integral since

$$(x-\xi)/r:(y-\eta)/r:(z-\zeta)/r \to \lambda:\mu:\nu: \to l:m:n$$

as $Q \to N$, $P \to N$.

With the above procedure (eqs (10)–(14)), eq. (9) in the limit $P \to N$, yields for the original hyper singular integral equation (5), the linear CPV singular integral equation

$$\Phi(P)\left[k^2 \int_S (\mathbf{n}\cdot\mathbf{v})\frac{e^{-ikr}}{r}dS + \int_C \left[\mathbf{n} \times \nabla\left(\frac{e^{-ikr}}{r}\right)\right]\cdot d\boldsymbol{\xi}\right.$$
$$+ \int_S [\Phi(Q)-\Phi(P)]\left[-\left\{(e^{-ikr}-1)\frac{3}{r^3} + e^{-ikr}\left(\frac{3ik}{r^2}-\frac{k^2}{r}\right)\right\}\cos(\mathbf{r},\mathbf{n})\cos(\mathbf{r},\mathbf{v})\right.$$
$$\left. + \left\{(e^{-ikr}-1)\frac{1}{r^3} + e^{-ikr}\frac{ik}{r^2}\right\}\cos(\mathbf{n},\mathbf{v})\right]dS$$
$$+3\int_S [\Phi(Q)-\Phi(P)][\{l(\eta-y)-m(\xi-x)\}\{\lambda(\eta-y)-\mu(\xi-x)\}$$
$$+\{m(\zeta-z)-n(\eta-y)\}\{\mu(\zeta-z)-\nu(\eta-y)\}+\{n(\xi-x)-l(\zeta-z)\}$$
$$\{\nu(\xi-x)-\lambda(\zeta-z)\}\Big]\frac{dS}{r^4} - 2\,\unicode{x2A0F}_S [\Phi(Q)-\Phi(P)]\frac{\mathbf{n}\cdot\mathbf{v}}{r^3}dS = \Psi(P), \quad P\to N. \tag{15}$$

A feature to note, is the occurence of $\Phi(P)$ inside integrals of the equation. This does not pose much numerical difficulty when the integrals are replaced by quadrature formulae.

4. A simple integral equation in tangential derivative

Integral equations of this type were first obtained by Maue [8] for the scattering problem. Using eq. (11)

$$\unicode{x2A0F}_S \Phi(Q)\frac{\partial^2 G}{\partial n \partial \nu}dS = k^2 \int_S (\mathbf{n}\cdot\mathbf{v})G\Phi(Q)dS$$
$$+ \unicode{x2A0F}_S \mathbf{v}\cdot\{\mathrm{curl}[\Phi(Q)(\mathbf{n}\times\nabla G)] - \nabla\Phi(Q)\times(\mathbf{n}\times\nabla G)\}dS.$$

By Stokes theorem, the second integral on the right hand side is equal to

$$\int_C \Phi(Q)(\mathbf{n}\times\nabla G)\cdot d\boldsymbol{\xi} = 0,$$

since $\Phi(Q) = \phi^+ - \phi^- = 0$ on the rim C. Hence eq. (5) becomes

$$\unicode{x2A0F}_S \nabla\Phi(Q)\times(\mathbf{n}\times\nabla r)e^{-ikr}\left(\frac{1}{r^2}+\frac{ik}{r}\right)dS$$
$$+k^2\int_S (\mathbf{n}\cdot\mathbf{v})\Phi(Q)\frac{e^{-ikr}}{r}dS = \Psi(P), \quad P\to N. \tag{16}$$

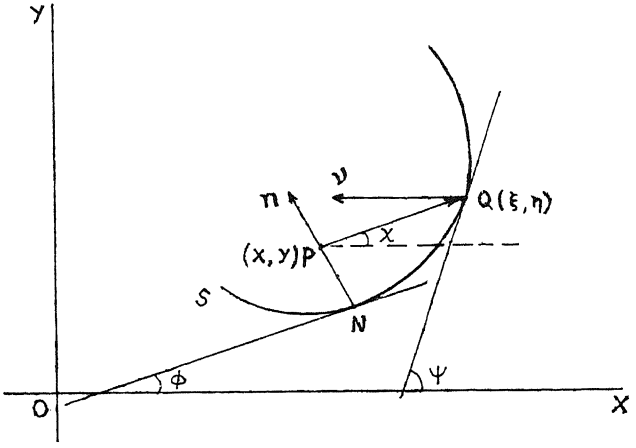

Figure 3. The geometry in two dimensions.

This equation does not contain $\Phi(P)$ inside the integrals.

5. Two dimensional case

In view of the fact that the surface integral degenerates into a line integral along a plane arc, the two dimensional case needs separate treatment. Denoting the inclinations of the tangents at N and Q to OX by ϕ, ψ and that of $\mathbf{r}$ by χ (figure 3),

$$\angle(\mathbf{r},\mathbf{n}) = \frac{\pi}{2} + \phi - \chi, \quad \angle(\mathbf{r},\mathbf{v}) = \frac{\pi}{2} + \psi - \chi, \quad \angle(\mathbf{n},\mathbf{v}) = \psi - \phi$$

in eq. (7). Also since

$$G = \frac{i\pi}{2} H_0^{(2)}(kr), \quad \frac{\partial G}{\partial r} = -\frac{i\pi k}{2} H_1^{(2)}(kr),$$

$$\frac{\partial^2 G}{\partial r^2} = \frac{i\pi k^2}{2}\left[\frac{1}{kr} H_1^{(2)}(kr) - H_0^{(2)}(kr)\right].$$

Thus, (7) takes the form

$$\frac{\partial^2 G}{\partial n \partial v} = \frac{i\pi k^2}{2}[H_0^{(2)}(Kr)\sin(\phi - \chi)\sin(\psi - \chi)$$
$$+ \frac{1}{kr} H_1^{(2)}(kr)\cos(\phi + \psi - 2\chi)]. \tag{17}$$

If we assume that the equation of S is given parametrically

$$x \to f(t), \quad y \to g(t); \quad \xi = f(\tau), \quad \eta = g(\tau), \quad a \leqslant t, \quad \tau \leqslant b \tag{18}$$

then the trigonometric expressions in the limit $P \to N$ can be written as

$$\sin(\phi - \chi)\sin(\psi - \chi) = \frac{K_1(t,\tau)}{\sqrt{x'^2 + y'^2}\sqrt{\xi'^2 + \eta'^2}} \tag{19}$$

$$\cos(\phi + \psi - 2\chi) = \frac{K_2(t,\tau)}{\sqrt{x'^2 + y'^2}\sqrt{\xi'^2 + \eta'^2}} \tag{19a}$$

where, $\tan\phi = y'/x'$, $\tan\psi = \eta'/\xi'$, primes denoting the derivatives and $\sin\chi = (\eta - y)/r$, $\cos\chi = (\xi - x)/r$. Also if $\phi < \pi/2$ and $\psi < \pi/2$,

$$K_1(t,\tau) = \frac{1}{r^2}[(\xi - x)^2\eta' y' + (\eta - y)^2\xi' x' - (\xi - x)(\eta - y)(\xi' y' + \eta' x')]. \tag{20}$$

$$K_2(t,\tau) = \frac{1}{r^2}[\{(\xi - x)^2 - (\eta - y)^2\}(\xi' x' - \eta' y') + 2(\xi - x)(\eta - y)(\xi' y' + \eta' x')]. \tag{20a}$$

If however, $\phi > \pi/2$ or $\psi > \pi/2, x', \xi'$ are to be replaced by $-x', -\xi'$. In the limiting case $P, Q \to N, r \to 0$ and $\psi \to \phi, \chi \to -((\pi/2) - \phi)$ and the left hand sides of (19) tend to 1 or -1 respectively. Hence, as $\tau \to t, K_1(t,\tau) = -K_2(t,\tau) \to x'^2 + y'^2$. With eqs (17), (19) and (20), eq. (5) becomes after a little rearrangement

$$\begin{aligned}&⨍_a^b \Phi(\tau)\left[H_0^{(2)}(kr)K_1(t,\tau) + \frac{1}{kr}\left\{H_1^{(2)}(kr) - \frac{2i}{\pi kr}\right\}K_2(t,\tau)\right.\\ &\left.\quad + \frac{2i}{\pi k^2 r^2}K_2(t,\tau)\right]\mathrm{d}\tau = -\frac{2i}{\pi k^2}\sqrt{x'^2 + y'^2}\,\Psi(P), \quad P \to N.\end{aligned} \tag{21}$$

The first term of the kernal within the square brackets is logarithmically singular and hence integrable; the second regular and the third hyper singular. In fact it can be shown from definition (20) that

$$\frac{K_2(t,\tau)}{r^2} \to -\tfrac{1}{2}(\tau - t)^{-2}, \quad \text{as } \tau \to t. \tag{22}$$

To extract the finite part of the hyper singular integral, we may use the following.

Lemma 2.

$$⨍_a^b \frac{f(t,\tau)}{(\tau - t)^2}\mathrm{d}\tau = \int_a^b \frac{f(t,\tau) - f(t,t)}{(\tau - t)^2}\mathrm{d}\tau + f(t,t)\frac{b - a}{(t - a)(t - b)}.$$

From definition of the finite part of the hyper singular integral

$$⨍_a^b \frac{g(\tau)}{(\tau - t)^2}\mathrm{d}\tau = \int_a^{t-\varepsilon} \frac{g(\tau)}{(\tau - t)^2} + \int_{t+\varepsilon}^b \frac{g(\tau)}{(\tau - t)^2}\mathrm{d}\tau + g(t)\int_{t-\varepsilon}^{t+\varepsilon} \frac{\mathrm{d}\tau}{(\tau - t)^2}, \quad \varepsilon \to 0. \tag{23}$$

The proof follows from the particular case $g(\tau) = 1$:

$$⨍_a^b \frac{\mathrm{d}\tau}{(\tau - t)^2} = \frac{1}{t - b} - \frac{1}{t - a}$$

(Kaya and Erdogan [4]).

The integral equation can thus be rewritten as

$$\Phi(t) + \frac{(b - t)(t - a)}{b - a}\left[\pi k^2 \int_a^b \Phi(\tau)\left\{H_0^{(2)}(kr)K_1(t,\tau)\right.\right.$$

$$+\frac{1}{kr}\left(H_1^{(2)}(kr)-\frac{2i}{\pi kr}\right)K_2(t,\tau)\bigg\} + 2\int_a^b \bigg\{\frac{\Phi(\tau)K_2(t,\tau)}{(\xi-x)^2+(\eta-y)^2}$$

$$+\frac{1}{2}\frac{\Phi(t)}{(\tau-t)^2}\bigg\}\mathrm{d}\tau\bigg]$$

$$= -2\frac{(b-t)(t-a)}{b-a}\sqrt{x'^2+y'^2}\,\Psi(P), \quad P\to N. \tag{24}$$

As in the three dimensional case, $\Phi(t)$ occurs inside the (CPV) integral, in this formulation. From the equation we find that in the neighbourhood of the extremities of the arc $t\to a$ or $t\to b$, $\Phi(t)\to 0$, but the law of approach is not apparent.

6. Alternative formula containing derivative

Lemma 3.

$$\int_a^b \frac{f(t,\tau)\Phi(\tau)}{(\tau-t)^2}\mathrm{d}\tau = \int_a^b \frac{f(t,\tau)-f(t,t)}{(\tau-t)^2}\Phi(\tau)\mathrm{d}\tau$$

$$+f(t,t)\left[\int_a^b \frac{\Phi'(\tau)}{\tau-t}\mathrm{d}\tau + \frac{\Phi(b)}{t-b} - \frac{\Phi(a)}{t-a}\right].$$

The proof follows from (Kaya and Erdogan [4])

$$\int_a^b \frac{g(\tau)}{(\tau-t)^2}\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \frac{g(\tau)}{\tau-t}\mathrm{d}\tau$$

$$= -\frac{g(a)}{t-a} + \frac{g(b)}{t-b} - \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b g'(\tau)\ln|\tau-t|\mathrm{d}\tau$$

$$= -\frac{g(a)}{t-a} + \frac{g(b)}{t-b} + \int_a^b \frac{g'(\tau)}{\tau-t}\mathrm{d}\tau \tag{25}$$

where the second line is obtained by splitting (a,b) into $(a,t-\varepsilon)$ and $(t+\varepsilon,b)$, $\varepsilon\to 0$ and integration by parts.

Remark. Equation (25) justifies the validity of (5) in two dimensional case.

With the aid of the above lemma and eq. (21), the hyper singular integral in it can be written as

$$\int_a^b \frac{K_2(t,\tau)}{r^2}\Phi(\tau)\mathrm{d}\tau = \int_a^b\left[\frac{K_2(t,\tau)}{r^2} + \frac{1}{2(\tau-t)^2}\right]\Phi(\tau)\mathrm{d}\tau - \frac{1}{2}\int_a^b \frac{\Phi(\tau)}{\tau-t}\mathrm{d}\tau. \tag{26}$$

The term contributed by $\Phi(a)$, $\Phi(b)$ drop out according to what has been stated at the end of the previous section. Equation (21) thus takes the form

$$\int_a^b \frac{\Phi'(\tau)}{\tau-t}\mathrm{d}\tau - 2\int_a^b \Phi(\tau)\left[\frac{K_2(t,\tau)}{(\xi-x)^2+(\eta-y)^2} + \frac{1}{2(\tau-t)^2}\right]\mathrm{d}\tau$$

$$+ i\pi k^2\int_a^b \Phi(\tau)\left[H_0^{(2)}(kr)K_1(t,\tau) + \frac{1}{kr}\left\{H_1^{(2)}(kr) - \frac{2i}{\pi kr}\right\}K_2(t,\tau)\right]\mathrm{d}\tau$$

$$= 2\sqrt{x'^2 + y'^2}\,\Psi(P), \quad P \to N. \tag{27}$$

This does not contain $\Phi(t)$ inside the integral.

The behaviour of $\Phi(t)$ at $t = a, b$ depends on the nature of solution of eq. (25) which is of the form

$$\int_a^b \frac{\Phi'(\tau)}{\tau - t} = F(t), \quad a \leqslant t \leqslant b. \tag{28}$$

If we think of $\Phi(t)$ as displacement discontinuity across S, $\Phi'(t)$ will represent a quantity proportional to stress and must become unbounded at $t = a, b$. Hence from the theory of singular integral equation of the above form (Sih [10]), the fundamental solution of (28) must be of the form

$$w(t) = \{(t-a)(b-t)\}^{-(1/2)}.$$

Thus, as $t \to a$ or b

$$\Phi'(t) = \{(t-a)(b-t)\}^{-(1/2)} \times \text{a bounded function}$$

or

$$\Phi(t) = \{(t-a)(b-t)\}^{1/2} \times \text{a bounded function,}$$

which verifies a well-known fact.

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Eigenvalue bounds for Orr–Sommerfeld equation 'No backward wave' theorem

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Abstract. Theoretical estimates of the phase velocity C_r of an arbitrary unstable, marginally stable or stable wave derived on the basis of the classical Orr–Sommerfeld eigenvalue problem governing the linear instability of plane Poiseuille flow ($U(z) = 1 - z^2$, $-1 \leqslant z \leqslant +1$), leave open the possibility of these phase velocities lying outside the range $U_{\min} < C_r < U_{\max}$, but not a single experimental or numerical investigation in this regard, which are concerned with unstable or marginally stable waves, has supported such a possibility as yet, $U_{\min}$ and $U_{\max}$ being respectively the minimum and the maximum value of $U(z)$ for $z \in [-1, +1]$. This gap between the theory on one side and the experiment and computation on the other has remained unexplained ever since Joseph derived these estimates, first, in 1968, and has even led to the speculation of a negative phase velocity (or rather, $C_r < U_{\min} = 0$) and hence the possibility of a 'backward' wave as in the case of the Jeffery–Hamel flow in a diverging channel with back flow ([1]). A simple mathematical proof of the non-existence of such a possibility is given herein by showing that the phase velocity C_r of an arbitrary unstable or marginally stable wave must satisfy the inequality $U_{\min} < C_r < U_{\max}$. It follows as a consequence stated here in this explicit form for the first time to the best of our knowledge, that 'overstability' and not the 'principle of exchange of stabilities' is valid for the problem of plane Poiseuille flow.

Keywords. Bounds; Orr–Sommerfeld equation.

1. Introduction

In the linear instability problem of parallel shear flow of a nonviscous fluid, Rayleigh showed that the phase velocity C_r of an arbitrary unstable wave must lie in the range $U_{\min} < C_r < U_{\max}$ and since then the problem of generalizing this result with the inclusion of the effect of viscosity of the fluid, which results in the basic flow being a Poiseuille one, has been much sought after. Joseph [3] discovered in the context of plane Poiseuille flow a lower and an upper bound of C_r of an arbitrary unstable, marginally stable or stable wave in the form

$$\frac{U_{\min} + 2\left[\dfrac{d^2U}{dz^2}\right]_{\min}}{\pi^2 + 4\alpha^2} = \frac{-4}{\pi^2 + 4\alpha^2} < C_r < U_{\max} = 1, \tag{1}$$

which leave open the possibility of these phase velocities lying outside the range $U_{\min} < C_r < U_{\max}$, but not a single experimental or numerical investigation in this regard, which are concerned with unstable or marginally stable waves has supported such a possibility as yet. This gap between the theory on one side and the experiment and the computation on the other has remained unexplained ever since Joseph derived these estimates and has even led to the speculation of a negative phase velocity (or, rather, $C_r < U_{\min} = 0$) and hence the possibility of a 'backward' wave as in the case of the Jeffery–Hamel flow in a diverging channel with back flow. A simple mathematical

proof of the non-existence of such a possibility is given herein by showing that the phase velocity C_r of an arbitrary unstable or marginally stable wave must satisfy the inequality $U_{\min} < C_r < U_{\max}$. It follows as a consequence, stated here in this explicit form for the first time to the best of our knowledge, that 'overstability' and not the 'principle of exchange of stabilities', is valid for the problem of plane Poiseuille flow.

2. Mathematical analysis

The classical Orr–Sommerfeld eigenvalue problem governing the linear instability of plane Poiseuille flow against two-dimensional perturbations is given by

$$\frac{1}{i\alpha R}(D^2-\alpha^2)^2\phi=(U-C)(D^2-\alpha^2)\phi-\frac{\mathrm{d}^2U}{\mathrm{d}z^2}\phi \tag{2}$$

and

$$\phi=0=D\phi \quad \text{at} \quad z=-1 \quad \text{and} \quad z=+1, \tag{3}$$

where z is the real independent variable such that $-1\leqslant z\leqslant +1$ and $D\equiv \mathrm{d}/\mathrm{d}z$; α is the wave number of the perturbation and is real; $R>0$ is the Reynold number of the flow; $C=C_r+iC_i$ is the complex wave velocity of the perturbation, C_r and C_i being respectively the phase velocity and the amplification factor; $\phi(z)$ is the amplitude of the stream function perturbation in the form $\phi(z)e^{i\alpha(x-ct)}$ and is a complex valued function of the real variable z while $U(z)=1-z^2$ is the basic background flow. Equations (2) and (3), thus define an eigenvalue problem for C for given values of α and R, and a perturbation is said to be unstable if $\alpha C_i>0$, marginally stable if $\alpha C_i=0$ for some values of α and R with the further condition that $\alpha C_i>0$ for any neighbouring values of α and R, and stable if $\alpha C_i\leqslant 0$.

We now prove the following theorems:

Theorem 1. *If (ϕ, C) with $\alpha C_i\neq 0$ is a solution of the Orr–Sommerfeld eigenvalue problem described by eqs* (2) *and* (3) *for prescribed values of α and R then the integral relation*

$$\begin{aligned}&2C_i\int_{-1}^{+1}(U-C_r)[|DF|^2+\alpha^2|F|^2]\mathrm{d}z\\&+\frac{1}{\alpha R}\int_{-1}^{+1}(U-C_r)[|D^2F|^2+2\alpha^2|DF|^2+\alpha^4|F|^2]\mathrm{d}z\\&-\frac{1}{\alpha R}\int_{-1}^{+1}\frac{d^2U}{\mathrm{d}z^2}[2|DF|^2+\alpha^2|F|^2]\mathrm{d}z+\frac{1}{2\alpha R}\int_{-1}^{+1}\frac{\mathrm{d}^4U}{\mathrm{d}z^4}|F|^2\mathrm{d}z=0,\end{aligned} \tag{4}$$

with $F=\phi/(U-C)$, is true.

Proof. We apply the transformation $\phi=(U-C)F$ which remains valid for all values of $z\in[-1,+1]$, since $\alpha C_i\neq 0$. Equations (2) and (3) then transform into

$$\frac{1}{i\alpha R}(D^2-\alpha^2)^2[(U-C)F]=D[(U-C)^2DF]-\alpha^2(U-C)^2F, \tag{5}$$

and

$$F=0=DF \quad \text{at} \quad z=-1 \quad \text{and} \quad +1. \tag{6}$$

Multiplying eq. (5) throughout by F^* (the complex conjugate of F) and integrating

the resulting equation over the range of z, we get

$$\frac{1}{i\alpha R}\int_{-1}^{+1} F^*(D^2-\alpha^2)^2[(U-C)F]\,\mathrm{d}z$$
$$=\int_{-1}^{+1} F^*D[(U-C)^2DF]\,\mathrm{d}z-\alpha^2\int_{-1}^{+1}(U-C)^2|F|^2\,\mathrm{d}z. \tag{7}$$

Equating the imaginary parts of both sides of (7), we have

$$-\frac{1}{\alpha R}\mathrm{Re}\int_{-1}^{+1} F^*(D^2-\alpha^2)^2[(U-C)F]\,\mathrm{d}z$$
$$=\mathrm{Im}\int_{-1}^{+1} F^*D[(U-C)^2DF]\,\mathrm{d}z-\alpha^2\,\mathrm{Im}\int_{-1}^{+1}(U-C)^2|F|^2\,\mathrm{d}z, \tag{8}$$

where the symbols Re and Im respectively denote the real and imaginary parts of the quantities that succeed them.

Now

$$\mathrm{Re}\int_{-1}^{+1} F^*(D^2-\alpha^2)^2[(U-C)F]\,\mathrm{d}z$$
$$=\mathrm{Re}\int_{-1}^{+1} F^*(D^4-2\alpha^2D^2+\alpha^4)[(U-C)F]\,\mathrm{d}z$$
$$=\mathrm{Re}\int_{-1}^{+1} F^*D^4[(U-C)F]\,\mathrm{d}z$$
$$-2\alpha^2\,\mathrm{Re}\int_{-1}^{+1}F^*D^2[(U-C)F]\,\mathrm{d}z+\alpha^4\,\mathrm{Re}\int_{-1}^{+1}(U-C)|F|^2\,\mathrm{d}z,$$

which upon integrating the first integral twice and the second integral once by parts and making use of the boundary conditions (6) yields

$$\mathrm{Re}\int_{-1}^{+1} D^2F^*\Big[(U-C)D^2F+2\frac{\mathrm{d}U}{\mathrm{d}z}DF+\frac{\mathrm{d}^2U}{\mathrm{d}z^2}F\Big]\mathrm{d}z$$
$$+2\alpha^2\,\mathrm{Re}\int_{-1}^{+1}DF^*\Big[(U-C)DF+\frac{\mathrm{d}U}{\mathrm{d}z}F\Big]\mathrm{d}z+\alpha^4\int_{-1}^{+1}(U-C_r)|F|^2\,\mathrm{d}z,$$

which upon rearranging yields

$$\int_{-1}^{+1}(U-C_r)[|D^2F|^2+2\alpha^2|DF|^2+\alpha^4|F|^2]\,\mathrm{d}z$$
$$+\mathrm{Re}\int_{-1}^{+1}D^2F^*\left[2\frac{\mathrm{d}U}{\mathrm{d}z}DF+\frac{\mathrm{d}^2U}{\mathrm{d}z^2}F\right]\mathrm{d}z$$
$$+2\alpha^2\,\mathrm{Re}\int_{-1}^{+1}DF^*\frac{\mathrm{d}U}{\mathrm{d}z}F\,\mathrm{d}z. \tag{9}$$

Integrating $2\int_{-1}^{+1}D^2F^*(\mathrm{d}U/\mathrm{d}z)DF\,\mathrm{d}z$, $\int_{-1}^{+1}D^2F^*(\mathrm{d}^2U/\mathrm{d}z^2)F\,\mathrm{d}z$ and $\int_{-1}^{+1}DF^*(\mathrm{d}U/\mathrm{d}z)F\,\mathrm{d}z$ by parts once, twice and once respectively and making use of the boundary conditions

(5), we derive

$$\mathrm{Re}\,2\int_{-1}^{+1} D^2F^*\frac{\mathrm{d}U}{\mathrm{d}z}DF\mathrm{d}z = -\int_{-1}^{+1}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}|DF|^2\mathrm{d}z, \tag{10}$$

$$\mathrm{Re}\int_{-1}^{+1} D^2F^*\frac{\mathrm{d}^2U}{\mathrm{d}z^2}F\mathrm{d}z = -\int_{-1}^{+1}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}|DF|^2\mathrm{d}z + \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}^4U}{\mathrm{d}z^4}|F|^2\mathrm{d}z, \tag{11}$$

and

$$\mathrm{Re}\int_{-1}^{+1} DF^*\frac{\mathrm{d}U}{\mathrm{d}z}F\mathrm{d}z = -\frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}|F|^2\mathrm{d}z. \tag{12}$$

It then follows from eqs (9), (10), (11) and (12) that

$$\begin{aligned}
&\mathrm{Re}\int_{-1}^{+1} F^*(D^2-\alpha^2)^2[(U-C)F]\mathrm{d}z\\
&= \int_{-1}^{+1}(U-C_r)[|D^2F|^2 + 2\alpha^2|DF|^2 + \alpha^4|F|^2]\mathrm{d}z\\
&\quad - \int_{-1}^{+1}\left(\frac{\mathrm{d}^2U}{\mathrm{d}z^2}\right)[2|DF|^2 + \alpha^2|F|^2]\mathrm{d}z + \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}^4U}{\mathrm{d}z^4}|F|^2\mathrm{d}z.
\end{aligned} \tag{13}$$

Further

$$\begin{aligned}
&\mathrm{Im}\int_{-1}^{+1} F^*D[(U-C)^2DF]\mathrm{d}z - \alpha^2\,\mathrm{Im}\int_{-1}^{+1}(U-C)^2|F|^2\mathrm{d}z\\
&= -\mathrm{Im}\int_{-1}^{+1}(U-C)^2[|DF|^2 + \alpha^2|F|^2]\mathrm{d}z\\
&= 2C_i\int_{-1}^{+1}(U-C_r)[|DF|^2 + \alpha^2|F|^2]\mathrm{d}z,
\end{aligned} \tag{14}$$

which follows by integrating the first integral by parts once and making use of the boundary conditions (5).

Combinding eqs (8), (13) and (14), we obtain the integral relation

$$\begin{aligned}
&2C_i\int_{-1}^{+1}(U-C_r)[|DF|^2 + \alpha^2|F|^2]\mathrm{d}z\\
&+\frac{1}{\alpha R}\int_{-1}^{+1}(U-C_r)[|D^2F|^2 + 2\alpha^2|DF|^2 + \alpha^4|F|^2]\mathrm{d}z\\
&-\frac{1}{\alpha R}\int_{-1}^{+1}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}[2|DF|^2 + \alpha^2|F|^2]\mathrm{d}z + \frac{1}{2\alpha R}\int_{-1}^{+1}\frac{\mathrm{d}^4U}{\mathrm{d}z^4}|F|^2\mathrm{d}z = 0,
\end{aligned} \tag{15}$$

and hence the theorem.

Theorem 2. *If (ϕ, C) with $\alpha C_i > 0$ is a solution of the Orr-Sommerfeld eigenvalue problem described by eqs* (2) *and* (3) *for prescribed values of α and R then C_r must satisfy the inequality*

$$U_{\min} < C_r < U_{\max}. \tag{16}$$

Proof. We write eq. (15), which is valid under the present conditions as

$$2\alpha C_i\int_{-1}^{+1}(U-C_r)[|DF|^2+\alpha^2|F|^2]\mathrm{d}z$$
$$+\frac{1}{R}\int_{-1}^{+1}(U-C_r)[|D^2F|^2+2\alpha^2|DF|^2+\alpha^4|F|^2]\mathrm{d}z$$
$$-\frac{1}{R}\int_{-1}^{+1}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}[2|DF|^2+\alpha^2|F|^2]\mathrm{d}z+\frac{1}{2R}\int_{-1}^{+1}\frac{\mathrm{d}^4U}{\mathrm{d}z^4}|F|^2\mathrm{d}z=0. \tag{17}$$

Now, for $U(z)=1-z^2$, we have $(\mathrm{d}^2U/\mathrm{d}z^2)=-2$ and $(d^4U/\mathrm{d}z^4)=0$ for all values of $z\in[-1,+1]$ and further since $\alpha C_i>0$, it follows from eq. (17) that

$$\int_{-1}^{+1}(U-C_r)\left[2\alpha C_i(|DF|^2+\alpha^2|F|^2)+\frac{1}{R}(|D^2F|^2+2\alpha^2|DF|^2+\alpha^4|F|^2)\right]\mathrm{d}z$$
$$+\frac{2}{R}\int_{-1}^{+1}(2|DF|^2+\alpha^2|F|^2)\mathrm{d}z=0. \tag{18}$$

The quantity within the square brackets under the first integral sign is a positive definite and therefore, for the validity of eq. (18), we must have for some z_s

$$U(z_s)-C_r<0,\quad z_s\in[-1,+1], \tag{19}$$

which implies that

$$C_r>U_{\min}. \tag{20}$$

Combining inequality (20) with Joseph's inequality given by (1) which holds good for $\alpha C_i \gtreqless 0$, we derive the result that

$$U_{\min}<C_r<U_{\max}, \tag{21}$$

and hence the theorem.

Theorem 3. *If* (ϕ, C) *with* $\alpha C_i=0$ $(\alpha\neq 0$ *since* $\alpha=0$ *corresponds to a trivial solution for* $\phi)$ *is a solution of the Orr–Sommerfeld eigenvalue problem described by eqs* (2) *and* (3) *for prescribed values of* α *and* R *then* C_r *must satisfy the inequality*

$$U_{\min}\leqslant C_r<U_{\max}. \tag{22}$$

Proof. Since $C_i=0$, it follows that the behaviour of $U-C=U-C_r$ must fall into one of the three mutually exclusive classes namely

(i) $U-C_r>0$ for all values of $z\in[-1,+1]$,
(ii) $U-C_r<0$ for all values of $z\in[-1,+1]$, and
(iii) $U-C_r=0$ for some value of $z=z_s\in[-1,+1)$.

If (i) is valid then under the present conditions the transformation $\phi=(U-C)F$ remains well defined for all values of $z\in[-1,+1]$ so that we derive from eq. (15) that

$$\int_{-1}^{+1}(U-C_r)[|D^2F|^2+2\alpha^2|DF|^2+\alpha^4|F|^2]\mathrm{d}z$$
$$+2\int_{-1}^{+1}(2|DF|^2+\alpha^2|F|^2)\mathrm{d}z=0. \tag{23}$$

A necessary condition for the validity of eq. (23) is that

$$U - C_r < 0 \text{ for some value of } z \in [-1, +1],$$

which clearly contradicts the starting hypothesis, namely (i). Thus (i) cannot be valid.

If (ii) is valid, then we must have

$$U - C_r < 0 \quad \text{for all values of } z \in [-1, +1],$$

from which it follows that

$$C_r > U(0) = U_{\text{max}},$$

which presents a contradiction, since by Joseph's estimate given by inequality (1) C_r satisfies $C_r < U_{\text{max}}$. Thus (ii) also cannot be valid.

Therefore (iii) must hold good so that we have

$$U - C_r = 0 \text{ for some value of } z = z_s \in [-1, +1],$$

which implies that

$$U(z_s) - C_r = 0,$$

from which it follows that

$$U_{\text{min}} \leqslant C_r \leqslant U_{\text{max}}. \tag{24}$$

Combining inequality (24) with inequality (1) established by Joseph, we derive that such values of C_r as given under conditions of Theorem 3 satisfies the inequality

$$U_{\text{min}} \leqslant C_r < U_{\text{max}}, \tag{25}$$

and hence the theorem.

Remarks. An arbitrary perturbation with $\alpha C_i = 0$ $(\alpha \neq 0)$ is a neutral perturbation and since $U_{\text{min}} = 0$, Theorem 3 clearly shows that stationary (i.e. $C_r = 0$) as well as oscillatory (i.e. $C_r \neq 0$) neutral perturbations are both allowed by inequality (25). However, for such a neutral perturbation to be a marginal or marginally stable perturbation it is necessary that it lies on the stability boundary i.e. a boundary or boundaries in the (α, R) – plane on crossing which, C_i changes sign, and this requires a more detailed analysis of the eigenvalue problem. In the next theorem we shall prove that all stationary non-neutral perturbations must of necessity decay which rules out the possibility of stationary neutral perturbations lying on the stability boundary which exists after the rigorous mathematical validation of Heisenberg's [2] results by Krylov [4]. Thus, it is the oscillatory neutral perturbations and not the stationary neutral ones that constitute the stability boundary or equivalently, in the Poincaré–Eddington terminology it is 'overstability' and not the 'principle of exchange of stabilities' that is valid for the problem of plane Poiseuille flow.

Theorem 4. *If* (ϕ, C) *with* $\alpha C_i \neq 0$ *and* $C_r = 0$ *is a solution of the Orr–Sommerfeld eigenvalue problem described by eqs* (2) *and* (3) *for prescribed values of* α *and* R *then* $\alpha C_i < 0$.

Proof. From eq. (15), which is valid under the present conditions we derive that

$$2\alpha C_i \int_{-1}^{+1} U[|DF|^2 + \alpha^2 |F|^2] \mathrm{d}z$$
$$+ \frac{1}{R} \int_{-1}^{+1} U[|D^2F|^2 + 2\alpha^2 |DF|^2 + \alpha^4 |F|^2] \mathrm{d}z$$
$$+ \frac{2}{R} \int_{-1}^{+1} (2|DF|^2 + \alpha^2 |F|^2) \mathrm{d}z = 0. \tag{26}$$

But, since $U(z) = 1 - z^2 \geqslant 0$ for all values of $z \in [-1, +1]$, we must have, for the validity of eq. (23), $\alpha C_i < 0$ and hence the theorem.

In view of the Remark mentioned above and Theorem 4 the following theorem is valid:

Theorem 5. *If* (ϕ, C) *with* $\alpha C_i = 0$ $(\alpha \neq 0)$ *and* $C_r \neq 0$ *is a solution of the Orr–Sommerfeld eigenvalue problem described by eqs* (2) *and* (3) *for prescribed values of* α *and* R *then* C_r *must satisfy the inequality*

$$U_{\min} < C_r < U_{\max}. \tag{27}$$

Theorem 2 and Theorem 5 show that the phase velocity of an arbitrary unstable or marginally stable wave must lie in the range $U_{\min} < C_r < U_{\max}$ while Theorem 2 and Theorem 3 show that the phase velocity of an arbitrary unstable or neutrally stable wave must lie in the range $U_{\min} \leqslant C_r < U_{\max}$. Thus, in both cases the possibility of C_r being, less than $U_{\min}$ and hence, negative is ruled out and therefore no 'backward' wave can exist in the instability of Poiseuille flow.

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Distributed computation of fixed points of ∞-nonexpansive maps

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Abstract. The distributed implementation of an algorithm for computing fixed points of an ∞-nonexpansive map is shown to converge to the set of fixed points under very general conditions.

Keywords. Distributed algorithm; fixed point computation; ∞-nonexpansive map; tapering stepsize; controlled Markov chains.

1. Introduction

Many problems in optimization can be cast as problems of finding a fixed point of a map $F:R^d \to R^d$ which is nonexpansive with respect to the ∞-norm, or a weighted version thereof. Examples are algorithms for solving dynamic programming equations for shortest path problems and Markov decision processes, certain network flow problems, etc. A standard approach then is to use the iteration

$$x_{n+1} = F(x_n)$$

or its 'relaxed' version

$$x_{n+1} = (1-a)x_n + aF(x_n), \quad a \in (0,1). \tag{1}$$

A comprehensive account of synchronous and asynchronous implementations of these algorithms and their applications to optimization and numerical analysis appears in [1], along with an extensive bibliography. A continuous time analog is studied in [4].

This work considers a distributed implementation of a variant of (1). We replace 'a' by a tapering stepsize $\{a(n)\} \subset (0,1)$ as in stochastic approximation theory, satisfying for some $r \in (0,1)$,

$$1 \geqslant a(n+1)/a(n) \to 1, \quad \sum_n a(n) = \infty, \quad \sum_n a(n)^{1+q} < \infty \tag{2}$$

for $q \geqslant r$. An example is $a(n) = (n+2)^{-1}$, $r = $ (say) 0·5. Our model of distributed computation is as follows: We postulate a set-valued random process which selects at each time a subset of $\{1, \ldots, d\}$ indicating the indices of the components which are to be updated. The update uses delayed information regarding other components with the delays being required to satisfy a mild conditional moment bound. We obtain a very general convergence theorem under these conditions (Theorem 1.1 below) which does not require F to be a contraction or a pseudocontraction as in the earlier literature.

Tapering stepsize as a means for suppressing the effects of delays was proposed in [3] in the case of algorithms whose continuous limits are differential equations admitting strict Liapunov functions. Equation (1) need not satisfy this condition. Furthermore, the model of distributed computing is more elaborate than in the above work.

The paper is organized as follows. The remainder of this section formulates the algorithm and states the main result. The next section studies an associated o.d.e. which this algorithm tracks in the limit. The third section proves the main result using this 'o.d.e. limit'. The last section gives some examples if nonexpansive maps arising in numerical analysis and optimization. A forthcoming companion paper studies the asynchronous version of this algorithm (i.e., one without a 'global clock').

Introduce the norms

$$\|x\|_p = \left(\frac{1}{d}\sum_{i=1}^{d} |x_i|^p\right)^{1/p}, \quad 1 \leqslant p < \infty,$$

$$\|x\|_\infty = \max_i |x_i|.$$

for $x = [x_1, \ldots, x_d]$. Let $F: R^d \to R^d$ be ∞-nonexpansive, i.e.,

$$\|F(x) - F(y)\|_\infty \leqslant \|x - y\|_\infty, \quad x, y \in R^d.$$

In particular, it is Lipschitz. Let $G = \{x | F(x) = x\}$ denote the (closed) set of fixed points of F, assumed nonempty.

Let $I = \{1, 2, \ldots, d\}$ and S a collection of nonempty subsets of I that cover I. Let $\{Y_n\}$ be an S-valued process and for each n, $\tau_{ij}(n)$, $i \neq j \in I$, random variables (delays) taking values $\{0, 1, \ldots, n\}$. We set $\tau_{ii}(n) = 0 \; \forall i, n$. The distributed version of (1) we consider is as follows: Given $X(0)$, compute $X(n) = [X_1(n), \ldots, X_d(n)]$ iteratively by

$$\begin{aligned} X_i(n+1) = X_i(n) + a(n)[F_i(X_1(n - \tau_{i1}(n)), \ldots, X_d(n - \tau_{id}(n))) \\ - X_i(n)] I\{i \in Y_n\} \end{aligned} \tag{3}$$

for $i \in I$, $n \geqslant 0$. Let $\mathscr{F}_n = \sigma(X(m), Y(m), m \leqslant n, \tau_{ij}(m), m < n, i, j \in I)$ and $\mathscr{G}_n = \sigma(X(m), Y(m), \tau_{ij}(m), m \leqslant n, i, j \in I)$, $n \geqslant 0$. We shall be making the following key assumptions:

(A1) There exists a $\delta > 0$ such that the following holds. For any $A, B \in S$, the quantity

$$P(Y_{n+1} = B / Y_n = A, \mathscr{G}_n) \tag{4}$$

is either always zero, a.s., or always exceeds δ, a.s. That is, having picked A at time n, picking B at the next instant is either improbable or probable with a minimum conditional probability of δ, regardless of n and the 'history' $\mathscr{G}_n$.

Furthermore, if we draw a directed graph with node set S and an edge from A to B when (4) exceeds δ a.s., then this graph is irreducible, i.e., there is a directed path from any $A \in S$ to any $B \in S$.

(A2) There exist $b \geqslant r/(1 - r)$ and $C > 0$ such that

$$E[(\tau_{ij}(n))^b / \mathscr{F}_n] \leqslant C \quad \text{a.s.} \quad \forall i, j, n.$$

(A3) If $\bar{n} =$ the integer part of $a(n)^{r-1}$, then $\bar{n}$ is $o(n)$ and moreover, $\limsup_{n \to \infty} a(n - \bar{n})/a(n) < \infty$. (Note that this condition is satisfied by our example $a(n) = (n+2)^{-1}$ with $r = 0{\cdot}5$.)

The condition $\tau_{ii}(n) = 0$ implies that for updating the ith component, its most recent value is immediately available. The idea is that each component is computed by a specific processor, and the different processors communicate with each other in conformity with (A2). We make the following immediate observation for later use.

Let $z \in G$. If $C_n = \max_{m \leqslant n} \|X(n) - z\|_\infty$, then by (3) and ∞-nonexpansivity F,$\{C_n\}$ is a nonincreasing sequence. Thus

$$\sup_n \|X(n) - z\|_\infty \leqslant \|X(0) - z\|_\infty.$$

Our main result is the following:

Theorem 1.1. *$X(n) \to G$ a.s.*

The proof will use the fact that (3) asymptotically tracks an o.d.e. The next section studies this o.d.e.

2. An associated o.d.e.

We start with some notation. For $A \in S$, define $F^A(\cdot) = [F_1^A(\cdot) \ldots, F_d^A(\cdot)]$: $R^d \to R^d$ by

$$F_j^A(x) = F_j(x) I\{j \in A\} + x_j I\{j \notin A\}$$

for $x = [x_1, \ldots, x_d]$. For any Polish space X, let $\mathscr{P}(X)$ denote the Polish space of probability measures on X with the Prohorov topology. In particular, $\mathscr{P}(S)$ is the space of probability vectors on S. For $\mu \in \mathscr{P}(S)$, define F^μ: $R^d \to R^d$ by

$$F^\mu(\cdot) = \sum \mu(A) F^A(\cdot).$$

Clearly, F^A, F^μ are ∞-nonexpansive. Given $\alpha > 0$, say that $\mu \in \mathscr{P}(S)$ is α-thick if $\min_A \mu(A) \geqslant \alpha$ and thick if it is α-thick for some $\alpha > 0$. The following is easily proved.

Lemma 2.1. *G is precisely the set of fixed points of F^μ for thick μ and the intersection of the sets of fixed points of F^A, $A \in S$ (resp., F^μ, $\mu \in \mathscr{P}(S)$).*

Let U denote the space of $\mathscr{P}(S)$-valued trajectories $\bar{\mu} = \{\mu_t, t \geqslant 0\}$ with the coarsest topology that renders continuous the maps $\bar{\mu} \to \int_0^T f(t)\mu_t(A)\mathrm{d}t$ for $T \geqslant 0$, $A \in S$, $f \in L_2[0, T]$. U then is compact metrizable. Let us say that $\bar{\mu} \in U$ is α-thick for a given $\alpha > 0$ if μ_t is for a.e. t, where the 'a.e.' may be dropped by taking an appropriate version. Say that it is thick if it is α-thick for some $\alpha > 0$.

Lemma 2.2. *Let $\bar{\mu}^n \to \bar{\mu}^\infty$ in U and $\bar{\mu}^n$ is α-thick, $\alpha > 0$, for $n = 1, 2, \ldots$. Then $\bar{\mu}^\infty$ is α-thick.*

Proof. For any $A \in S$, $t > s \geqslant 0$, $n \geqslant 1$,

$$\int_s^t \mu_y^n(A)\mathrm{d}y \geqslant \alpha(t - s).$$

The inequality is preserved in the limit $n \to \infty$. The rest is easy. □

Given $\bar{\mu} \in U$, consider the o.d.e.

$$\dot{x}(t) = F^{\mu_t}(x(t)) - x(t), \; x(0) = x. \tag{5}$$

Lemma 2.3. *The map $(\bar{\mu}, x) \in U \times R^d \to x(\cdot) \in C([0, \infty); R^d)$ defined by* (5) *is continuous.*

Proof. Let $(\bar{\mu}^n, x_n) \to (\bar{\mu}^\infty, x_\infty)$. For $n \geqslant 1$, let $x^n(\cdot)$ satisfy

$$\dot{x}^n(t) = F^{\mu_t^n}(x^n(t)) - x^n(t), \; x^n(0) = x_n.$$

Using the Gronwall lemma and Arzela-Ascoli theorem, one verifies that $\{x^n(\cdot)\}$ is relatively compact in $C([0,\infty);R^d)$. By dropping to a subsequence if necessary, suppose that $x^n(\cdot)\to x^\infty(\cdot)$. Then $x^\infty(0)=x_\infty$. Also, for $t\geqslant 0$, $n\geqslant 1$,

$$\begin{aligned} x^n(t)=x_n&+\int_0^t (F^{\mu^n_s}(x^n(s))-F^{\mu^n_s}(x^\infty(s)))\mathrm{d}s\\ &+\int_0^t (F^{\mu^n_s}(x^\infty(s))-F^{\mu^\infty_s}(x^\infty(s)))\mathrm{d}s\\ &+\int_0^t (F^{\mu^\infty_s}(x^\infty(s))-x^\infty(s))\mathrm{d}s+\int_0^t (x^\infty(s)-x^n(s))\mathrm{d}s. \end{aligned}\tag{6}$$

The first and the fourth integral go to zero as $n\to\infty$ because $x^n(\cdot)\to x^\infty(\cdot)$. (We use the ∞-nonexpansivity of F^μ in the former case.) So does the second integral in view of our topology on U. Thus $x^\infty(\cdot)$ satisfies

$$\dot{x}^\infty(t)=F^{\mu^\infty_t}(x^\infty(t))-x^\infty(t),\ x^\infty(0)=x_\infty.$$

The claim follows. □

The main result of this section is the following:

Theorem 2.1. *Given a thick $\bar{\mu}\in U$, the solution $x(\cdot)$ of (5) converges to a point in G which may depend on the initial condition. Furthermore, $t\to\|x(t)-x^*\|_\infty$ is nonincreasing for any $x^*\in G$.*

The proof will closely follow that of Theorem 3.1 [4], except for the additional complications caused by the fact that (5) is nonautonomous. We split the proof into several lemmas. Let $x^*\in G$ and $\bar{\mu}$ be α-thick for $\alpha>0$.

Lemma 2.4. *The map $t\to\|x(t)-x^*\|_\infty$ is nonincreasing and hence converges to some $a>0$.*

Proof. A straightforward computation shows that for $p\in(1,\infty)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)-x^*\|_p=-\|x(t)-x^*\|_p+\|x(t)-x^*\|_p^{1-p}g(t),$$

where

$$\begin{aligned} g(t)&=\frac{1}{d}\sum_{i=1}^d|x_i(t)-x_i^*|^{p-1}\,sgn(x_i(t)-x_i^*)(F_i^{\mu_t}(x(t))-F_i^{\mu_t}(x^*))\\ &\leqslant\|x(t)-x^*\|_p^{p-1}\|F^{\mu_t}(x(t))-F^{\mu_t}(x^*)\|_p. \end{aligned}\tag{7}$$

Thus for $t>s$,

$$\begin{aligned}\|x(t)-x^*\|_p\leqslant\|x(s)-x^*\|_p&+\int_s^t(-\|x(y)-x^*\|_p\\ &+\|F^{\mu_y}(x(y))-F^{\mu_y}(x^*)\|_p)\mathrm{d}y.\end{aligned}$$

Let $p\to\infty$ to obtain

$$\begin{aligned}\|x(t)-x^*\|_\infty\leqslant\|x(s)-x^*\|_\infty&+\int_s^t(-\|x(y)-x^*\|_\infty\\ &+\|F^{\mu_y}(x(y))-F^{\mu_y}(x^*)\|_\infty)\mathrm{d}y.\end{aligned}$$

The claim follows in view of the ∞-nonexpansivity of F^{μ}. □

If $a=0$, we are done. Suppose $a>0$. Define $B_a=\{x\in R^d \mid \|x-x^*\|_\infty=a\}$. Introduce the terminology: An m-face for $m\leqslant d$ is a set of the type

$$\{x=[x_1,\ldots,x_d] \mid x_{i_k}\in[a_{i_k},b_{i_k}],\ k\leqslant m,\ x_{i_k}=c_{i_k},\ k>m\},$$

where $\{i_1,\ldots,i_d\}$ is a permutation of $\{1,\ldots,d\}$ and $b_k>a_k$, c_k are scalars. Then B_a is the union of $(d-1)$-faces of the type $\{x \mid x_i-x_i^*=a$ (or $-a$) and $|x_j-x_j^*|<a$ for $j\neq i\}$. For any $(d-1)$-face of this type (say, H), define $G_H=\{x\in H \mid F(x)\in H\}$. Then G_H is closed, possibly empty. Since $\|x(t)-x^*\|_\infty\to a$, $x(t)\to B_a$. Now the trajectories $x(t+\cdot)$, $t\geqslant 0$, form a relatively compact set in $C([0,\infty);R^d)$. Thus any limit point $\tilde{x}(\cdot)$ thereof as $t\to\infty$ must lie in B_a. By Lemmas 2.2 and 2.3, $\tilde{x}(\cdot)$ satisfies

$$\dot{\tilde{x}}(t)=F^{\bar{\mu}_t}(\tilde{x}(t))-\tilde{x}(t)$$

for some α-thick $\bar{\mu}\in U$. Let $\{\tilde{x}(\cdot)\}=\{\tilde{x}(t) \mid t\in R^+\}$.

Lemma 2.5. $\{\tilde{x}(\cdot)\}\cap H\subset G_H$.

Proof. If both sets are empty, there is nothing to prove. Suppose $\{\tilde{x}(\cdot)\}\cap H\neq\phi$. For simplicity, let $H=\{x \mid x_1-x_1^*=a, |x_i-x_i^*|\leqslant a, i>1\}$. Suppose $\{\tilde{x}(t) \mid t\in[0,\Delta]\}\subset H$. Then $\tilde{x}_1(t)=x_1^*+a$, $t\in[0,\Delta]$, leading to $0=\dot{\tilde{x}}_1(t)=F_1^{\bar{\mu}_t}(\tilde{x}(t))-\tilde{x}_1(t)$. Thus $F_1^{\bar{\mu}_t}(\tilde{x}(t))=\tilde{x}_1(t)=x_1^*+a=F_1(\tilde{x}(t))$ in view of α-thickness of $\bar{\mu}$, for $t\in[0,\Delta]$. Since F is ∞-nonexpansive and x^* is a fixed point of it, we also have

$$\|F(\tilde{x}(t))-x^*\|_\infty\leqslant\|\tilde{x}(t)-x^*\|_\infty=a.$$

Thus we must have $F(\tilde{x}(t))\in H$, $t\in[0,\Delta]$, implying $\tilde{x}(t)\in G_H$. It follows that all connected segments of $\{\tilde{x}(\cdot)\}\cap H$ that contain more than one point must be in G_H. On the other hand, those containing a single point must clearly be in the relative boundary ∂H of H, which is the union of its $(d-2)$-faces. Let $x\in\{\tilde{x}(\cdot)\}\cap\partial H$. It suffices to show that $F(x)\in\partial H$. If not, $F(x)-x$ and therefore $F^{\mu}(x)-x$ for any thick μ would be transversal to ∂H at x. This is not possible because $\tilde{x}(\cdot)$ is a differentiable trajectory confined to B_a and cannot make 'sharp turns' around corners. This completes the proof. □

Consider a fixed $(d-1)$-face H of B_a for the time being. Let $H=\{x \mid x_1=x_1^*+a_1, |x_j-x_j^*|\leqslant a, j>1\}$ for simplicity.

Lemma 2.6. *If $G_H\neq\phi$, the map $F:G_H\to H$ can be extended to an ∞-nonexpansive map $\tilde{F}:H\to H$ which has a fixed point $\tilde{x}\in H$.*

Proof. The second claim follows from the first by the Brouwer fixed point theorem. Fix $1<i\leqslant d$ and define

$$g_i(x)=\inf_{y\in G_H}(F_i(y)+\|x-y\|_\infty),\ x\in H.$$

Then $g_i(x)\leqslant F_i(x)$, $x\in G_H$. For $x,y\in G_H$, ∞-nonexpansivity of F leads to

$$F_i(y)+\|x-y\|_\infty\geqslant F_i(x).$$

Thus $g_i(x) \geqslant F_i(x)$, implying $g_i = F_i$ on G_H. Now for $x, z \in H$,

$$g_i(x) \leqslant \inf_{y \in G_H} (F_i(y) + \|y - z\|_\infty + \|z - x\|_\infty)$$

$$\leqslant g_i(z) + \|z - x\|_\infty. \tag{8}$$

Similarly, $g_i(z) \leqslant g_i(x) + \|z - x\|_\infty$. Hence

$$|g_i(x) - g_i(z)| \leqslant \|z - x\|_\infty.$$

Let $\tilde{F}_i(x) = \max(x_i^* - a, \min(g_i(x), x_i^* + a))$. Then

$$|\tilde{F}_i(x) - \tilde{F}_i(z)| \leqslant \|x - z\|_\infty.$$

Let $\tilde{F}_1(x) = x_1^* + a$, $x \in H$. Then $\tilde{F}(\cdot) = [\tilde{F}_1(\cdot), \ldots, \tilde{F}_d(\cdot)]$ is the desired map. □

Proof of Theorem 2.1. The same argument as in Lemma 2.6 can be used to extend $\tilde{F}$ to an ∞-nonexpansive map $\bar{F}: R^d \to R^d$ that restricts to $\tilde{F}$ on H and to F on $\cup_A G_A$. Define $\bar{F}^A$, $\bar{F}^\mu$ in analogy with F^A, F^μ using $\bar{F}$ in place of F. Then $\tilde{x}(\cdot)$ satisfies

$$\dot{\tilde{x}}(t) = \bar{F}^{\bar{\mu}_t}(\tilde{x}(t)) - \tilde{x}(t).$$

We conclude that $t \to \|\tilde{x}(t) - \tilde{x}\|_\infty$ is nonincreasing in t and hence decreases to $b \geqslant 0$. If $b = 0$, we are done. If not, let $B_b = \{x \mid \|x - \tilde{x}\|_\infty = b\}$. Then $\tilde{x}(t) \to B_b$. Also, it is clear that no $(d-1)$-face H of B_b is coplanar with H. This argument can now be repeated for each $(d-1)$-face H of B_a that intersects $\{\tilde{x}(\cdot)\}$, leading to possibly more $\|.\|_\infty$-spheres $B_c, B_d, \ldots$ defined analogously to B_a such that $\tilde{x}(t) \to B_a \cap B_b \cap B_c \cap \ldots$. The above remarks also imply that this intersection is a union of m-faces with m at most $(d-2)$. Now consider a limit point $x'(\cdot)$ of $\tilde{x}(t + .)$, $t \geqslant 0$, in $C([0, \infty); R^d)$ as $t \to \infty$. Repeat the above argument to conclude that $x'(t)$ converges to a union of m-faces with m at most $(d-3)$. Iterating this argument at most d times, we are left with a union of finitely many points to one of which $\tilde{x}(\cdot)$, $x'(\cdot) \ldots$ and hence $x(\cdot)$ must converge and which then must be a fixed point of F^μ for some thick μ, hence of F. This completes the proof. □

COROLLARY 2.1

Given $\varepsilon, b > 0$, there exist $T = T(\varepsilon, b) > 0$, $\eta = \eta(\varepsilon, b) > 0$ such that for any solution $x(\cdot)$ of (5) *satisfying $\|X(0)\|_\infty \leqslant K < \infty$ and*

(i) *$\bar{\mu}$ is b-thick,*
(ii) *$\{x(t) \mid t \in [0, T]\} \cap G^\varepsilon = \phi$ where $G^\varepsilon = \{x \mid \inf_{y \in G} \|x - y\| < \varepsilon\}$, we have*

$$\inf_{y \in G} \|x(t) - y\|_\infty \leqslant \inf_{y \in G} \|x(0) - y\|_\infty - \eta \quad \text{for} \quad t \geqslant T.$$

Proof. Suppose that the claim is false. Then there exist $x^n \in R^d$, $\bar{\mu}^n \in U$ such that for some $b > 0$, $\{\bar{\mu}^n\}$ are b-thick and $x^n(\cdot)$ satisfy (i) $\dot{x}^n(t) = F^{\mu_t^n}(x^n(t)) - x^n(t)$, $x^n(0) = x^n$, $\|x^n\|_\infty \leqslant K$, (ii) $x^n(t) \notin G^\varepsilon$ for $t \in [0, n]$ and

$$\inf_{y \in G} \|x^n - y\|_\infty \geqslant \sup_{t \in [0,n]} \inf_{y \in G} \|x^n(t) - y\|_\infty \geqslant \inf_{y \in G} \|x^n - y\|_\infty - 1/n,$$

for $n \geqslant 1$. By dropping to a subsequence if necessary, we may then suppose that

$x^n \to x^\infty \notin G^\varepsilon$, $\bar{\mu}^n \to \bar{\mu}^\infty$ in U and $x^n(\cdot) \to x^\infty(\cdot)$ in $C([0,\infty); R^d)$. By Lemmas 2.2 and 2.3, $\bar{\mu}^\infty$ is b-thick and $x^\infty(\cdot)$ satisfies.

$$\dot{x}^\infty(t) = F^{\mu_t^\infty}(x^\infty(t)) - x^\infty(t),\ x^\infty(0) = x^\infty.$$

Also, $\inf_{y \in G} \|x^\infty - y\|_\infty = \inf_{y \in G} \|x^\infty(t) - y\|_\infty$, $t \geq 0$. This contradicts Theorem 2.1, establishing the claim. □

For $\alpha, \eta, T > 0$, call a trajectory $y(\cdot): R^+ \to R^d$ an (α, η, T)-perturbation of (5) if there exist $0 = T_0 < T_1 < T_2 < \cdots$ in $[0, \infty)$ with $T_{j+1} - T_j \geq T$, such that for some $x^j(\cdot)$ satisfying (5) for some α-thick $\bar{\mu}^j$ in place of $\bar{\mu}$, we have

$$\sup_{t \in [T_j, T_{j+1}]} \|y(t) - x^j(t)\|_\infty < \eta, j \geq 0.$$

COROLLARY 2.2

For any $\alpha, \varepsilon > 0$, $T > 0$ as in Corollary 2.1 and $\gamma > 0$ sufficiently small, any (α, γ, T)-perturbation $y(\cdot)$ of (5) converges to G^ε.

Proof. In view of Corollary 2.1, this is a straightforward adaptation of Theorem 1, p. 339 of [5]. □

3. Proof of Theorem 1.1

As a first step towards establishing that $\{X(n)\}$ tracks (5) asymptotically, we analyze the S-valued process $\{Y_n\}$. In particular, we shall show that it may be viewed as a controlled Markov chain.

For $A \in S$, let $D_A = \{B \in S|$ (4) exceeds δ, a.s.$\}$ and $V_A = \{u \in \mathscr{P}(D_A) | u(B) \geq \delta \,\forall B \in D_A\}$. Let $V = \Pi_A V_A$. Define $p: S \times S \times V \to [0, 1]$ by

$$p(A, B, \mu) = \mu_A(B),$$

where μ_A is the Ath component of μ. Define V-valued random variables $\{Z^n\}$ as follows. The Ath component of Z^n, denoted by Z^n_A is given by

$$Z^n_A(B) = P(Y_{n+1} = B/\mathscr{G}_n) I\{Y_n = A) + \Psi_A I\{Y_n \neq A\},$$

where Ψ_A are fixed elements of V_A, $A \in S$. Then (4) equals $p(A, B, Z^n)$ and $\{Y_n)$ may be viewed as an S-valued controlled Markov chain with action space V and transition probability function p. It should be kept in mind, however, that this is purely a technical convenience and it is in no way implied that $\{Z^n\}$ is actually a control process. In particular, this allows us to conceive of a 'stationary control policy π' associated with a map $\pi: S \to V$ wherein $Z^n = \pi(Y_n)$, $n \geq 0$. The latter part of (A1) implies that $\{Y_n\}$ will be an ergodic Markov chain under any stationary policy π with a corresponding unique stationary distribution $v_\pi \in \mathscr{P}(S)$.

Let $t_0 = 0$, $t_n = \Sigma^n_{m=1} a(m)$, $n \geq 1$. Define $y(\cdot): [0, \infty) \to S$ by

$$y(t) = Y_n, t_n \leq t < t_{n+1}, n \geq 0.$$

Define $\bar{\mu} \in U$ by $\mu_t(A) = I\{y(t) = A\}$, $t \geq 0$ and $\bar{\mu}^s \in U$, $s \geq 0$, by $\mu^s_t = \mu_{s+t}$, $t \geq 0$.

Lemma 3.1. *There exists an $\alpha > 0$ such that any limit point of $\{\bar{\mu}^s\}$ as $s \to \infty$ is α-thick, a.s.*

Proof. Let $A \in S$. Then

$$M_n = \sum_{m=1}^{n} a(m)[I\{Y_m = A\} - \sum_B p(B, A, Z^{m-1}) I\{Y_{m-1} = B\}]$$

is a zero mean bounded increment martingale with respect to $\{\mathscr{G}_n\}$, whose quadratic variation process converges a.s. in view of (2). By Proposition VII-2-3(c), pp. 149–150, [6], $\{M_n\}$ converges a.s. For $n \geqslant 0$, let $\bar{n}(s) = \min\{m > n | \sum_{j=n+1}^{m} a(j) \geqslant s\}$, $s > 0$. Then $\lim_{n\to\infty}(M_{\bar{n}(s)} - M_n) = 0$ a.s. and $\sum_{m=n}^{\bar{n}(s)} a(m) \geqslant s$ together imply

$$\frac{\sum_{m=n}^{\bar{n}(s)} a(m) I\{Y_m = A\}}{\sum_{m=n}^{\bar{n}(s)} a(m)} - \frac{\sum_{m=n}^{\bar{n}(s)} a(m) \sum_B p(B, A, Z^{m-1}) I\{Y_{m-1} = B\}}{\sum_{m=n}^{\bar{n}(s)} a(m)} \to 0 \qquad (9)$$

a.s. Define $\Phi_{n,s} \in \mathscr{P}(S \times V)$ by

$$\Phi_{n,s}(C \times J) = \frac{\sum_{m=n}^{\bar{n}(s)} a(m) I\{Y_m \in C, Z^m \in J\}}{\sum_{m=n}^{\bar{n}(s)} a(m)}$$

for $C \subset S$, $J \subset V$ Borel. Then from (9), it follows that a.s., any limit point Φ of $\Phi_{n,s}$ in $\mathscr{P}(S \times V)$ as $n \to \infty$ must satisfy

$$\Phi(\{A\} \times V) = \int p(.,A,.) \mathrm{d}\Phi, \ A \in S.$$

Thus Φ must be of the form

$$\Phi(\{A\} \times J) = v(A) \varphi_A(J), \ A \in S, J \subset V \text{ Borel},$$

where $A \to \varphi_A : S \to \mathscr{P}(V)$ defines a stationary randomized policy and v the corresponding stationary distribution (see e.g. [2], pp. 55–56). By Lemma 1.2, p. 56 and Lemma 2.1, p. 60 of [2], it follows that

$$\min_A v(A) \geqslant \min_{A,\pi} v_\pi(A) \triangleq \alpha > 0.$$

Thus

$$\liminf_{n\to\infty} \frac{\sum_{m=n}^{\bar{n}(s)} a(m) I\{Y_m = A\}}{\sum_{m=n}^{\bar{n}(s)} a(m)} \geqslant \alpha \text{ a.s.}$$

From our definition of $\{\bar{\mu}^t\}$, it then follows that

$$\liminf_{t\to\infty} \frac{1}{s} \int_0^s \mathrm{d}y \mu_y^t(A) \geqslant \alpha \text{ a.s.} \forall A \in S.$$

Fix a sample point outside the zero probability set on which this claim fails for any $A \in S$, $s > 0$ rational. For this sample point, the claim follows easily in view of our topology on U. □

Now rewrite (3) as

$$X(n+1) = X(n) + a(n)(W(n) - X(n))$$

for appropriately defined $W(n) = [W_1(n), \ldots, W_d(n)]$ and set $\hat{W}(n) = E[W(n)/\mathscr{F}_n]$, $n \geqslant 0$, the conditioning being componentwise. Write $\hat{W}(n) = [\hat{W}_1(n), \ldots, \hat{W}_d(n)]$.

Lemma 3.2. *There exist $K > 0$, $N \geqslant 1$ such that for $n \geqslant N$,*

$$\|F^{Y_n}(X(n)) - \hat{W}(n)\|_\infty < Ka(n)^r.$$

Proof. W.l.o.g., let $X(0)$ be deterministic. Then $\forall A \subset I$, $z \in G$,

$$\begin{aligned}\|F^A(X(n))\|_\infty &\leqslant \|X(n) - z\|_\infty + \|F(z)\|_\infty V \|z\|_\infty \\ &\leqslant 3\|z\|_\infty + \|F(z)\|_\infty + \|X(O)\|_\infty \triangleq M < \infty.\end{aligned}$$

For each i, $1 \leqslant i \leqslant d$, and $c = 1 - r$,

$$\begin{aligned}|F^{Y_n}(X(n)) - \hat{W}_i(n)| &\leqslant E[|F_i^{Y_n}(X(n)) - W_i(n)| I\{\tau_{ij}(n) \leqslant a(n)^{-c} \text{ for all } i,j\}/\mathscr{F}_n] \\ &\quad + E[|F_i^{Y_n}((X(n)) - W_i(n)| I\{\tau_{ij}(n) > a(n)^{-c} \text{ for some } i,j\}/\mathscr{F}_n].\end{aligned}$$

The second term is bounded by $2MCa(n)^{bc}$ in view of (A2) and the conditional Chebyshev inequality. Let $\bar{n}$ be the integer part of $a(n)^{-c}$. Since $a(n)^{-c}$ is $o(n)$, we may pick n large enough so that $n > \bar{n}$. Then for $m \leqslant \bar{n}$, (A3) leads to

$$\|X(n) - X(n-m)\|_\infty \leqslant 2M \sum_{k=n-\bar{n}}^{n} a(k) \leqslant \bar{K}a(n)^{1-c}$$

for a suitable $\bar{K} > 0$. Thus the first term is bounded by $\bar{K}a(n)^r$. Since $b \geqslant r/(1-r)$, the claim follows. □

Let $T > 0$. Define $T_0 = 0, T_n = \min\{t_m | t_m \geqslant T_{n-1} + T\}$, $n \geqslant 1$. Then $T_n = t_{m(n)}$ for a strictly increasing sequence $\{m(n)\}$. Let $I_n = [T_n, T_{n+1}]$, $n \geqslant 0$. Define $\bar{x}^n(t)$, $t \in I_n$, by $\bar{x}^n(T_n) = X(m(n))$ and

$$\begin{aligned}\bar{x}^n(t_{m(n)+k+1}) &= \bar{x}^n(t_{m(n)+k}) + F^{Y_{m(n)+k}}(\bar{x}^n(t_{m(n)+k})) \\ &\quad - \bar{x}^n(t_{m(n)+k}))(t_{m(n)+k+1} - t_{m(n)+k}),\end{aligned}$$

with linear interpolation on each interval $[t_{m(n)+k}, t_{m(n)+k+1}]$. Define $x(t)$, $t \geqslant 0$, by $x(t_n) = X(n)$ with linear interpolation on each interval $[t_n, t_{n+1}]$.

Lemma 3.3.

$$\limsup_{n\to\infty} \sup_{t \in I_n} \|x(t) - \bar{x}^n(t)\|_\infty = 0 \text{ a.s.}$$

Proof. Let $n \geqslant 1$. For $i \geqslant m(n)$, we have

$$\begin{aligned}x(t_{i+1}) &= x(t_i) + a(i)(F^{Y_i}(x(t_i)) - x(t_i)) + a(i)(\hat{W}(i) - F^{Y_i}(x(t_i))) \\ &\quad + a(i)(W(i) - \hat{W}(i)).\end{aligned}$$

Let $\bar{M}_m = \Sigma_{i=0}^m a(i)(W(i) - \hat{W}(i))$ and $\xi_i = \bar{M}_i - \bar{M}_{m(n)}$ for $i \geqslant m(n)$. Then $\{\bar{M}_m, \mathscr{F}_m\}$ is a zero mean bounded increment vector martingale and the quadratic variation process of each of its component martingales converges a.s. by virtue of (2). Thus by Proposition VII-2-3(c), pp. 149–150 of [6], $\{\bar{M}_m\}$ converges a.s. Fix a sample point for which this convergence holds and let $\delta > 0$. Then

$$\sup_{i \geqslant m(n)} \|\xi_i\|_\infty < \delta/2$$

for n sufficiently large.

Let $\hat{x}_{i+1} = x(t_{i+1}) - \xi_i, i \geqslant m(n)$ with $\hat{x}_{m(n)} = X_{m(n)}$ (i.e., $\xi_{m(n)-1} \triangleq 0$). Then for $i \geqslant m(n)$,

$$\hat{x}_{i+1} = \hat{x}_i + a(i)(F^{Y_i}(\hat{x}_i) - \hat{x}_i) + a(i)(F^{Y_i}(\hat{x}_i + \xi_{i-1}) - (\hat{x}_i + \xi_{i-1}) - F^{Y_i}(\hat{x}_i) + \hat{x}_i) + a(i)(\hat{W}(i) - F^{Y_i}(x(t_i)).$$

Also

$$\bar{x}^n(t_{i+1}) = \bar{x}^n(t_i) + a(i)(F^{Y_i}(\bar{x}^n(t_i)) - \bar{x}^n(t_i)).$$

Subtracting and using the preceding lemma, we have, for n sufficiently large,

$$\|\hat{x}_{i+1} - \bar{x}^n(t_{i+1})\|_\infty \leqslant (1 + 2a(i))\|\hat{x}_i - \bar{x}^n(t_i)\|_\infty + 2a(i)\|\xi_{i-1}\|_\infty + Ka(i)^{1+r}.$$

By increasing n if necessary, we may suppose that

$$\sum_{i \geqslant n} a(i)^{1+r} < \delta/2.$$

Then using the inequality $1 + 2a(i) \leqslant \exp(2a(i))$ and iterating, we have for n sufficiently large,

$$\sup_{m(n) \leqslant i \leqslant m(n+1)} \|\hat{x}_i - \bar{x}^n(t_i)\|_\infty \leqslant 2e^{2(T+1)}(K + T + 1)\delta.$$

Also

$$\sup_{m(n) \leqslant i \leqslant m(n+1)} \|\hat{x}_i - x(t_i)\|_\infty \leqslant \delta/2,$$

for sufficiently large n. Since $\delta > 0$ was arbitrary, the claim follows on observing that $x(\cdot), \bar{x}^n(\cdot)$ are linearly interpolated from their values at $\{t_i\}$. □

Next define $\tilde{x}^n(t), t \in I_n$, by $\tilde{x}^n(t_{m(n)}) = x(t_{m(n)})$ and

$$\dot{\tilde{x}}^n(t) = F^{y(t)}(\tilde{x}^n(t)) - \tilde{x}^n(t), t \in I_n.$$

Lemma 3.4.

$$\lim_{n \to \infty} \sup_{t \in I_n} \|\bar{x}^n(t) - \tilde{x}^n(t)\|_\infty = 0.$$

Proof. This follows easily from the Gronwall lemma. □

Let $\alpha > 0$ be as in Lemma 3.1.

Lemma 3.5. *Almost surely, the following holds. There exists an α-thick sequence $\bar{\mu}^n \in U, n \geqslant 0$, such that if $\hat{x}^n(t), t \in I_n$, is defined by $\hat{x}^n(t_{m(n)}) = x(t_{m(n)})$ and*

$$\dot{\hat{x}}^n(t) = F^{\bar{\mu}^n_t}(\hat{x}^n(t)) - \hat{x}(t), t \in I_n,$$

for $n \geqslant 1$, then

$$\lim_{n \to \infty} \sup_{t \in I_n} \|\hat{x}^n(t) - \tilde{x}^n(t)\|_\infty = 0.$$

Proof. This is immediate from Lemmas 2.3 and 3.1. □

Proof of Theorem 1.1. Let $\varepsilon > 0$. Let $b = \alpha$ above in Corollary 2.1 and pick $T = T(\varepsilon, \alpha)$ accordingly. Pick $\gamma > 0$ as in Corollary 2.2. Combining Lemmas 3.3–3.5, we have

$$\lim_{n\to\infty} \sup_{t\in I_n} \|\hat{x}^n(t) - x(t)\|_\infty = 0 \text{ a.s.}$$

Thus $x(t_n + \cdot)$ is a (α, T, γ)-perturbation of (5) for sufficiently large n. By Corollary 2.2, it follows that $x(t) \to G^\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the claim follows. □

Observe that the foregoing can be easily extended to the following relaxation of the latter half of (A2). The directed graph formed therein need not be irreducible, but each communicating class in it must correspond to elements of S which together cover I. Also, extension to nonexpansive F with respect to the weighted ∞-norm is straighforward.

4. Examples

This section sketches some important instances of fixed point problems for ∞-nonexpansive maps. A general reference for these is [1].

(i) *Shortest path problems*: Given $d+1$ locations $\{0, 1, \ldots, d\}$ and the distances $\{d_{ij}, 0 \leqslant i,j \leqslant d, i \neq j\}$ between them, the problem is to find the shortest path from location $i \neq 0$ to location 0. Letting $V(i) =$ the length of the shortest path from i to 0, one has the dynamic programming equations

$$V(i) = \min\left(d_{i0}, \min_{j\neq i,0} (d_{ij} + V(j))\right), \quad 1 \leqslant i \leqslant d.$$

Letting $V = [V(1), \ldots, V(d)]^T$, this has the form $V = F(V)$ for an ∞-nonexpansive F.

(ii) *Markov decision processes*: Consider a controlled Markov chain $\{X_n\}$ on a finite state space S, with a compact metric action space A and a continuous transition probability function $p: S \times S \times A \to [0,1]$. The aim is to choose an A-valued sequence $\{Z_n\}$ that does not anticipate future, to minimize a suitable total expected cost. Thus

$$P(X_{n+1} = j / X_i, Z_i, i \leqslant n) = p(X_n, j, Z_n) \,\forall n.$$

Let B be a proper subset of S and $a \in (0,1)$. Consider two cost functionals: For $k \in C(S \times A)$,

(1) cost up to a first passage time:

$$E\left[\sum_{n=0}^{\tau-1} k(X_n, Z_n)\right],$$

where $\tau = \min\{n \geqslant 0 | X_n \in B\}$,

(2) infinite horizon discounted cost:

$$E\left[\sum_{n=0}^{\infty} a^n k(X_n, Z_n)\right].$$

Letting $V(i)$ denote the minimum cost when $X_0 = i$, the dynamic programming equations in the two cases are, respectively,

$$V(i) = \min_u \left(k(i,u) + \sum_{j\notin B} p(i,j,u)\, V(j)\right), \quad i \in S \backslash B,$$

and

$$V(i) = \min_u \left(k(i,u) + a \sum_j p(i,j,u)\, V(j) \right), \quad i \in S.$$

Both can be cast as fixed point equations $V = F(V)$ for ∞-nonexpansive V.
(iii) *Systems of linear equations*: The problem of solving a system of linear equations $Ax = b$ can be cast as finding the fixed point of $F(x) = x - a(Ax - b), a \in (0,1)$. If $\|I - A\|_\infty \leqslant 1$, F is ∞-nonexpansive.
(iv) *Strictly convex network flow problems*: The following problem arises in network flow optimization:

$$\text{minimize} \sum_{(i,j)\in A} a_{ij}(f_{ij})$$

subject to

$$\sum_{\{j|(i,j)\in A\}} f_{ij} - \sum_{\{j|(j,i)\in A\}} f_{ji} = S_i \, \forall i \in N, b_{ij} \leqslant f_{ij} \leqslant c_{ij} \forall (i,j) \in A,$$

where $a_{ij}(\cdot)$ are strictly convex. This problem can be cast as that of finding a fixed point of a pseudononexpansive map, i.e., a map F satisfying $\|F(x) - y\|_\infty \leqslant \|x - y\|_\infty$ whenever y is a fixed point of F. (See (1), § 7.2 for details). Our analysis applies here as well.

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Moduli for principal bundles over algebraic curves: I

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Abstract. We classify principal bundles on a compact Riemann surface. A moduli space for semistable principal bundles with a reductive structure group is constructed using Mumford's geometric invariant theory.

Keywords. Principal bundles; compact Riemann surface; geometric invariant theory; reductive algebraic groups.

1. Introduction

Let X be a projective nonsingular irreducible curve over $\mathbb{C}$ (or equivalently a connected compact Rieman surface) of genus $\geqslant 2$ and G a connected reductive algebraic group over $\mathbb{C}$. Our problem is to classify algebraic principal G-bundles on X. When $G = GL(r, \mathbb{C})$, i.e. for vector bundles, this has been done by Mumford, Narasimhan and Seshadri ([13], [15], [19]).

In [14] we have defined the notion of stable and semistable G-bundles on X (Definition 2.12) and have proved that a G-bundle is stable if and only if it is associated to certain representations of $\pi_1(X - x_0)$ (cf. Definition 3.14), and have constructed by local analytic methods, a moduli space, which is a normal complex space, for stable G-bundles on X ([14], Theorems 7.7 and 4.3). In this thesis we use global algebraic methods depending on Mumford's theory of stable and semistable points for actions of reductive groups on algebraic schemes to construct a moduli space, which is a normal projective algebraic variety, for semistable principal G-bundles under a suitable equivalence; see Definitions 3.1, 3.9 and Theorem 5.9 (in part II: Editor).

In § 2 we explain some notations and recall some preliminary results. In § 3 we prove a kind of Jordan–Hölder theorem for semistable G-bundles.

Professor Annamalai Ramanathan, who was a Fellow of the Academy and a co-editor of the Proceedings, passed away on 12 March 1993, at the young age of 46. For some reason his doctoral thesis (written in 1976) was never published. A manuscript had in fact been prepared for publication, but apparently he wanted to revise it, which unfortunately was not to be.

The results in the thesis have been found very useful by researchers in the area, especially more recently in view of the remarkable connection between Conformal Field Theory and moduli spaces of principal bundles on curves. It was suggested by his teachers Professors M S Narasimhan and S Ramanan that the thesis be published, in the available form, and the idea also found enthusiastic support among other mathematicians, as this would provide a much needed reference article for the material.

Professor Ramanathan, who was on the Faculty of the Tata Institute of Fundamental Research, Bombay, was an accomplished mathematician, a recipient of the Shanti Swarup Bhatnagar prize, and a fine person very helpful to students and colleagues. As a tribute to his memory we are publishing the thesis in the Proceedings, convinced that he would have appreciated it if he were to be with us.

For convenience the publication will be in two parts, the second being scheduled for the next issue.

– Editor

Since the unipotent radical U of a parabolic subgroup $P = M \cdot U$, M a maximal reductive subgroup of P, can be shrunk to identity (Lemma 3.5.12), it follows that given a P-bundle we can make it 'jump' to the M-bundle obtained from it by the extension of structure group $P \to M$ (Propositions 3.5 (i) and 3.24 (ii)). For constructing moduli space, this then makes it necessary to identify a G-bundle E and a G-bundle E' obtained from E through a reduction of structure group to a parabolic subgroup $P = M \cdot U$ and followed by the extensions of structure group $P \to M$ and $M \subsetneq G$. If E is semistable and the reduction to P is admissible (Definition 3.3), we prove that the G-bundle E' obtained by this process is again semistable (Lemma 3.5.11) and define E and E' to be equivalent (Definition 3.6).

We then prove that given a semistable G-bundle E, there is an admissible reduction of structure group to a suitable parabolic subgroup $P = M \cdot U$ such that the M-bundle obtained by the extension of structure group $P \to M$ is a stable M-bundle (which is a Jordan–Holder series for E). Further the G-bundle obtained from this stable M-bundle by the extension of structure group $M \subsetneq G$ depends only on E and is denoted by $\mathrm{gr}E$ (Proposition 3.12). Also E_1 is equivalent to E_2 if and only if $\mathrm{gr}E_1 \approx \mathrm{gr}E_2$ (Proposition 3.12 (iii)).

Next we prove a result (Proposition 3.24) which shows that the equivalence on semistable G-bundles introduced above is the right one in the sense that a semistable G-bundle can only tend to an equivalent bundle in the limit. Lemmas 3.21 and 3.23 provide the essential tools for proving this proposition.

The method of proof generally is to reduce the problem to a proper reductive subgroup M of G of maximal rank (a Levi component of a parabolic subgroup P) and to use induction on the semisimple rank of the structure group. Lemma 2.11 which says that given two reductions of structure group σ_1, σ_2 of a semistable G-bundle to the parabolic subgroups P_1, P_2 respectively (with P_1, P_2 in general position (cf. Remark 3.5.6)), we can get a common reduction $\sigma_1 \cap \sigma_2$ to the subgroup $P_1 \cap P_2$ which gives both σ_1 and σ_2 under the extension $P_1 \cap P_2 \subsetneq P_i$, $i = 1, 2$, is quite useful in this context.

In §4 we first show that there is a family of G-bundles $\xi \to T \times X$ with a group H acting on T and on ξ as a group of G-bundle isomorphisms compatible with its action on T with the properties that any other family of G-bundles is locally induced from ξ and for any two morphisms $t_1, t_2 : S \to T$ from any scheme S, the induced G-bundle $(t_i \times id_X)^* \xi$ on $S \times X$ are isomorphic if and only if there is a morphism $h : S \to H$ such that $t_1 = h[t_2]$ (where $h[t_2]$ is the composite $S \xrightarrow{h \times t_2} H \times T \xrightarrow{\alpha} T$, α being the action of H on T). Hence a good quotient of T modulo H (Definition 4.1) would give a moduli scheme for G-bundles. (Actually it is sufficient if such a G-bundle ξ existed locally with respect to the faithfully flat topology on T; see Definition 4.4 and Proposition 4.5.) Let us call the family of G-bundles $\xi \to T \times X$ with the above properties, a universal family of G-bundles (cf. Definition 4.6). Seshadri has constructed a universal family of semistable vector bundles (say, of rank r) $\mathscr{V} \to R \times X$, R being a subscheme of the Quot scheme so that we have a surjective homomorphism $I_n \to \mathscr{V} \to 0$, where I_n is the trivial vector bundle of rank n, with the group $GL(n, \mathbb{C})$ acting naturally on $\mathscr{V}$ and R ([19], § 6; [20]; Prop. 4.11.5). To construct a universal family for G-bundles we take an embedding of G in a $GL(k, \mathbb{C})$ and look upon a G-bundle as a vector bundle (of rank k) with a G-structure, i.e. a $GL(k, \mathbb{C})$-bundle $E \to X$ with a reduction of structure group to G given by a section $X \to E/G$ of $E/G \to X$ so that the space S of sections of $\mathscr{V}_r \to r \times X = X$, as r varies through R, gives a family of

G-bundles $\xi \to S \times X$ (§4·8; Lemma 4.8.1). The group $GL(n, \mathbb{C})$ acts naturally on ξ and S and it can be shown that ξ is a universal family (Lemma 4.10). That the space S can be constructed as a scheme with suitable universal properties follows from the existence theorems on Hilbert schemes (Lemma 4.8.1). Then we have to prove the existence of a good quotient of S modulo $GL(n, \mathbb{C})$. For this it is convenient to take the adjoint representation $\mathrm{Ad}: G \to GL(\mathscr{G})$, $\mathscr{G}$ the Lie algebra of G. However, Ad is not injective in general and hence we construct first, as outlined above, a universal family $\xi' \to R' \times X$ for Ad G-bundles and then from ξ' a universal family for G-bundles. (To be more precise the construction is in three steps: from vector bundles to Aut G-bundles to Ad G-bundles to G-bundles.) To get from Ad G-bundles to G-bundles, the idea is to look upon a G-bundle E as an Ad G-bundle E' together with certain line bundles of suitable types on the associated bundle $E'/\bar{B}(=E/B)$ where B is a Borel subgroup of G and $\bar{B}$ its image in Ad G. This is analogous to the fact that a vector bundle V is determined by the projective bundle $\mathbb{P}(V)$ and the tautological line bundle on $\mathbb{P}(V)$ corresponding to V (Lemma 4.15.1). This involves the existence of the space S of line bundles on the fibres of the composite $\xi'/\bar{B} \to R' \times X \to R'$, and for this we make use of the existence theorems on Picard schemes ([TDTE, V]). The Picard functors in general are representable only after 'sheafification' with respect to the faithfully flat topology ([TDTE, V], §1; [1]) and hence a universal family of line bundles will exist only locally in the faithfully flat topology. This means that we will be able to construct universal families for G-bundles only locally in the faithfully flat topology by this method. However, this is sufficient for our purposes (cf. Definition 4.4 and Proposition 4.5).

In §5 we complete the proof of the existence of a coarse moduli scheme for semistable G-bundles by showing that a good quotient of S modulo $GL(n, \mathbb{C})$ exists. It follows from Lemma 5.1, that it is enough to show the existence of a good quotient of R' modulo $GL(n, \mathbb{C})$. For this we adopt the method of Mumford and Seshadri in the case of vector bundles where they reduce it to a problem on a product of Grassmannians as follows ([15], §5; [19], §§6, 7). The surjection $I_n \to \mathscr{V} \to 0$ makes the fibres of $\mathscr{V}$ points of the Grassmannian $G_{n,r} = Z$ of r-dimensional quotients of I_n and by evaluating at points $x_1, \ldots, x_N \in X$ we get a morphism $R \to Z^N$. Seshadri has proved that for a suitable choice of $x_1, \ldots, x_N$, $N \gg 0$, R maps into the set Z^N_{ss} of semistable points of Z^N for the natural action of $SL(n, \mathbb{C})$ on Z^N, and that the morphism $R \to Z^N_{ss}$ is a proper injection ([20], §3, Lemma 2). Since Z^N_{ss} has a good quotient modulo $GL(n, \mathbb{C})$ ([10], Theorem 1.10, p. 38; [22], Theorem 1.1(B)), it follows that R has a good quotient modulo $GL(n, \mathbb{C})$. In the case of G-bundles for any point $(r', x) \in R' \times X$ we not only have a fibre of $\mathscr{V}$ over (r, x), where $r \in R$ is the image of $r' \in R'$, which gives a point of $G_{n,r}$, but also a 'G-structure' on this quotient. Since we have taken the adjoint representation, this 'G-structure' is actually a Lie algebra structure. Let $Y = GL(\mathscr{G})/\mathrm{Ad}\, G$ and $Q \to G_{n,r}$ be the universal quotient bundle on the Grassmannian $G_{n,r}$. We then have, for any point $x \in X$, a morphism $R' \to Q(Y)$ where $Q(Y)$ is the associated bundle of Q, which is considered as a $GL(\mathscr{G})$-bundle, with fibre Y. We prove that for suitable choice of $x_1, \ldots, x_N \in X$ under the natural morphism $R' \to Q(Y)^N$, the image of R' is contained in $Q(Y)^N_{ss}$ (Lemma 5.5.3), the set of semistable points of $Q(Y)^N$ for the natural action of $SL(n, \mathbb{C})$ on $Q(Y)^N$ (and suitable polarization). This is a consequence of the fact that the tensor in $\mathscr{G}^* \otimes \mathscr{G}^* \otimes \mathscr{G} = \mathrm{Hom}(\mathscr{G} \otimes \mathscr{G}, \mathscr{G})$ corresponding to the Lie algebra structure is semistable for the action of $SL(\mathscr{G})$ (Lemma 5.5.1). We also prove $R' \to Q(Y)^N_{ss}$ is proper (Lemma 5.6). (By Lemma 5.1 it follows that R' has a good quotient modulo $GL(n, \mathbb{C})$ as required.) For this, the crucial fact needed is that if a sequence of isomorphic semisimple Lie algebra

structures on a vector space V (considered as elements of $\mathrm{Hom}(V \otimes V, V)$) tends to a semisimple Lie algebra structure on V, then the limit also gives the same Lie algebra structure on V. In other words, if Y is the $GL(\mathscr{G})$ orbit of $x \in \mathrm{Hom}(V \otimes V, V)$ such that $x\colon V \otimes V \to V$ makes V into a semisimple Lie algebra and if $x_0 \in \bar{Y}$, the closure of Y in $\mathrm{Hom}(V \otimes V, V)$, is such that $x_0\colon V \otimes V \to V$ also makes V into a semisimple Lie algebra, then $x_0 \in Y$. (This rigidity result is a consequence of the vanishing theorem, $H^2(\mathscr{G}, \mathscr{G}) = 0$ (cf. [16], §§ 3, 4).)

We make essential use (cf. Lemmas 5.5.2 and 5.5.3) of the fact that if we take a representation $G \to GL(n, \mathbb{C})$ such that center of G maps into scalars then the associated vector bundle of a semistable G-bundle is a semistable vector bundle (Proposition 3.17). We deduce this as a consequence of the result that stable G-bundles are unitary bundles. This is one of the reasons why we restrict to working over $\mathbb{C}$, and not over fields of arbitrary characteristic.

2. Notation and preliminaries

2.1. By a *scheme* we mean a separated scheme of finite type over the complex numbers $\mathbb{C}$ and by a point of a scheme we mean a closed (or $\mathbb{C}$-valued) point of the scheme. Terms such as open, closed, dense, etc. are used with reference to the Zariski topology.

By an algebraic group we mean an *affine* algebraic group.

2.2. Let A be an algebraic group and Y a scheme. A principal bundle over Y with structure group A (or an A-bundle over Y, for short) is a scheme E on which A operates (from the right) and an A-invariant morphism $\pi\colon E \to Y$ such that for any point $y \in Y$ there is a neighbourhood U and a faithfully flat morphism $f\colon U' \to U$, and an A-equivariant isomorphism $f^*(E) \xrightarrow{\approx} U' \times A$, over U', where A operates on $U' \times A$ by right translations on the second factor ([SGA, I], expose XI, Definition 4.1). Since A is affine and hence a linear group, it follows ([TDTE, I], Proposition 7.1 and the paragraph following it) that under these conditions $\pi\colon E \to Y$ is locally isotrivial i.e. we can take $f\colon U' \to U$ above to be an étale covering.

2.3. Let $\pi\colon E \to Y$ be an A-bundle. Let F be a quasi-projective scheme on which A operates from the left. Let the group A act on the product $E \times F$ by $a(e, f) = (e \cdot a, a^{-1} f)$, $a \in A$, $e \in E$, $f \in F$. From local isotriviality and the fact that any finite set of points of F is contained in an affine subset it follows by descent that there exists an unique scheme $E(F)$ and a morphism $E \times F \to E(F)$ which makes $E \times F$ an A-bundle over $E(F)$ for this action of A on $E \times F$ (cf. [17], § 3.2, Proposition 4, pp. 15–16; [SGA, I], exposé V, § 1). There is a natural morphism $E(F) \to Y$ and we call $E(F)$ the fiber bundle associated to E for the action of A on F.

If B is a subgroup of A we denote by E/B the fiber bundle associated to E for the action of A on A/B by left translations.

Let $\rho\colon A \to A'$ be a homomorphism of algebraic groups and let A act on A' by $a \cdot a' = \rho(a)a'$, $a \in A$, $a' \in A'$. The group A' acts on $E \times A'$ by right translation on the second factor and this action goes down to $E(A')$. This makes $E(A')$ into an A'-bundle. We sometimes denote $E(A')$ by $\rho_*(E)$.

2.4. Suppose A operates on the quasi projective schemes F_1 and F_2 and $j\colon F_1 \to F_2$ is an A-equivariant morphism. Then $id_E \times j\colon E \times F_1 \to E \times F_2$ induces a morphism

$E(j){:}E(F_1)\to E(F_2)$. Using local isotriviality it follows easily that if j is an open (respectively closed, locally closed) immersion then so is $E(j)$ ([EGA, IV], Proposition 2.7.1 (x), (xii)).

2.5. Let B be an algebraic subgroup of A. A pair (E', φ) where E' is a B-bundle and φ is an isomorphism $i_*E'\to E$ of A-bundles is said to give a reduction of structure group of E to B. Reductions of structure group of E to B and the sections of the fiber bundle $E/B\to Y$ are in natural one-to-one correspondence ([17], Proposition 9; cf. also § 4.8 and Lemma 4.8.1 below). $E\to E/B$ is a B-bundle over E/B, and to a section $\sigma\colon Y\to E/B$ we associate the B-bundle $\sigma^*(E)$ in this correspondence. We sometimes denote the B-bundle $\sigma^*(E)$ by $E[B]$. Moreover, if $\rho{:}B\to B'$ is a homomorphism we denote $\rho_*\sigma^*(E)$ by $E[B,B']$ also.

2.6. If $E\to Y$ is a $GL(n,\mathbb{C})$-bundle we often denote by the same letter E the associated vector bundle $E(\mathbb{C}^n)$ and if $V\to Y$ is a vector bundle again we denote by the same letter V the corresponding $GL(n,\mathbb{C})$-bundle (which can be constructed back from V) (cf. [TDTE, I], p. 28).

For a vector bundle $V\to Y$ we denote by $\mathbb{P}(V)$ the associated projective bundle of 1-dimensional sub-spaces of V.

If X is a projective non-singular irreducible algebraic curve and $V\to X$ is a vector bundle of rank n, we mean by the degree of V the degree of the line bundle $\overset{n}{\Lambda}\, V$. We denote by $\mu(V)$ the number $\deg V/\mathrm{rk}\, V$.

2.7. We let X always stand for a projective non-singular irreducible curve of genus $g\geqslant 2$ (over $\mathbb{C}$) and G a connected reductive algebraic group (over $\mathbb{C}$). We denote by $\mathscr{G}$ the Lie algebra of G and by $\mathscr{G}'$ the commutator subalgebra $[\mathscr{G},\mathscr{G}]$ of $\mathscr{G}$. The center of $\mathscr{G}$ is denoted by $\mathscr{z}$ so that $\mathscr{G}=\mathscr{z}\oplus\mathscr{G}'$.

For any group M, we let $Z[M]$ stand for its center and $Z_0[M]$ the connected component of identity of $Z[M]$. Let $Z=Z[G]$ and $Z_0=Z_0[G]$.

2.8. A subgroup P of G is a parabolic subgroup of G if G/P is complete. It is convenient for us to consider G also as a parabolic subgroup. However, by a maximal parabolic subgroup of G, we mean a parabolic subgroup of G which is maximal among proper parabolic subgroups. (For comparison note that in [14] we had reserved the term parabolic to proper parabolic subgroups.)

2.9. For a parabolic subgroup P we generally use the notation $M\cdot U$ for a Levi decomposition of P with U the unipotent radical of P and M a maximal reductive subgroup of P. We call a maximal reductive subgroup of P a Levi component of P.

If P_1 is a parabolic subgroup of M, then $P_1\cdot U$ is a parabolic subgroup of G. For any parabolic subgroup P' of G contained in P, $P'\cap M$ is a parabolic subgroup of M. This gives a bijective correspondence between the set of parabolic subgroups of M and those of G contained in P ([3], Proposition 4.6, p. 86).

2.10. Since G is a linear algebraic group ([17], § 6.3) any analytic G-bundle on the compact Riemann surface X has a unique algebraic G-bundle structure and any analytic morphism between G-bundles is an algebraic morphism. The equivalence of the categories of algebraic and analytic G-bundles on X justifies our use of results from [14] though we work in the algebraic category now.

2.11. Let $k(X)$ be the function field of X. Then it follows from a result of Springer that any connected linear algebraic group defined over $k(X)$ has a Borel subgroup defined over $k(X)$ (cf. [23], Theorem 1.9, p. 57 and the remarks following it.) From this it can be deduced that any G-bundle over the curve X is locally trivial in the Zariski topology.

2.12. We denote by (Sch) the category of algebraic schemes (over $\mathbb{C}$). We use the faithfully flat topology on (Sch) and for any functor $\mathbf{F}$: (Sch) $\to$ (Sets) we mean by the sheaf $\tilde{\mathbf{F}}$ associated to $\mathbf{F}$ the 'sheafification' of the presheaf $\mathbf{F}$ with respect to the faithfully flat topology ([1], Theorem 1.1, Chapter II, p. 24; [TDTE, V], p. 3).

We now recall a few things from [14].

2.13. DEFINITION

A G-bundle $E \to X$ is called *stable* (resp. *semistable*) if for any reduction of structure group $\sigma: X \to E/P$ to any maximal parabolic subgroup P of G we have $\deg \sigma^*(T_{E/P}) > 0$ (resp. $\geqslant 0$), where $T_{E/P}$ is the tangent bundle along fibers of $E/P \to X$ (cf. [14], Definition 1.1).

2.14. DEFINITION

Let P be a proper parabolic subgroup of G. A character $\chi: P \to \mathbb{C}^*$ is called *dominant* if it is given by a positive linear combination of fundamental weights for some choice of a Cartan subalgebra and positive system of roots (cf. [14], p. 131).

A dominant character is trivial on Z_0.

2.15. *Lemma. The G-bundle $E \to X$ is stable* (*resp. semistable*) *if and only if for any reduction $\sigma: X \to E/P$ to any proper parabolic subgroup, not necessarily maximal, we have* $\deg(\chi_* \sigma^* E) < 0$ (*resp.* $\leqslant 0$) *for any nontrivial dominant character χ of P.*

For proof see ([14], Lemma 2.1, pp. 131–132).

3. Equivalence on semistable bundles

We will be interested in studying the following functor.

3.1. DEFINITION

Let $\mathbf{F}_{ss}$:(Sch)$\to$(Sets) be the functor which associates to a scheme S the set of isomorphism classes of G-bundles $\xi \to S \times X$ such that for every point $s \in S$ the restriction $\xi_s \to s \times X = X$ of ξ to $s \times X$ is a semistable G-bundle. For a morphism $f: S' \to S$, $\mathbf{F}_{ss}(f)(\xi)$ is defined to be the pull back $(f \times id_X)^*(\xi)$.

Given a topological G-bundle τ on X we denote by $\mathbf{F}_{ss}^{\tau}$ the sub-functor of $\mathbf{F}_{ss}$ defined by

$$\mathbf{F}_{ss}^{\tau}(S) = \{\xi \in \mathbf{F}_{ss}(S) \mid \xi_s \to X \text{ is topologically isomorphic to } \tau \; \forall \, s \in S\}.$$

We refer to a $\xi \in F_{ss}(S)$ as a family of semistable G-bundles parameterized by S.

3.1.1. *Remark.* We will prove later that for an arbitrary G-bundle $\xi \to S \times X$ the set $\{s \in S \mid \xi_s \text{ is semistable}\}$ is an open subset of S (Proposition 5.8, cf. also [14]; Proposition 4.1, p. 138).

We recall the definition of a coarse moduli scheme (cf. [10], Definition 5.6, p. 99).

3.2. DEFINITION

Let $\mathbf{F}$:(Sch)$\rightarrow$(Sets) be a functor. A scheme M and a morphism of functors φ from F to h_M, the functor represented by M (i.e. $h_M(S) = \mathrm{Hom}(S, M)$), is called a *coarse moduli scheme* for $\mathbf{F}$ if (i) the map $\varphi_{\mathbb{C}}:\mathbf{F}(\mathrm{Spec}\,\mathbb{C}) \rightarrow h_M(\mathrm{Spec}\,\mathbb{C})$ is a bijection; (ii) given any scheme N and any morphism $\psi:\mathbf{F} \rightarrow h_N$ there is an unique morphism $\chi:h_M \rightarrow h_N$ such that $\psi = \chi o \varphi$.

Proposition 3.5 below (see also Proposition 3.24) shows that $\mathbf{F}_{\mathrm{ss}}$ cannot have a separated coarse moduli scheme and suggests an equivalence relation between semistable G-bundles to obtain a coarse moduli scheme.

3.3. DEFINITION

Let $\xi \rightarrow S \times X$ be a G-bundle. A reduction σ of structure group of ξ to a parabolic subgroup P is called *admissible* if for any character χ on P which is trivial on Z_0 and any point $s \in S$ the line bundle $\chi_* \sigma_s^*(\xi_s)$, given by the reduction σ_s of structure group of ξ_s induced by σ and the character χ, has degree zero.

3.4. *Remark.* For a $GL(n, \mathbb{C})$-bundle $E \rightarrow X$ a reduction σ of structure group to the parabolic subgroup P defined by a flag $0 = V_0 \subset \ldots \subset V_r = \mathbb{C}^n$ is equivalent to giving the sub-bundles $(\sigma^* E)(V_i)$, $i = 0, \ldots, r$. The reduction σ is admissible if and only if $\mu(\sigma^* E(V_i/V_{i-1})) = \mu(E(\mathbb{C}^n))$, $i = 1, \ldots, r$.

3.5. PROPOSITION

Let $\xi \rightarrow S \times X = Y$ be a G-bundle. Let σ be a reduction of structure group to a parabolic subgroup $P = M \cdot U$. Then there is a G-bundle $\xi' \rightarrow \mathbb{C} \times Y$ such that

i) $\xi'|_{\mathbb{C}^* \times Y} \approx (\pi_Y^* \xi)|_{\mathbb{C}^* \times Y}$, *where $\pi_Y:\mathbb{C} \times Y \rightarrow Y$ is the projection and $\xi'_0 \rightarrow Y$, the restriction of ξ' to $0 \times Y = Y$ is isomorphic to $j_* p_* \sigma^*(\xi)$ where $p:P \rightarrow M$ is the projection and $j:M \hookrightarrow G$ is the inclusion ($\mathbb{C}^*$ denotes $\mathbb{C} - (0)$).*

ii) *if $\xi \rightarrow S \times X$ is a family of semistable G-bundles and σ is an admissible reduction then $\xi'_0 \rightarrow S \times X$ is also a family of semistable G-bundles.*

Before giving a proof of this proposition we note down several remarks and lemmas.

3.5.1. DEFINITION

Let $\rho:G \rightarrow GL(V)$ be a representation. Let W be a subspace of V such that the stabilizer $\{g \in G | \rho(g)\, W = W\}$ of W in G is a parabolic subgroup. We call any subspace of V of the form $\rho(g)\, W$, for some $g \in G$, a *subspace of type W* (cf. [14], Definition 3.1, p. 135).

Note that the subspaces of type W form the orbit $\mathrm{Gr}(W)$ of W under the natural action of G on the Grassmannian Gr of all subspaces of V of rank = rank W. Since the stabilizer of W is a parabolic subgroup, $\mathrm{Gr}(W)$ is closed in Gr. Suppose $E \rightarrow X$ is a G-bundle. Taking $\mathrm{Gr}(W)$ with the canonical reduced subscheme structure we have a closed immersion $E(\mathrm{Gr}(W)) \hookrightarrow E(\mathrm{Gr})$ (§ 2.4). A section σ of $E(\mathrm{Gr}) \rightarrow X$ gives a sub-bundle of $E(V)$ in a natural way.

3.5.2. DEFINITION

If $\sigma:X \rightarrow E(\mathrm{Gr})$ factors through $E(\mathrm{Gr}(W)) \hookrightarrow E(\mathrm{Gr})$ we call the sub-bundle given by σ a *sub-bundle of type W.*

3.5.3. *Remark.* If P is the stabilizer of W in G then $\mathrm{Gr}(W) = G/P$. Therefore, such a section σ of $E(\mathrm{Gr}(W)) = E/P$ gives a reduction of structure group to P. The sub-bundle corresponding to σ is then $(\sigma E)(W)$. If P is a proper parabolic subgroup and E is stable (resp. semistable) then it follows, as in the proof of Lemma 3.3 of [14], that $\mu((\sigma^* E)(W)) <$ (resp. $\leqslant$) $\mu(E(V))$.

3.5.4. *Remark.* If a sub-bundle of $E(V)$ is generically of type W (i.e. for a nonempty open subset U, $\sigma(U) \subset E(\mathrm{Gr}(W))$) it is actually of type W everywhere (i.e. $\sigma(X) \subset E(\mathrm{Gr}(W))$) since $E(\mathrm{Gr}(W))$ is a closed subvariety of $E(\mathrm{Gr})$.

3.5.5. *Remark.* Let $V \to X$ be a vector bundle. Let W, W' be sub-bundles of V. We often identify a vector bundle with the sheaf of its sections, which is a locally free sheaf. That W is a sub-bundle of V is equivalent to saying that the sheaf W is a subsheaf of V such that the quotient sheaf V/W is torsion free (or equivalently locally free, since X is a curve). We denote by $W \cap W'$ the subsheaf of V which is the kernel of the natural homomorphism $W \to V/W'$ (or $W' \to V/W$). We denote by $\overline{W \cap W'}$ the inverse image of the torsion subsheaf of $V/W \cap W'$ under the projection $V \to V/W \cap W'$. Then $\overline{W \cap W'}$ is a sub-bundle of V. We call it the sub-bundle generated by $W \cap W'$. For $x \in X$, let V_x denote the fiber of V at x. Then $W_x \cap W'_x \subset (\overline{W \cap W'})_x$. There is a non-empty open subset $U \subset X$ such that for $x \in U$, $W_x \cap W'_x = (\overline{W \cap W'})_x$. Moreover $W \cap W' = \overline{W \cap W'}$ if and only if $\dim(W_x \cap W'_x)$ is constant as x varies over X.

3.5.6. *Remark.* Let σ_1, σ_2 be two reductions of structure group of $E \to X$ to the parabolic subgroups P_1, P_2 respectively. Let U be an open subset of X on which E is trivial (§ 1.11). Identifying $p^{-1}(U)$ with $U \times G$ the reductions σ_i give rise to morphisms $\sigma_i: U \to G/P_i$. Let $\varphi: U \to G/P_1 \times G/P_2$ be defined by $\varphi(x) = (\sigma_1(x), \sigma_2(x))$, $x \in U$. Since P_i is its own normalizer G/P_i can be identified with the set of conjugates of P_i in G by associating to the coset gP_i the conjugate gP_ig^{-1}. The group G acts diagonally on $G/P_1 \times G/P_2$ and an orbit of G on $G/P_1 \times G/P_2$ can be thought of as giving a relative position of conjugates of P_1 and P_2. Therefore, as can be seen by using Bruhat's lemma and the configuration of standard parabolic subgroups ([3], § 4) the number of orbits for this action is finite. Let $O_1, \ldots, O_r$ be the orbits of G in $G/P_1 \times G/P_2$. Then O_i are locally closed ([2], p. 98) and hence $\varphi^{-1}(O_i)$ are locally closed in U_i. Since U is 1-dimensional, the locally closed subsets are either open sets or finite set of points. Therefore, there is a unique orbit O_i such that $\varphi^{-1}(O_i)$ is a nonempty open subset of U. If $(P_1, P_2) \in O_i$, then P_1 and P_2 are in the relative position corresponding to the generic relative position determined by σ_1 and σ_2 and we then say that (P_1, P_2) is *compatible* with (σ_1, σ_2). It is easy to see that this notion does not depend on U and the trivialization of E over U.

3.5.7. *Remark.* For $g \in G$ let $R_g: E \to E$ be the action of g on $E: R_g(e) = e \cdot g$ for $e \in E$. Then R_g induces a morphism $E/P \to E/g^{-1}Pg$. Hence any reduction σ of structure group to the subgroup P gives rise, by composition with $E/P \to E/g^{-1}Pg$, to a reduction σ_g of structure group to $g^{-1}Pg$. The $g^{-1}Pg$-bundle $\sigma_g^* E$ is obtained from the P-bundle $\sigma^* E$ by the extension of structure group $P \to g^{-1}Pg$, $p \mapsto g^{-1}pg$, $p \in P$. If $\chi^g: g^{-1}Pg \to \mathbb{C}^*$ is the character defined by $\chi^g(g^{-1}pg) = \chi(p)$ then $\chi_* \sigma^* E \approx \chi_*^g \sigma_g^* E$. Therefore the stability (resp. semistability) condition for the reduction σ is satisfied if and only if it is satisfied for σ_g.

3.5.8. *Lemma. Let $E \to X$ be a G-bundle and σ an admissible reduction of structure group to the proper parabolic subgroup P. Let $\rho: G \to GL(V)$ be a representation such that Z_0 acts by scalars. Let $W \subset V$ be a nonzero subspace left invariant by P. Then $\mu(\sigma^* E(W)) = \mu(E(V))$.*

Proof. Let $\det_V: GL(V) \to \mathbb{C}^*$ and $\det_W: GL(W) \to \mathbb{C}^*$ be the determinant characters. The representation $\rho: G \to GL(V)$ induces $\rho: P \to GL(W)$. Define $\chi_1: G \to \mathbb{C}^*$ to be $\det_V \circ \rho$ and $\chi_2: P \to \mathbb{C}^*$ to be $\det_W \circ \rho$. We then have $\deg(E(V)) = \deg(\chi_{1*}E)$ and $\deg(\sigma^* E(W)) = \deg(\chi_{2*}\sigma^* E)$. Also $\chi_{1*}E \approx \chi_{2*}\sigma^* E$. Let $r_1 = \operatorname{rank} V$ and $r_2 = \operatorname{rank} W$. Since Z_0 acts by scalars the character $\chi = \chi_1^{r_2} \cdot \chi_2^{-r_1}$ of P is trivial on Z_0. Therefore, σ being admissible, $\deg(\chi_* \sigma^* E) = 0$, i.e. $r_2 \deg(\chi_{1*}\sigma^* E) - r_1 \deg(\chi_{2*}\sigma^* E) = 0$, i.e. $\mu(\sigma^* E(W)) = \mu(E(V))$.

3.5.9. *Lemma. Let $P = M \cdot U$ be a parabolic subgroup of G. Let P_1 be a proper parabolic subgroup of M.*

i) *Let χ_1 be a dominant character of P_1. Extend the character χ_1 to a character χ_1' on $P_1 \cdot U$ by defining it to be trivial on U. Then there exists an integer $n > 0$ such that on $P_1 \cdot U$ we have $\chi_1'^n = \chi' \cdot \chi^{-1}$ where χ' is a dominant character of $P_1 \cdot U$ and χ is a dominant character of P.*
ii) *Let μ_1 be a character on $P_1 \cdot U$, trivial on Z_0. Then there exists an integer $n > 0$ such that $\mu_1^n = \mu' \cdot \mu$ where μ' is a character on $P_1 \cdot U$ which is trivial on $Z_0[M]$ and μ is a character on P which is trivial on Z_0.*

Proof. Let $\mathscr{h}$ be a Cartan subalgebra of $\mathscr{G}'$ and $\mathscr{G} = \mathscr{z} \oplus \mathscr{h} \oplus \Sigma_{\alpha\in\Delta} \mathscr{G}^\alpha$ be a root space decomposition. Let $\alpha_1, \ldots, \alpha_l$ be a system of simple roots and $\lambda_1, \ldots, \lambda_l$ the corresponding fundamental weights. We can assume, by conjugating if necessary that the Lie algebras of P, $P_1 \cdot U$ and M are $\mathscr{z} \oplus \mathscr{h} \oplus \Sigma_{\alpha\in D} \mathscr{G}^\alpha$, $\mathscr{z} \oplus \mathscr{h} \oplus \Sigma_{\alpha\in D_1} \mathscr{G}^\alpha$ and $\mathscr{z} \oplus \mathscr{h} \oplus \Sigma_{\alpha\in D^s} \mathscr{G}^\alpha$ respectively where

$$D = \{\alpha\in\Delta \mid \alpha = \Sigma m_i\alpha_i \quad \text{with } m_i \geqslant 0 \quad \text{for } i = 1, \ldots, r\}$$

$$D_1 = \{\alpha\in\Delta \mid \alpha = \Sigma m_i\alpha_i \quad \text{with } m_i \geqslant 0 \quad \text{for } i = 1, \ldots, s\}$$

and

$$D^s = \{\alpha\in\Delta \mid \text{both } \alpha \text{ and } -\alpha \text{ are in } D\}.$$

Since $P_1 U \subset P$ we have $D_1 \subset D$. The roots $\alpha_{r+1}, \ldots, \alpha_l$ constitute a simple system of roots for $\mathscr{M}$, the Lie algebra of M. Since the vector space spanned by $\lambda_1, \ldots, \lambda_r$ is the orthogonal complement of the space spanned by $\alpha_{r+1}, \ldots, \alpha_l$, we can determine constants a_{ij} such that $\lambda_i' = \lambda_i + \Sigma_{j=1}^r a_{ij}\lambda_j$, $i = r+1, \ldots, l$, belong to the space spanned by $\alpha_{r+1}, \ldots, \alpha_l$. Clearly then λ_i', $i = r+1, \ldots, l$ are the fundamental weights of $\mathscr{M}$ corresponding to the simple system of roots $\alpha_{r+1}, \ldots, \alpha_l$. Therefore, $\tilde{\chi}_1$, being a dominant form of P_1 in M, is of the form $\tilde{\chi}_1 = \Sigma_{k=r+1}^l b_k \lambda_k'$, $b_k \geqslant 0$. Clearly the form $\tilde{\chi}_1'$ corresponding to the character χ_1' on $P_1 \cdot U$ is also $\Sigma_{k=r+1}^l b_k \lambda_k' = \tilde{\chi}_1$. Substituting for λ_k' in terms of λ_i we have

$$\tilde{\chi}_1' = \sum_{k=r+1}^{s} b_k \lambda_k + \sum_{j=1}^{r} a_j \lambda_j.$$

Let

$$\tilde{\chi}' = \sum_{k=r+1}^{s} b_k \lambda_k + \sum_{a_j \geqslant 0} a_j \lambda_j$$

and

$$-\tilde{\chi} = \sum_{a_j < 0} a_j \lambda_j$$

so that

$$\tilde{\chi}'_1 = \tilde{\chi}' - \tilde{\chi}.$$

Note that since $\tilde{\chi}'_1$ corresponds to a character on $P_1 \cdot U$, $b_k, a_j \in \mathbb{Z}$. The linear forms $\tilde{\chi}'$ and $\tilde{\chi}$ may not go down to give characters on the corresponding groups since G may not be simply connected. However, we can find an integer $n > 0$ such that $n\tilde{\chi}'$ and $n\tilde{\chi}$ give rise to characters χ' and χ on $P_1 U$ and P respectively. Then $\chi_1'^n = \chi' \cdot \chi^{-1}$ gives the required decomposition.

To prove (ii) note that if the finite group $Z_0[M] \cap [M, M]$ is of order, say n then the character μ_1^n restricted to $Z_0[M]$ is trivial on $Z_0[M] \cap [M, M]$ and hence can be extended to a character μ of P by defining it to be trivial on $[M, M] \cdot U$. Then the character $\mu' = \mu_1^n \cdot \mu^{-1}$ of $P_1 \cdot U$ is trivial on $Z_0[M]$ and gives the required decomposition $\mu_1^n = \mu' \cdot \mu$.

3.5.10. *Lemma. Let $P = M \cdot U$ be a parabolic subgroup of G and F a P-bundle on X. Let $F' = p_*(F)$ where $p: P \to M$ is the projection. Let P_1 be a parabolic subgroup of M. Then*

i) *if σ_1 is a reduction of structure group of F' to the subgroup P_1 of M then there exists a reduction σ of structure group of F to $P_1 \cdot U$ such that $p_{1*}\sigma^* E \simeq \sigma_1^* F'$ where $p_1: P_1 \cdot U \to P_1$ is the projection.*
ii) *if τ is a reduction of structure group of F to $P_1 \cdot U$ then there exists a reduction of structure group τ_1 of F' to P_1 such that $\tau_1^* F' \approx p_{1*}\tau^* F$.*

Suppose E is a G-bundle and $F = \sigma'^* E$ for an admissible reduction σ' of structure group of E to P. Let $q: (\sigma'^* E)/P_1 \cdot U = \sigma'^*(E/P_1 \cdot U) \to E/P_1 \cdot U$ be the projection. Then

i)$'$ *in the notation of* (i) *if σ_1 is admissible then $q \circ \sigma$ is an admissible reduction of structure group of E to $P_1 \cdot U$.*
ii)$'$ *in the notation of* (ii) *if $q \circ \tau$ is admissible then τ_1 is also admissible.*

Proof. The natural morphism $\psi: P/P_1 U \to M/P_1$ has a section $\varphi: M/P_1 \to P/P_1 \cdot U$ induced by the inclusion $M \subsetneq P$. The morphism ψ induces $\tilde{\psi}: F/P_1 \cdot U \to F'/P_1$ and φ induces a section $\tilde{\varphi}: F'/P_1 \to F/P_1 \cdot U$ for $\tilde{\psi}$ since $F/P_1 \cdot U$ and F'/P_1 are the associated fiber bundles of F with fibers $P/P_1 \cdot U$ and M/P_1 respectively.

For (i) take $\sigma = \tilde{\varphi} \circ \sigma_1$ and for (ii) take $\tau_1 = \tilde{\psi} \circ \sigma$.

(i)$'$ Let χ be a character of $P_1 \cdot U$. By 3.5.9(ii) for some $n > 0$ we can write $\chi^n = \chi' \cdot \chi$ with χ' a character of P_1, trivial on $Z_0[M]$ and χ a character of P, trivial on Z_0. We have therefore,

$$\chi^n_*(q \circ \sigma)^* E \approx (\chi' \cdot \chi)_*(q \circ \sigma)^* E \approx (\chi'_* \sigma^* F) \otimes (\chi_* F)$$
$$\approx (\chi'_* \sigma_1^* F') \otimes (\chi_* F).$$

Since σ_1 is admissible, $\deg(\chi'_* \sigma_1^* F') = 0$. Since σ' is admissible, $\deg(\chi_* F) = 0$. Therefore $\deg(\chi_*(q \circ \sigma)^* E) = 0$ which proves that $q \circ \sigma$ is admissible.
(ii)$'$ Let χ_1 be a character of P_1, trivial on $Z_0[M]$. Let χ'_1 be the character on $P_1 \cdot U$ got by extending χ_1 to $P_1 \cdot U$ by setting it to be trivial on U. We then have

$\chi_{1*}\tau_1^* F' \approx \chi_{1*} p_1^* \tau^* F$ by (ii). Also $\chi_{1*} p_1^* \tau^* F \approx \chi'_{1*} \tau^* F \approx \chi'_1 (q \circ \tau)^* E$. Since $q \circ \tau$ is admissible $\deg(\chi'_{1*}(q \circ \tau)^* E) = 0$, which proves τ_1 is admissible.

3.5.11. *Lemma. Let $E \to X$ be a G-bundle and σ an admissible reduction of structure group of E to a proper parabolic subgroup $P = M \cdot U$. Let $p: P \to M$ be the projection, and $j: M \subsetneq G$ the inclusion. Then*

i) *E is a semistable G-bundle if and only if $p_* \sigma^* E$ is a semistable M-bundle.*
ii) *if E is semistable then $j_* p_* \sigma^* E$ is semistable.*

Proof. Assume $p_* \sigma^* E$ is a semistable M-bundle. Let σ' be a reduction of structure group of E to a proper parabolic subgroup P' of G. We have to show that σ' satisfies the condition for semistability. We can assume by conjugating that (P, P') is compatible with (σ, σ') (see 3.5.6 and 3.5.7).

Let $\mathscr{P}, \mathscr{P}'$ be the subalgebras of $\mathscr{G}$ corresponding to the subgroups P, P' respectively. By ([14], Remark 2.2, p. 132) semistability condition is equivalent to $\deg((\sigma'^* E)(\mathscr{P}')) \leqslant 0$.

Let $0 = V_0 \subset V_1 \subset \cdots \subset V_r = \mathscr{G}$ be a flag in $\mathscr{G}$ such that V_j is invariant under P and U acts trivially on V_j/V_{j-1} for $j = 1, \ldots, r$. The stabilizer of $\operatorname{Im}(\mathscr{P}' \cap V_j \to V_j/V_{j-1})$ in M is a parabolic subgroup of M since $P \cap P'$ leaves $\mathscr{P}' \cap V_j$ invariant and $(P \cap P') \cap M$ is a parabolic subgroup of M (§ 2.9). Since (P, P') is compatible with (σ, σ') the sub-bundle W_j generated by $\operatorname{Im}((\sigma'^* E(\mathscr{P}')) \cap (\sigma^* E(V_j)) \to \sigma^* E(V_j/V_{j-1}))$ is a sub-bundle (generically and hence everywhere cf. 3.5.4) of type $\operatorname{Im}(\mathscr{P}' \cap V_j \to V_j/V_{j-1})$. Note that since U acts trivially on V_j/V_{j-1}, $\sigma^* E(V_j/V_{j-1}) \approx p_* \sigma^* E(V_j/V_{j-1})$ where $p: P \to M$ is the projection. Since $p_* \sigma^* E$ is a semistable M-bundle

$$\mu(W_j) \leqslant \mu(p_* \sigma^* E(V_j/V_{j-1})). \tag{1}$$

(cf. Remark 3.5.3; if the stabilizer of $\operatorname{Im}(\mathscr{P}' \cap V_j \to V_j/V_{j-1})$ is M itself, use the fact that σ is admissible). Since σ is admissible, by Lemma 3.5.8

$$\mu(p_* \sigma^* E(V_j/V_{j-1})) = \mu(E(\mathscr{G})) = 0.$$

Therefore,

$$\mu(W_j) \leqslant 0 \quad \text{for all } j. \tag{2}$$

Denote by $\underline{V}_j$ the sub-bundle $\sigma^* E(V_j)$ of $E(\mathscr{G})$ and by $\underline{\mathscr{P}}'$ the sub-bundle $\sigma'^* E(\mathscr{P})$ of $E(\mathscr{G})$.

We shall show that for $1 \leqslant j \leqslant r-1$

$$\deg(\overline{\underline{\mathscr{P}}' \cap \underline{V}_j}) \leqslant 0 \Rightarrow \deg(\overline{\underline{\mathscr{P}}' \cap \underline{V}_{j+1}}) \leqslant 0 \tag{$*$}$$

(see Remark 3.5.5 for notation). Since $\overline{\underline{\mathscr{P}}' \cap \underline{V}_1} = W_1$ and $\deg W_1 \leqslant 0$ by (2) and $\underline{\mathscr{P}}' \cap \underline{V}_r = \underline{\mathscr{P}}'$ it will then follow by induction that $\deg(\underline{\mathscr{P}}') \leqslant 0$ as required to be shown.

To prove $(*)$ factorize the natural homomorphism $\overline{\underline{\mathscr{P}}' \cap \underline{V}_{j+1}} \to \underline{V}_{j+1}/\underline{V}_j$ (cf. [13], § 4, p. 547)

$$\begin{array}{ccccccccc}
O & \longrightarrow & \overline{\underline{\mathscr{P}}' \cap \underline{V}_j} & \longrightarrow & \overline{\underline{\mathscr{P}}' \cap \underline{V}_{j+1}} & \longrightarrow & Q & \to & 0 \\
 & & & & & & \downarrow & & \\
O & \longleftarrow & Q' & \longleftarrow & \underline{V}_{j+1}/\underline{V}_j & \longleftarrow & W_{j+1} & \longleftarrow & 0.
\end{array}$$

Since $Q \to W_{j+1}$ is a generic isomorphism $\deg Q \leqslant \deg W_{j+1}$. Using (2) $\deg Q \leqslant 0$. By the hypothesis of $(*)$ $\deg(\overline{\underline{\mathscr{P}}' \cap \underline{V}_j}) \leqslant 0$. Therefore $\deg(\overline{\underline{\mathscr{P}}' \cap \underline{V}_{j+1}}) \leqslant 0$, as was to be shown.

Now suppose, conversely, E is a semistable G-bundle. Let σ_1 be a reduction of structure group of $p_*\sigma^*E=E'$ to a parabolic subgroup P_1 of M. Let χ_1 be a dominant character on P_1. By Lemma 3.5.10 there is a reduction of structure group σ' of σ^*E to the parabolic subgroup $P_1\cdot U$ such that $\sigma_1^*E'\approx p_{1*}\sigma'^*\sigma^*E$ where $p_1:P_1\cdot U\to P_1$ is the projection. By Lemma 3.5.9

(i) for the character χ_1 extended to $P_1\cdot U$ we have $\chi_1^n=\chi'\cdot\chi^{-1}$ for some $n>0$ with χ' a dominant character on $P_1\cdot U$ and χ a dominant character on P. We have $\chi_*^n\sigma_1^*E'\approx(\chi'\cdot\chi^{-1})_*(q\circ\sigma')^*E$ (where $q:\sigma^*(E/P_1\cdot U)\to E/P_1\cdot U$ is the projection). Also $\chi_*(q\circ\sigma')^*E\approx\chi_*\sigma^*E$. Since σ is admissible $\deg(\chi_*\sigma^*E)=0$ and since E is semistable $\deg(\chi'_*(q\circ\sigma')^*E)\leqslant 0$. Therefore $\deg(\chi_*\sigma_1^*E')=(1/n)\deg(\chi_*^n\sigma_1^*E')=(1/n)\,\{\deg\,(\chi'_*(q\circ\sigma')^*E)-\deg(\chi_*\sigma^*E)\}\leqslant 0$. This shows $p_*\sigma^*E$ is a semistable M-bundle.
(ii) By (i) the M-bundle $p_*\sigma^*E$ is semistable. But $p_*\sigma^*E$ gives a reduction of structure group of $j_*p_*\sigma^*E$ to the subgroup M. Again by (i) it follows $j_*p_*\sigma^*E$ is a semistable G-bundle.

3.5.12. *Lemma. Let $P=M\cdot U$ be a proper parabolic subgroup of G. Then there is a 1-parameter subgroup $\lambda:\mathbb{C}^*\to Z_0[M]$ such that the morphism $\mathbb{C}^*\times P\to P$ defined by $(z,p)\mapsto\lambda(z)p\lambda(z)^{-1}$, $z\in\mathbb{C}^*$, $p\in P$, extends to a morphism $\varphi:\mathbb{C}\times P\to P$ such that $\varphi(0,p)=m$ where $p=m\cdot u$, $m\in M$, $u\in U$.*

Proof. We use the notation of [3], §4. Let $P=P_\theta$ where θ is a subset of a system of simple roots of G with respect to a maximal torus T. Then $M=Z_\theta$, $U=V_\theta$ and $p:\Pi_{b\in\alpha_\theta}U_b\to V_\theta$ given by group multiplication is an isomorphism of algebraic varieties ([2], p. 327, § 14.4) where U_b is the radical group corresponding to the root b, i.e. there is an isomorphism of algebraic groups $\theta_b:\mathbb{C}\to U_b$ such that

$$t\theta_b(x)t^{-1}=\theta_b(t^bx)(*)$$

([3], §2.3, p. 64).

It follows from ([3], Proposition 3.6, p. 75) that we can find a 1-parameter subgroup $\lambda:\mathbb{C}^*\to Z_0(M)$ such that $\langle\lambda,b\rangle>0$ for every $b\in\alpha_\theta$, where $\langle\lambda,b\rangle$ is the integer such that the composite $\mathbb{C}^*\to T\xrightarrow{b}\mathbb{C}$ is given by $z\mapsto z^{\langle\lambda,b\rangle}$.

Let $\varphi_b:\mathbb{C}^*\times U_b\to U_b$ be given by $\varphi_b(z,u)=\lambda(z)u\lambda(z)^{-1}$, $z\in\mathbb{C}^*$, $u\in U_b$. Define $\varphi'_b:\mathbb{C}^*\times\mathbb{C}\to\mathbb{C}$ such that the diagram

$$\begin{array}{ccc}\mathbb{C}^*\times U_b & \xrightarrow{\varphi_b} & U_b\\ {\scriptstyle id\times\theta_b}\uparrow & & \uparrow{\scriptstyle\theta_b}\\ \mathbb{C}^*\times\mathbb{C} & \xrightarrow[\varphi'_b]{} & \mathbb{C}\end{array}$$

commutes.

Using $(*)$ we get $\varphi'_b(z,\zeta)=z^{\langle\lambda,b\rangle}\cdot\zeta$, $z\in\mathbb{C}^*$, $\zeta\in\mathbb{C}$. Since $\langle\lambda,b\rangle>0$, $\mathbb{C}^*\times\mathbb{C}\to\mathbb{C}$ can be extended as a morphism $\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ such that $(0,\zeta)\mapsto 0$. Therefore, φ_b extends to a morphism $\mathbb{C}\times U_b\to U_b$, again denoted by φ_b, such that $\varphi_b(0,u)=1$, $\forall u\in U_b$. Using the isomorphism $p:\Pi_{b\in\alpha_\theta}U_b\to V_\theta$ we can write $P=M\cdot\Pi_{b\in\alpha_\theta}U_b$. Then the morphism

$\varphi:\mathbb{C}\times(M\cdot\Pi_{b\in\alpha_0}U_b)\to M\cdot\Pi_{b\in\alpha_0}U_b$ defined by $\varphi(z,m\cdot\Pi_{b\in\alpha_0}u_b)=m\cdot\Pi_{b\in\alpha_0}\varphi_b(z,u_b)$ $z\in\mathbb{C}$, $m\in M$, $u_b\in U_b$ satisfies the requirements of the lemma.

We can now give the proof of Proposition 3.5.

Proof of Proposition 3.5. Consider $\mathbb{C}\times Y\times P$ as the trivial group scheme over $\mathbb{C}\times Y$ determined by P. Then $\mathbb{C}\times(\sigma^*E)$ is a principal homogeneous space over $\mathbb{C}\times Y$ under the group scheme $\mathbb{C}\times Y\times P$, in the obvious way (in the sense of [SGA, I], expose XI, Definition 4.1). Let $\tilde{\varphi}:\mathbb{C}\times Y\times P\to\mathbb{C}\times Y\times P$ be the homomorphism of group schemes defined by $\varphi(z,y,p)=(z,y,\varphi(z,p))$ where $\varphi:\mathbb{C}\times P\to P$ is the morphism given by Lemma 3.5.12. Then taking the associated principal homogeneous space of $\mathbb{C}\times\sigma^*E\to\mathbb{C}\times Y$ under the extension $\tilde{\varphi}$ (which exists, since $\mathbb{C}\times Y\times P$ is an affine algebraic group scheme over $\mathbb{C}\times Y$, see [SGA I], expose XI, § 4, p. 11) we get a principal homogeneous space ξ'' over $\mathbb{C}\times Y$ under the trivial group scheme $\mathbb{C}\times Y\times P$. We can consider ξ'' as a P-bundle over $\mathbb{C}\times Y$ in the obvious way. Let $i:P\subsetneq G$ be the inclusion. Then $\xi'=i_*\xi''$ gives a family of G-bundles for which the assertion (i) of Proposition 3.5 is easily seen to hold. Then (ii) follows from 3.5.11 (ii).

3.6. DEFINITION

Two families of semistable G-bundles $\xi\to S\times X$ and $\xi'\to S\times X$ parametrized by the scheme S are said to be *related* if there is an admissible reduction of structure group σ (resp. σ') of ξ (resp. ξ') to a parabolic subgroup $P=M\cdot U$ (resp. $P'=M'\cdot U'$) such that the G-bundles $j_*p_*\sigma^*\xi$ and $j'_*p'_*\sigma'^*\xi'$ are isomorphic, where $p:P\to M$, $p':P'\to M'$ are the projections and $j:M\subsetneq G$, $j':M'\subsetneq G$ are the inclusions. We say ξ is *equivalent* to ξ' if there exist families of semistable G-bundles $\xi_i\to S\times X$, $i=1,\dots,r$ such that ξ is related to ξ_1, ξ_r is related to ξ' and ξ_i is related to ξ_{i+1} for $i=1,\dots,r-1$.

3.7. *Lemma. If the semistable G-bundles $E\to X$ and $E'\to X$ are equivalent then they are topologically isomorphic.*

Proof. It is enough to prove that if σ is a (admissible) reduction of structure group of E to the parabolic subgroup $P=M\cdot U$ then E and $j_*p_*\sigma^*E$, where $p:P\to M$ is the projection and $j:M\subsetneq G$ is the inclusion, are topologically isomorphic.

By Proposition 3.5 (i) there is a family of G-bundles $\xi\to\mathbb{C}\times X$ such that $\xi_z\approx E$ for $z\neq 0$ and $\xi_0\approx j_*p_*\sigma^*E$. Since $\mathbb{C}$ is connected, E and $j_*p_*\sigma^*E$ are topologically isomorphic.

3.8. *Remark.* We can also prove the lemma directly without using Proposition 3.5 by using the topological classification of bundles on X ([14], § 5, pp. 142–143).

3.9. DEFINITION

Let $\tilde{\mathbf{F}}_{ss}$:(Sch)$\to$(Sets) be the functor which associates to $S\in$(Sch) the set of equivalence classes of families of semistable G-bundles parametrized by S. On morphisms $\tilde{\mathbf{F}}_{ss}$ is defined in the obvious way. Given a topological G-bundle τ on X let $\tilde{\mathbf{F}}^{\tau}_{ss}$ be the sub-functor of $\tilde{\mathbf{F}}_{ss}$ which associates to S the set of equivalence classes of families of semistable G-bundles of the topological type τ parametrized by S.

We prove in this thesis that the functors $\tilde{\mathbf{F}}^{\tau}_{ss}$ have coarse moduli schemes which are projective (see Theorem 5.9).

We shall now study the equivalence of semistable G-bundles on X and pick out representatives for the equivalence classes. We shall need some more lemmas.

3.10. *Lemma. Let* $P_i = M_i \cdot U_i$ *be proper parabolic subgroups of* G, $i = 1,2$. *Given a character* χ *on the parabolic subgroup* $P_1 \cap P_2 \cdot U_1$ (§ 1.9; [3], Proposition 4.4, p. 86) *which is trivial on* Z_0, *there is an integer* $n > 0$ *such that* $\chi^n = \chi_1 \cdot \chi_2$ *with* χ_i *a character on* P_i, *trivial on* Z_0, $i = 1,2$.

Proof. By Bruhat's lemma there is a maximal torus $T \subset P_1 \cap P_2$. Let $\mathcal{T}$ be the Lie algebra of T and $\mathcal{G} = z \oplus h \oplus \Sigma_{\alpha \in \Delta} \mathcal{G}^\alpha$ be a root space decomposition with $z + h = \mathcal{T}$.

Let L be the lattice $\ker(\exp: h \to T)$ and L^* the lattice of linear forms on h which take integral values on L. Then the characters of a parabolic subgroup P containing T correspond to the linear forms in L^* which are orthogonal (with respect to a Weyl group invariant form which can be taken to be Q-valued on L^*) to all the roots α such that both $\mathcal{G}^\alpha$ and $\mathcal{G}^{-\alpha}$ are contained in the Lie algebra of P. Let V_i (resp. V) be the Q-vector space spanned by the elements of L^* which are orthogonal to those roots α for which both $\mathcal{G}^\alpha$ and $\mathcal{G}^{-\alpha}$ are contained in the Lie algebra of P_i (resp. $P_1 \cap P_2 \cdot U_1$), $i = 1,2$. Then clearly $V = V_1 + V_2$. Let $\tilde{\chi}$ be the linear form corresponding to the character χ. Then $\tilde{\chi} \in V$. Write $\tilde{\chi} = \tilde{\chi}_1 + \tilde{\chi}_2$ with $\tilde{\chi}_i \in V_i$. We can find an integer $n > 0$ such that $n\tilde{\chi}_i \in L^*$, $i = 1,2$. Then $n\tilde{\chi}_i$ give characters χ_i on P_i and we have $\chi^n = \chi_1 \cdot \chi_2$.

3.11. *Lemma. Let* E *be a semistable* G-*bundle and* σ_i *an admissible reduction of structure group of* E *to the parabolic subgroup* $P_i = M_i \cdot U_i$, $i = 1,2$. *Let* $p_i : P_i \to M_i$ *be the projections. Suppose* (P_1, P_2) *is compatible with* (σ_1, σ_2). *Then*

i) *there is a reduction of structure group* $\sigma = \sigma_1 \cap \sigma_2$ *of* E *to the subgroup* $P_1 \cap P_2$ *such that* $\sigma_i = \pi_i \circ \sigma$ where $\pi_i : E/P_1 \cap P_2 \to E/P_i$, $i = 1,2$ *is the natural morphism.*
ii) *letting* $q_i : E/P_1 \cap P_2 \to E/P_1 \cap P_2 \cdot U_i$ *be the natural morphism, the reductions* $q_i \circ \sigma$ $(i = 1,2)$ *are admissible.*
iii) *if* $p_{1*}\sigma_1^* E$ *is a stable* M_1-*bundle then* $P_1 \cap P_2$ *contains a Levi component of* P_1.
iv) *let* $P_1 \cap P_2 = M_3 \cdot U_3$ *be a Levi decomposition of* $P_1 \cap P_2$ *such that* $M_3 \subset M_1$. *Note that* M_3 *is a Levi component for both* $P_1 \cap P_2 \cdot U_1$ *and* $P_2 \cap M_1$. *By* 3.5.10 (ii) *and* (ii)′ *the admissible reduction of structure group* $q_1 \circ \sigma$ *of* E *induces an admissible reduction of structure group* σ' *of the* M_1-*bundle* $p_{1*}\sigma^* E = F'$ *to the parabolic sub-group* $P_2 \cap M_1$. *We then have* $p_{3*}\sigma^* E \approx p'_{3*}\sigma'^* F'$ *where* $p_3 : P_1 \cap P_2 \to M_3$ *and* $p'_3 : P_2 \cap M_1 \to M_3$ *are the projections.*

Proof. Let $\mathcal{P}_i$ be the subalgebra of $\mathcal{G}$ corresponding to the sub-group P_i, $i = 1,2$. Let $0 = V_0 \subset V_1 \subset \cdots \subset V_p = \mathcal{P}_1 \ldots V_r = \mathcal{G}$ be a flag in $\mathcal{G}$ such that each V_j is invariant under P_1 and U_1 acts trivially on V_j/V_{j-1}, $j = 1, \ldots, r$. Let $0 = W_0 \subset W_1 \subset \cdots \subset W_q = \mathcal{P}_2 \subset \cdots \subset W_s = \mathcal{G}$ be a flag with the same properties with respect to P_2. We denote by $\underline{V}_j$ (resp. $\underline{W}_j$) the sub-bundle $\sigma_1^* E(V_j)$ (resp. $\sigma_2^* E(W_j)$) of $E(\mathcal{G})$. We shall show that $\underline{\mathcal{P}_1 \cap \mathcal{P}_2}$ is a sub-bundle of $E(\mathcal{G})$, i.e. $\underline{\mathcal{P}_1} \cap \underline{\mathcal{P}_2} = \underline{\mathcal{P}_1 \cap \mathcal{P}_2}$ (cf. Remark 3.5.5). We note that since σ_1 and σ_2 are admissible by Lemma 3.5.8. $\mu(\underline{V}_j/\underline{V}_{j-1}) = \mu(E(\mathcal{G}))$ and $\mu(\underline{W}_j/\underline{W}_{j-1}) = \mu(E(\mathcal{G}))$. Moreover $\mu(E(\mathcal{G})) = 0$ (cf. [14], Remark 2.2).

We prove that each $\underline{V}_i \cap \underline{W}_j$ is a sub-bundle by induction. Assume that $\underline{V}_i \cap \underline{W}_j$ is a sub-bundle with $\deg(\underline{V}_i \cap \underline{W}_j) = 0$ for (i,j) such that either $i \leqslant m - 1$ and j arbitrary or $i = m$ and $j \geqslant n + 1$. Under this assumption we shall prove that $\underline{V}_m \cap \underline{W}_n$ is a

sub-bundle of degree zero. It will then follow by induction that $\underline{V}_i \cap \underline{W}_j$ is a sub-bundle of degree zero for all i,j.

If $\underline{V}_m \cap \underline{W}_{n+1} = 0$ there is nothing to prove since $\underline{V}_m \cap \underline{W}_n = 0$ in that case. Suppose $\underline{V}_m \cap \underline{W}_{n+1} \neq 0$. Factorize the natural homomorphism $\underline{V}_m \cap \underline{W}_{n+1} \to \underline{W}_{n+1}/\underline{W}_n$.

$$\begin{array}{c} 0 \to A_1 \to \underline{V}_m \cap \underline{W}_{n+1} \to A_2 \to 0 \\ \downarrow \\ 0 \leftarrow A_4 \leftarrow \underline{W}_{n+1}/\underline{W}_n \leftarrow A_3 \leftarrow 0. \end{array}$$

Since $\underline{W}_{n+1}/\underline{W}_n = p_{2*}\sigma_2^* E(W_{n+1}/W_n)$ and $p_{2*}\sigma_2^* E$ is a semistable M_2-bundle (Lemma 3.5.11(i)) and A_3 is a sub-bundle of type $\operatorname{Im}(V_m \cap W_{n+1} \to W_{n+1}/W_n)$ which has as stabilizer a parabolic subgroup in M_2, we have $\deg A_3 \leqslant 0$ (cf. proof of Lemma 3.5.11). Therefore $\deg A_2 \leqslant 0$ and hence

$$\deg A_1 \geqslant 0, \tag{1}$$

noting that $\deg(\underline{V}_m \cap \underline{W}_{n+1}) = 0$ by the induction hypothesis. Now consider the sub-bundle A_1 of $E(\mathscr{G})$. If $A_1 = 0$ it is easy to see that $V_m \cap W_n = A_1 = 0$ and we are through. If $A_1 \neq 0$ we can find a $t \leqslant m$ such that $A_1 \subset \underline{V}_t$ and $A_1 \not\subset \underline{V}_{t-1}$. Then factorize the non-zero homomorphism $A_1 \to \underline{V}_t/\underline{V}_{t-1}$

$$\begin{array}{c} 0 \to B_1 \to A_1 \to B_2 \to 0 \\ \downarrow \\ 0 \leftarrow \mathrm{B}_4 \leftarrow \underline{V}_t/\underline{V}_{t-1} \leftarrow B_3 \leftarrow 0. \end{array}$$

Since by the induction hypothesis $\underline{V}_{t-1} \cap \underline{W}_n$ is a sub-bundle, we see that $B_2 \approx B_3$ and $B_1 = \underline{V}_{t-1} \cap \underline{W}_n$. Since σ_1 is admissible as before we have $\deg B_3 \leqslant 0$. Therefore $\deg B_2 \leqslant 0$. Also $\deg B_1 = 0$. Therefore

$$\deg A_1 \leqslant 0. \tag{2}$$

By (1) and (2) $\deg A_1 = 0$. This implies $\deg A_2 = \deg A_3 = 0$. Then $A_2 \to A_3$ being a generic isomorphism of vector bundles of the same degree becomes an isomorphism. Therefore $A_1 = \underline{V}_m \cap \underline{W}_n$ which shows that $\underline{V}_m \cap \underline{W}_n$ is a sub-bundle.

In particular we have proved that $\underline{\mathscr{P}}_1 \cap \underline{\mathscr{P}}_2$ is a sub-bundle of $E(\mathscr{G})$.

(i) Since (P_1, P_2) is compatible with (σ_1, σ_2) there is a nonempty open subset U of X over which E is trivial such that, choosing a trivialization, the morphism $\sigma_1 \times \sigma_2 : U \to G/P_1 \times G/P_2$ given by the sections σ_1 and σ_2 has its image in the G orbit O of $(P_1, P_2) \in G/P_1 \times G/P_2$ (Remark 3.5.6). Since the stabilizer in G of $(P_1, P_2) \in G/P_1 \times G/P_2$ is $P_1 \cap P_2$ we have that O is naturally isomorphic to $G/P_1 \cap P_2$. It then follows easily that there is a section σ of $E/P_1 \cap P_2 \to X$ over U such that $\sigma_i | U = \pi_i \circ \sigma$, $i = 1,2$. The complement $X - U$ of U in X is a set of a finite number of points. Let $x \in X - U$. Let U' be a neighbourhood of x in X, over which E is trivial, and choose a trivialization. Then σ_1, σ_2 and σ give morphisms $\sigma_1' \times \sigma_2' : U' \to G/P_1 \times G/P_2$ and $\sigma' : U \cap U' \to G/P_1 \cap P_2$. By our choice of U, $(\sigma_1' \times \sigma_2')(U \cap U') \subset O$. Therefore $(\sigma_1' \times \sigma_2')(x)$ is in the closure $\bar{O}$ of O in $G/P_1 \times G/P_2$. Suppose $(\sigma_1' \times \sigma_2')(x) \notin O$. Since $\bar{O} - O$ is a union of orbits of dimension strictly less than that of O ([2], §(1.8), p. 98), the stabilizer of $(\sigma_1' \times \sigma_2')(x)$ in G will have dimension strictly greater than $P_1 \cap P_2$. However, letting $\underline{\mathscr{P}}_{i_x}$ be the fiber over x of the vector bundle $\underline{\mathscr{P}}_i$ it is easy to see that the Lie algebra of this stabilizer corresponds to $\underline{\mathscr{P}}_{1_x} \cap \underline{\mathscr{P}}_{2_x}$, which has the same dimension as $P_1 \cap P_2$ since $\underline{\mathscr{P}}_1 \cap \underline{\mathscr{P}}_2$ is a sub-bundle (Remark 3.5.6). This contradiction proves that $(\sigma_1' \times \sigma_2')(x) \in \bar{O}$. This implies that we can extend the section σ over U to x and hence, by

the same argument at other points of $X - U$, to the whole of X. Then $\sigma_i = \pi_i \circ \sigma$ on the whole of X since both the sides agree on the dense open subset U.
(ii) Let χ be a character on $P_1 \cap P_2 \cdot U_1$ which is trivial on Z_0. By Lemma 3.10 for some integer $n > 0$, we can write $\chi^n = \chi_1 \cdot \chi_2$ where χ_i is a character on P_i trivial on Z_0. Clearly $\chi^n_*(q_1 \circ \sigma)^* E \approx (\chi_{1*} \sigma_1^* E) \otimes (\chi_{2*} \sigma_2^* E)$. Since both σ_1 and σ_2 are admissible $\deg(\chi_{i*} \sigma_i^* E) = 0$, $i = 1, 2$. Therefore $\deg(\chi_*(q_1 \circ \sigma)^* E) = 0$.
(iii) The reduction σ gives rise to a reduction of structure group of the P_1-bundle $\sigma_1^* E$ to the subgroup $P_1 \cap P_2$ and hence to $P_1 \cap P_2 \cdot U_1$. By Lemma 3.5.10 (ii) and (ii)′ we have a reduction of structure group of $p_{1*} \sigma_1^* E$ to $(P_1 \cap P_2 \cdot U_1) \cap M_1$ which is admissible. If $p_{1*} \sigma_1^* E$ is a stable M_1-bundle then we should have $(P_1 \cap P_2 \cdot U_1) \cap M_1 = M_1$. This implies that $P_1 \cap P_2$ contains a Levi component of P_1 ([3], Proposition 4.4, p. 86).
(iv) Let $F = \sigma_1^* E$. By Lemma 3.5.10 (ii) $\sigma'^* F' \approx p_*(q_1 \circ \sigma)^* F$ where $p: (P_1 \cap P_2) \cdot U_1 = (P_2 \cap M_1) \cdot U_1 \to P_2 \cap M_1$ is the projection. Therefore applying p'_{3*} to both sides $p'_{3*} \sigma'^* F' \approx p'_{3*} p_*(q_1 \circ \sigma)^* F$. But the latter is isomorphic to $p_{3*} \sigma^* E$.

3.12. PROPOSITION

Let E be a semistable G-bundle on X. Then there exists a semistable G-bundle grade E, denoted by gr E, *uniquely determined up to isomorphism by the condition that there exists an admissible reduction of structure group σ of E to a parabolic subgroup $P = M \cdot U$ such that $p_* \sigma^* E$ (where $p: P \to M$ is the projection) is a stable M-bundle and* $\operatorname{gr} E \approx j_* p_* \sigma^* E$ *(where j: $M \subsetneq G$ is the inclusion). We then have* i) *E and* $\operatorname{gr} E$ *are equivalent,* ii) $\operatorname{gr}(\operatorname{gr} E) \approx \operatorname{gr} E$ *and* iii) *two semistable G-bundles E_1 and E_2 are equivalent if and only if* $\operatorname{gr} E_1 \approx \operatorname{gr} E_2$.

3.12.1. COROLLARY

The set of isomorphism classes of semistable G-bundles E such that $E \approx$ gr *E forms a set of representatives for the equivalence classes of semistable G-bundles.*

Proof. The corollary follows immediately from the proposition.

We shall first prove that given E there exists a semistable G-bundle gr E satisfying the above condition. If E is stable we have only to take $\operatorname{gr} E = E$. If E is not stable then there exists an admissible reduction of structure group of E to a proper parabolic subgroup $P = M \cdot U$. By Lemma 3.5.11(i) $E' = E[P, M]$ (cf. §2.5 for notation) is a semistable M-bundle. To prove the existence of gr E we now use induction on the semi-simple rank (i.e. the rank of the commutator subgroup) of the structure group. If the semisimple rank is zero then the group is a torus group and any bundle with a torus group as structure group is stable and hence we can start the induction. Since the semisimple rank of M is strictly less than that of G by the induction hypothesis there exists an admissible reduction of structure group of E' to a parabolic subgroup $P_1 = M_1 \cdot U_1$ of M such that $E'[P_1, M_1]$ is a stable M_1-bundle. Then by Lemma 3.5.10 (ii) and (ii)′ there is an admissible reduction of structure group of E to the parabolic subgroup $P_1 \cdot U$ of G such that $E[P_1 U, P_1] \approx E'[P_1]$. Therefore $E[P_1 U, P_1](M_1) \approx E'[P_1, M_1]$. Note that M_1 is a Levi component of $P_1 \cdot U$ also and $E[P_1 U, P_1](M_1) \approx E[P_1 U, M_1]$. Therefore we can set $\operatorname{gr} E = E[P_1 \cdot U, M_1](G)$.

We now proceed to show the uniqueness of a bundle satisfying the condition of the proposition.

First note that for any reduction of structure group of E to $P = M \cdot U$ the isomorphism class of $E[P, M](P)$ and hence of $E[P, M](G)$, does not depend on the choice of

the Levi component of P since any two Levi components are conjugate by an element of U ([3], §0.8, p. 59).

If E is stable, uniqueness is obvious. On the other hand suppose σ_1 and σ_2 are two admissible reductions of structure group of E to the proper parabolic subgroups $P_1 = M_1 \cdot U_1$ and $P_2 = M_2 \cdot U_2$ respectively such that $E[P_i, M_i]$ is a stable M_i-bundle, $i = 1, 2$. Then we have to show that the G-bundles $E_i = E[P_i, M_i](G)$ are isomorphic.

We can assume, by conjugating if necessary, that (P_1, P_2) is compatible with (σ_1, σ_2) (cf. Remark 3.5.7). Then by Lemma 3.11, it follows that there is a reduction of structure group σ of E to the subgroup $P_1 \cap P_2$ such that $\sigma_i = p_i \circ \sigma$ where $p_i : E/P_1 \cap P_2 \to E/P_i$, $i = 1, 2$. By Lemma 3.11 (iii) $P_1 \cap P_2$ contains a Levi component of both P_1 and P_2. Let $P_1 \cap P_2 = M \cdot U$ be a Levi decomposition for $P_1 \cap P_2$. Then M is a Levi component of both P_1 and P_2. Also $U \subset U_i$, $i = 1, 2$. Therefore both E_1 and E_2 are isomorphic to $E[P_1 \cap P_2, M](G)$ proving the uniqueness.

The fact that if E_1 and E_2 are equivalent then $\operatorname{gr} E_1 \approx \operatorname{gr} E_2$ follows from Lemma 3.13 (i) below. The other assertions of the proposition are clear.

3.13. *Lemma. Let σ (resp. σ') be an admissible reduction of structure group of the semistable G-bundle E (resp. E') to the parabolic subgroup $P = M \cdot U$. Then we have the following.*

i) $\operatorname{gr} E \approx (\operatorname{gr}(E[P, M]))(G)$
ii) $\operatorname{gr} E \approx \operatorname{gr}(E[P, M](G))$
iii) *If* $\operatorname{gr}(E[P, M]) \approx \operatorname{gr}(E'[P, M])$ *then* $\operatorname{gr} E \approx \operatorname{gr} E'$.
iv) *If the M-bundles $E[P, M]$ and $E'[P, M]$ are equivalent then the G-bundles E and E' are equivalent.*

Proof. Let σ_1 be an admissible reduction of structure group of E to a parabolic subgroup $P_1 = M_1 \cdot U_1$ such that the M_1-bundle $E[P_1, M_1]$ is stable. Then $\operatorname{gr} E \approx E[P_1, M_1](G)$. We can assume that (P_1, P) is compatible with (σ_1, σ). Then by Lemma 3.5.10 (i) there is a reduction of structure group s of E to $P_1 \cap P_2$ such that $\sigma_1 = p_1 \circ s$ and $\sigma = p \circ s$ where $p_1 : E/P_1 \cap P \to E/P_1$ and $p : E/P_1 \cap P \to E/P$ are the natural morphisms. Also by Lemma 3.5.10(iii) $P_1 \cap P$ contains a Levi component of P_1. Therefore, conjugating if necessary, we can assume $M_1 \subset M$. We have a Levi decomposition $P_1 \cap P = M_1 \cdot U'$ for $P_1 \cap P$ where U' is the unipotent radical of $P_1 \cap P$. Also $U' \subset U$ and $(P_1 \cap P) \cdot U = M_1 \cdot U$ is a Levi decomposition for $(P_1 \cap P) \cdot U$. Therefore,

$$\operatorname{gr}(E[P_1, M_1]) \approx E[P_1 \cap P, M_1] \tag{1}$$

and M_1 is a Levi component of $P_1 \cap M$. It follows from Lemma 3.5.10 (iv) that

$$F'[P_1 \cap M, M_1] \approx E[P_1 \cap P, M_1]. \tag{2}$$

Using (1), $F'[P_1 \cap M, M_1]$ is a stable M_1-bundle. Therefore

$$\operatorname{gr} F' \approx F'[P_1 \cap M, M_1](M) \approx E[P_1 \cap P, M_1](M). \tag{3}$$

From (1) and (3) $(\operatorname{gr} F')(G) \approx \operatorname{gr} E$ which proves (i). The reduction of structure group of F' to $P_1 \cap M$ gives in the obvious way a reduction of structure group of $F'(G)$ to $P_1 \cap M$ such that

$$F'[P_1 \cap M] \approx (F'(G))[P_1 \cap M]. \tag{4}$$

Since M_1 is a Levi component of $(P_1 \cap M) \cdot U$, considering the reduction of structure

group of $F'(G)$ to $P_1 \cap M$ as a reduction to the subgroup $(P_1 \cap M)\cdot U$ we have from (4),

$$(F'(G))[P_1 \cap M)\cdot U, M_1] \approx F'[P_1 \cap M, M_1].$$

Since $F'[P_1 \cap M, M_1]$ is a stable M_1-bundle, $\mathrm{gr}(F'(G)) \approx F'[P_1 \cap M, M_1](G) \approx (\mathrm{gr}\, F')(G)$ and the last bundle is gr E by (i). This proves (ii), and (iii) follows immediately from (i). Proposition 3.12 (iii) and part (iii) above imply (iv).

We will now relate the algebraic notion of equivalence classes of semistable bundles to the transcendental notion of unitary bundles. We recall the definition of unitary bundles on the compact Riemann surface X (cf. [14], § 1, § 6). Let $x_0 \in X$ be a fixed point. Let $\Gamma = \pi_1(X - x_0)$ be the fundamental group of $X - x_0$ (with respect to some base point, the base point does not count since we will be concerned only with the isomorphism classes in the constructions below). Let $\gamma \in \Gamma$ be the element corresponding to the 'loop around x_0' ([14], p. 144). Let $\rho: \Gamma \to G$ be a homomorphism such that $\rho(\gamma) = C \in Z_0$. Let $\tilde{Z}_0$ be the Lie algebra of Z_0 and $\tilde{C} \in \tilde{Z}_0$ such that $\exp(\tilde{C}) = C$. Let E_ρ be the G-bundle on $X - x_0$ associated to the universal covering $\widetilde{X - x_0} \to X - x_0$, which is a Γ-bundle, by the homomorphism ρ. Let D be a neighbourhood of x_0 in X isomorphic to the unit disc. Let H be the upper half plane. Then $H \to D - x_0$, $z \mapsto \exp 2\pi i z$, is the universal covering and $\psi: H \to Z_0 \subset G$ defined by $\psi(z) = \exp(-z\tilde{C})$ gives a section of E_ρ over $D - x_0$. We define $E(\rho, \tilde{C})$ to be the G-bundle obtained by patching up the trivial G-bundle on D with E_ρ on $X - x_0$ with the help of the trivialization of E_ρ on $D - x_0$ given by ψ (cf. [14], § 6 for more details).

3.14. DEFINITION

We call a G-bundle a *unitary G-bundle* if it is isomorphic to a G-bundle $E(\rho, \tilde{C})$ constructed as above for some $\rho: \Gamma \to G$ such that $\rho(\Gamma) \subset K$, a maximal compact subgroup of G.

3.15. PROPOSITION

A semistable G-bundle E is unitary if and only if $E \approx \mathrm{gr}\, E$.

Proof. Let $E = E(\rho, \tilde{C})$ be an unitary bundle. Suppose ρ is irreducible, i.e. the only elements of $\mathscr{G}$ fixed by ad $\rho(h)$ for every $h \in \Gamma$ are those of the center of $\mathscr{G}$ ([14, § 1). Then by ([14], Theorem 7.1) E is stable. Therefore in this case $E \approx \mathrm{gr}\, E$. Suppose ρ is not irreducible. Then by ([14], Proposition 2.1) $\rho(\Gamma)$ leaves a proper parabolic subalgebra $\mathscr{P}$ of $\mathscr{G}$ invariant and hence $\rho(\Gamma) \subset P$, the subgroup corresponding to $\mathscr{P}$. Since ρ is unitary $\rho(\Gamma) \subset K \cap P \subset M$ where M is a Levi component of P. This implies that $E(\rho, \tilde{C})$ has a reduction of structure group to M. This reduction considered as a reduction to P is admissible. For let $\chi: P \to \mathbb{C}^*$ be a character on P which is trivial on Z_0. Then by ([14], Remark 6.1, p. 145), $\chi_* E(\rho, \tilde{C}) \approx E(\chi \circ \rho, \tilde{\chi}(\tilde{C}))$ where $\tilde{\chi}$ is the morphism induced by χ on the universal coverings. Since $\tilde{C} \in \tilde{Z}_0$ and χ is trivial on Z_0, $\tilde{\chi}(\tilde{C}) = 0$. Therefore $\deg(\chi_*(E(\rho, \tilde{C})) = 0$. We have for this reduction $E[P, M](G) \approx E$. If $E[P, M]$ is a stable M-bundle we are through. If not we use induction on the semisimple rank of the structure group to conclude that $\mathrm{gr}\, E(\rho, \tilde{C}) \approx E(\rho, \tilde{C})$.

Conversely suppose $E \approx \mathrm{gr}\, E$. If E is stable by ([14], Theorem 7.1) E is an unitary G-bundle. If E is not stable, let σ be an admissible reduction of structure group of E to the proper parabolic subgroup $P = M \cdot U$, such that $E' = E[P, M]$ is a stable M-bundle.

Then $\operatorname{gr} E \cong E'(G)$ (Proposition 3.12). By ([14], Theorem 7.1) $E' \cong E'(\rho', \tilde{C}')$ for some $\rho':\Gamma \to K' \subset M$, where K' is a maximal compact subgroup of M, $\rho(\gamma) = C' \in Z_0[M]$ and $\tilde{C}' \in \tilde{Z}_0[M]$ such that $\exp \tilde{C}' = C'$. We then make the following

Claim. $\tilde{C}' \in \tilde{Z}_0 \subset \tilde{Z}_0[M]$.

Since the reduction σ is admissible $\deg(\chi_* E[P,M]) = 0$ for any character χ on P which is trivial on Z_0. But $\chi_*(E[P,M]) \approx E'(\chi \circ \rho', \tilde{\chi}(\tilde{C}'))$ by ([14], Remark 6.1). Therefore $\tilde{\chi}(\tilde{C}') = 0$. Since any character of $Z_0[M]$ which is trivial on the finite group $Z_0[M] \cap [M,M]$ extends to a character of M and hence of P we see that the group of characters of P (resp. the group of characters of P which are trivial on Z_0) is a subgroup of finite index in the group of characters of $Z_0[M]$ (resp. in the group of characters of $Z_0[M]$ which are trivial on Z_0). This shows that if $\tilde{C}' \notin \tilde{Z}_0$ there will be a character χ on P, trivial on Z_0, such that $\tilde{\chi}(\tilde{C}') \neq 0$. This contradiction proves the claim.

Let $\rho = i \cdot \rho'$ where $i: M \subsetneq G$ is the inclusion. Then $E \approx \operatorname{gr} E \approx E'(G) \approx E(\rho, \tilde{C})$ where $\tilde{C} = \tilde{\imath}(\tilde{C}')$.

3.15.1. COROLLARY

For any semistable G-bundle E the G-bundle gr *E is unitary. Associating E to* gr *E gives a bijection between the set of equivalence classes of semistable G-bundles and the set of isomorphism classes of unitary G-bundles.*

Proof. This follows immediately from Proposition 3.12 (ii) and the preceding proposition.

3.16. *Lemma. Let P be a maximal parabolic subgroup of G and χ the dominant character of P which generates the group of characters of P/Z_0. Then there is an irreducible representation $\rho: G \to SL(V)$ such that $\rho(Z_0) = 1$. There is a line $\{v\} \subset V$ whose stabilizer is precisely P and P acts by the character χ on the line $\{v\}$.*

Proof. Let $G' = G/Z_0$ and P' = image of P in G'. Let $\pi: \tilde{G}' \to G'$ be the universal covering group of G'. Then $\tilde{P}' = \pi^{-1}(P')$ is a maximal parabolic subgroup of $\tilde{G}'$. Let $\rho': \tilde{G}' \to SL(W)$ be the fundamental representation corresponding to $\tilde{P}'$. Then there is a line $\{w\} \subset W$ whose stabilizer is $\tilde{P}'$ and $\tilde{P}'$ acts on $\{w\}$ by the dominant character $\tilde{\chi}$ which generates the character group of $\tilde{P}'$. Since χ is dominant $\chi \circ \pi = \tilde{\chi}^n$, for an integer $n > 0$. Then the irreducible G-subspace V generated by $v = w \otimes w \otimes \cdots \otimes w$ in $W \otimes W \otimes \cdots \otimes W$ (n factors) is actually a representation space for G' (since the highest weight $\tilde{\chi}^n$ goes down to G') and hence for G. Thus v and V satisfy the conditions of the lemma.

3.17. PROPOSITION

Let $\rho: G_1 \to G_2$ be a homomorphism between the reductive, connected algebraic groups G_1 and G_2 such that $\rho(Z_0[G_1]) \subset Z_0(G_2)$. Then

(i) *if E is a semistable G_1-bundle then the associated G_2-bundle $\rho_* E$ is semistable.*
(ii) *if the kernel of the homomorphism $G_1/Z_0[G_1] \to G_2/Z_0[G_2]$ induced by ρ is finite, then E is semistable if and only if $\rho_* E$ is semistable.*

Proof. (i) Let E be a semistable G_1-bundle. We first prove the case when $G_2 = GL(V)$. So let $\rho: G_1 \to GL(V)$ be a representation such that $Z_0[G_1]$ acts on V through the

character $\chi: Z_0[G_1] \to \mathbb{C}^*$. Since E is semistable there is an admissible reduction of structure group of E to a parabolic subgroup $P = M \cdot U$ such that $E[P, M]$ is a stable M-bundle. Then by ([14], Theorem 7.1, p. 146) $E[P, M] \approx E'(h, \tilde{C})$ for an unitary representation $h: \Gamma \to K', K'$ a maximal compact subgroup of M and $\tilde{C} \in \tilde{Z}_0[M]$. Since σ is admissible $\tilde{C} \in \tilde{Z}_0[G_1]$ (as in the claim in the proof of Proposition 3.15).

Let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ be the flag in V defined by the condition that V_i/V_{i-1} is the largest subspace of $V/V_{i-1}, i = 1, \ldots, n$ on which U acts trivially. This flag is invariant under the action of P. Since σ is admissible by Lemma 3.5.8

$$\mu(E[P](V_i/V_{i-1})) = \mu(E(V)). \tag{1}$$

Let $P' = M' \cdot U'$ be the parabolic subgroup of $GL(V)$ which is the stabilizer of the flag $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$. We can assume (by taking a conjugate of M' if necessary) that $\rho(M) \subset M'$. The flag of sub-bundles $E[P](V_0) \subset E[P](V_1) \subset \ldots \subset E(V)$ gives a reduction of structure group of the $GL(V)$-bundle $E_2 = \rho_* E$ to the subgroup P' which is admissible because of (1) (Remark 3.4). Now $E_2[P', M']$ is isomorphic to $E[P, M](M')$, the M'-bundle got from $E[P, M]$ by the extension of structure group $\rho: M \to M'$. By ([14], Remark 6.1, p. 145) $E_2[P', M'] \approx E(\rho \circ h, \tilde{\rho}(\tilde{C}))$. (Note that since $\tilde{C} \in \tilde{Z}_0[G_1]$ and $Z_0[G_1]$ acts by scalars $\tilde{\rho}(\tilde{C}) \in \tilde{Z}_0[GL(V))]$). Therefore $E_2[P', M']$ is semistable by ([14], Proposition 2.2). Therefore by Lemma 3.5.11, E_2 is semistable.

Now let G_2 be arbitrary, and E a semistable G_1-bundle. We shall show $E_2 = \rho_* E$ is semistable. Let σ be a reduction of structure group of E_2 to a maximal parabolic subgroup P of G_2. Let λ be the dominant character of P which generates the character group of $P/Z_0[G_1]$. By Lemma 3.16, there is a representation $h: G_2 \to SL(V)$ of G_2 on a vector space V such that $Z_0[G_2]$ acts trivially on V, there is a line $\{v\} \subset V$ whose stabilizer is precisely P, and P acts on the line $\{v\}$ by the character λ. Then as proved above $h_* E_2 = h_* \rho_* E$ is a semistable vector bundle (of degree 0 since $h(G_2) \subset SL(V)$). Further $\lambda_* E_2[P]$ is a sub-bundle of $h_* E_2$ (corresponding to $\{v\} \subset V$). Therefore $\deg(\lambda_* E_2[P]) \leqslant 0$, which shows that σ satisfies the semistability condition.

(ii) Now suppose $G_1/Z_0[G_1] \to G_2/Z_0[G_2]$ has finite kernel and $E_2 = \rho_* E$ is semistable. Let $h: G_2 \to SL(V)$ be a representation such that $\ker h = Z_0[G_2]$. Let $P = M \cdot U$ be a maximal parabolic subgroup of G_1 and σ a reduction of structure group of E to P. Clearly there is a G_2-invariant subspace V_1 of V such that V_1 is neither 0 nor V and U acts nontrivially on V_1. Let V_2 be the largest subspace of V_1 on which U acts trivially. Then $V_2 \neq 0$ by Lie–Kolchin theorem ([2], (10.5), p. 243) and $V_2 \neq V_1$. Also V_2 is stable under P. Let $r = $ rank of V_2. Let $W = \overset{r}{\wedge} V_1$ and $\{w\} \subset W$, the line $\overset{r}{\wedge} V_2$ corresponding to $V_2 \subset V_1$. Then P acts on the line $\{w\}$ by a character λ^n for some $n > 0$, where λ is the dominant character of P which generates the group of characters of $P/Z_0[G_1]$. Now since E_2 is semistable by part (i) $E_2(W)$ is a semistable vector bundle (of degree 0).

Also $E_2(W) \approx E[P](W)$ and $\lambda_*^n E[P] \approx E[P](\{w\}) \subset E[P](W)$. Since $E[P](W)$ is semistable we have $\deg \lambda_*^n E[P] \leqslant 0$ which shows that σ satisfies the semistability condition.

3.18. COROLLARY

Let E be a G-bundle and $\mathrm{Ad}\, E$ *be its adjoint bundle. Then E is semistable if and only if* $\mathrm{Ad}\, E$ *is semistable.*

Proof. This follows immediately from the preceding proposition.

3.19. *Lemma. Let A and B be $\mathbb{C}$-algebras of finite type. Let J be an ideal of* $A \otimes_{\mathbb{C}} B$. *Then there is an unique ideal J_1 of A such that for any $\mathbb{C}$-algebra C and any homomorphism* $f: A \to C$, $\ker f$ *contains J_1 if and only if* $\ker(f \otimes id_B)$ *contains J.*

Moreover, if A is an $\mathbb{N}^r$-graded algebra ($\mathbb{N}$ = set of natural numbers) and J is a homogeneous ideal of $A \otimes_{\mathbb{C}} B$ where $A \otimes_{\mathbb{C}} B$ is given the gradation taking B as an algebra with trivial gradation, then J_1 is a homogeneous ideal of A.

Proof. Let $\{I_i\}$ be the collection of all ideals of A with the property $I_i \otimes_{\mathbb{C}} B \supset J$. Let $J_1 = \bigcap_i I_i$. For any collection of subspaces $\{S_j\}$ of A, $\left(\bigcap_j S_j\right) \otimes_{\mathbb{C}} B = \bigcap_j \left(S_j \otimes_{\mathbb{C}} B\right)$ we have $J_1 \otimes_{\mathbb{C}} B \supset J$. It is easy to check that J_1 has the required properties.

If A is graded by the subspaces $\{A_n\}_{n \in \mathbb{N}^r}$ and J is homogeneous then $J = \bigoplus_n J \cap \left(A_n \otimes_{\mathbb{C}} B\right)$. Since $J \cap \left(A_n \otimes_{\mathbb{C}} B\right) \subset \left(J_1 \otimes_{\mathbb{C}} B\right) \cap \left(A_n \otimes_{\mathbb{C}} B\right) = (J_1 \cap A_n) \otimes_{\mathbb{C}} B$, for the homogeneous ideal $J'_1 = \bigoplus_n J_1 \cap A_n \subset J_1$ we have $J'_1 \otimes_{\mathbb{C}} B \supset J$. Therefore $J'_1 = J_1$, proving the homogeneity of J_1.

3.20. *Lemma. Let H and Y be schemes and D a closed subscheme of $H \times Y$. Then there exists a unique closed subscheme H_1 of H such that, for any morphism $f: S \to H$ from a scheme S, the morphism $f \times id_Y: S \times Y \to H \times Y$ factors through $D \subsetneq H \times Y$ if and only if f factors through $H_1 \subsetneq H$.*

Proof. We can assume H, Y to be affine and then apply the previous lemma.

We shall now prove a lemma which is essentially the same as ([14], Lemma 4.1). We need it in this form.

Let Y be a projective scheme and L a very ample line bundle on Y. Let T be a scheme and $V_i \to T \times X$, $i = 1, \ldots, r$, vector bundles on $T \times X$. Let $\underline{V}_i$ be the sheaf of sections of V_i. Then the scheme V_i is $\mathrm{Spec}(S(\underline{V}_i^*)$ where $\underline{V}_i^*$ is the dual sheaf of $\underline{V}_i$ and $S(\underline{V}_i^*)$ is the symmetric algebra of $\underline{V}_i^*$. Let $\underline{V} = \oplus_{i=1}^r \underline{V}_i$. Then $V = \mathrm{Spec}(S(\underline{V}_1^*) \otimes \cdots \otimes S(\underline{V}_r^*))$, since $S(\underline{V}^*) = S(\underline{V}_1^*) \otimes \cdots \otimes S(\underline{V}_r^*)$. The $S(\underline{V}_i^*)$ are $\mathbb{N}$-graded algebras and hence $S(\underline{V}^*)$ has a natural $\mathbb{N}^r$-gradation. Let $C \subset V$ be a closed subscheme of V and $\mathscr{I}$ the sheaf of ideals of C in V. We call C a *multicone* if $\mathscr{I}$ is homogeneous with respect to the $\mathbb{N}^r$-gradation of $S(\underline{V}^*)$.

Let (Sch/T) be the category of schemes over T. Let $\pi = \pi(C/T \times Y): (\mathrm{Sch}/T) \to (\mathrm{Sets})$ be the functor which associates to an $(S \to T) \in (\mathrm{Sch}/T)$ the set $\mathrm{Hom}_{T \times Y}(S \times Y, C)$ of $T \times Y$-morphisms of $S \times Y$ into C. For a T-morphism $f: S' \to S$, $\pi(f)$ is defined to be pulling back by f.

Let $p_T: T \times Y \to T$ and $p_y: T \times Y \to Y$ be the projections. Fix an integer m sufficiently large such that $H^1(Y, V_{j,t}(m)) = 0$ for all $t \in T$, where $V_{j,t}(m)$ is the restriction of $V_j(m) = V_j \otimes p_Y^*(L^m)$ to $t \times Y = Y$([11], Lecture 7, p. 49).

Let $\mathscr{H}_j = p_{T*}(V_j(m))$ and $\mathscr{H} = p_{T*}(V(m)) = \oplus_{j=1}^r \mathscr{H}_j$. Then $\mathscr{H}_j$ and $\mathscr{H}$ are locally free ([11], Lecture 7, p. 51). Let $H_j = \mathrm{Spec}(S(\mathscr{H}_j^*))$ be the vector bundle on T corresponding to the locally free sheaf $\mathscr{H}_j$ and $H = \mathrm{Spec}(S(\mathscr{H}^*)) = \mathrm{Spec}(S(\mathscr{H}_1^*) \ldots S(\mathscr{H}_r^*))$.

3.21. *Lemma. The functor π defined above is representable by a closed subscheme N of the scheme H.*

If C is a multicone in $V = \oplus_{j=1}^{r} V_j$ then N is a multicone in $H = \oplus_{j=1}^{r} H_j$.

3.21.1. COROLLARY

If C is a multicone then the set

$$\left\{ t \in T \,\middle|\, \begin{array}{l} \exists(\sigma_1, \ldots, \sigma_r) \in \pi(t \hookrightarrow T) \subset H^0(Y, V_t) = \bigoplus_{j=1}^{r} H^0(Y, V_{j,t}) \text{ such} \\ \text{that } \sigma_j \neq 0 \text{ as a section of } V_{j,t} \forall j = 1, \ldots, r \end{array} \right\}$$

is a closed subset of T.

Proof. Let $s \in H^0(Y, L^m)$ be a non-zero section and D the divisor of zeros of s. We then have an exact sequence

$$0 \to O \to L^m \to L^m \otimes O_D \to 0, \tag{1}$$

where O (resp. O_D) is the structure sheaf of Y (resp. D) (see [11], p. 63). Pulling back (1) by p_Y and tensoring by V we get the following exact (p_Y is flat, V is locally free) sequence on $T \times Y$.

$$0 \to V \to V(m) \to \mathcal{T} \to 0, \tag{2}$$

where $\mathcal{T} = V(m) \otimes p_Y^* O_D$.

Let $\pi_m = \pi(V(m)/T \times Y)$ (resp. $\pi_0 = \pi(V/T \times Y)$) be the functor from (Sch/T) to (Sets) which associates to $(f: S \to T)$ the set $\mathrm{Hom}_{T \times Y}(S \times Y, V(m)) = H^0(S \times Y, (f \times id_Y)^*(V(m))$ (resp. $\mathrm{Hom}_{T \times Y}(S \times Y, V) = H^0(S \times Y, (f \times id_Y)^*(V))$).

Pulling back (2) by $f \times id_Y$ to $S \times Y$ (which is equivalent to pulling back (1) by the flat morphism $p_Y \circ (f \times id_Y)$ and tensoring by the locally free sheaf $(f \times id_Y)(V)$) we get the following exact sequence on $S \times Y$

$$0 \to (f \times id_Y)^*(V) \xrightarrow{\alpha} (f \times id_Y)^*(V(m)) \xrightarrow{\beta} (f \times id_Y)^* \mathcal{T} \to 0. \tag{3}$$

The cohomology exact sequence of (3) gives $\pi_0(S) \subset \pi_m(S)$ so that π_0 is a subfunctor of π_m. Since C is a subscheme of V clearly $\pi(S) \subset \pi_0(S)$. We shall prove that π_m is represented by the T-scheme $H = \mathrm{Spec}(S(\mathcal{H}^*)) \xrightarrow{p} T$ and π_0 and π are represented by closed subschemes of H.

Since p_T is a projective morphism and $V(m)$, being locally free, is flat over T and $H^1(Y, V_t(m)) = 0$ for every $t \in T$ we have ([11], Lecture 7, pp. 50–51) $\mathcal{H}$ to be locally free and for any morphism $f: S \to T$ the natural morphism

$$f^*(\mathcal{H}) \to p_{S*}((f \times id_Y)^*(V(m))) \tag{*}$$

is an isomorphism. We then have the following sequence of natural isomorphisms:

$$\begin{aligned} \pi_m(S) &\simeq H^0(S \times Y, (f \times id_Y)^*(V(m))) \\ &\simeq H^0(S, p_S((f \times id_Y)^*(V(m))) \\ &\simeq H^0(S, f^*(\mathcal{H})), \text{ (by (*))} \end{aligned}$$

$$\simeq \mathrm{Hom}_S(S, f^*(H))$$
$$\simeq \mathrm{Hom}_T(S, H).$$

It follows that π_m is represented by H.

Let σ be the universal section in $H^0(H \times Y, (p \times id_Y)^*(V(m)))$ (i.e. the element corresponding to identity in $\mathrm{Hom}_T(H, H)$ in the above identifications). Let σ_D be the restriction of σ to $H \times D$. On $H \times Y$ we have the exact sequence (corresponding to (3))

$$0 \to (p \times id_Y)^*(V) \xrightarrow{\alpha'} (p \times id_Y)^*(V(m)) \xrightarrow{\beta'} (p \times id_Y)^*(\mathscr{T}) \to 0. \tag{3$'$}$$

The sheaf $(p \times id_Y)^*\mathscr{T}$ has support on $H \times D$ and its restriction to $H \times D$ is a vector bundle which we denote by W. For the vector bundle $W \to H \times D$, $\beta' \circ \sigma_D$ gives a section. Let M be the closed subscheme of $H \times D$ which is the pull back of the zero section of W by $\beta' \circ \sigma_D$. Apply Lemma 3.20 to get a closed subscheme N_0 of H such that any morphism $f: S \to H$ factors through N_0 if and only if $f \times id_D: S \times D \to H \times D$ factors through M. It is easy to see that the ideal sheaf of M in $H \times D$ is homogeneous with respect to the natural $\mathbb{N}^r$-gradation induced from $S(\mathscr{H}^*)$. Therefore by Lemma 3.19 the ideal sheaf of N_0 in H is also homogeneous.

We claim that N_0 represents the functor π_0. It is enough to show that the element $(f \times id_Y)^*(\sigma)$ in $\pi_m(S)$ corresponding to $f \in \mathrm{Hom}(S, H)$ lies in $\pi_0(S)$ if and only if f factors through N_0. Since the restriction of $\beta' \circ \sigma$ to $N_0 \times Y$ (i.e. pull back by $N_0 \times Y \subsetneq H \times Y$) is zero we have that if f factors through N_0 then $(f \times id_Y)^*(\sigma)$ is zero. This implies by (3), $(f \times id_Y)^*(\sigma) \in \pi_0(S)$. Conversely if $(f \times id_Y)^*(\sigma) \in \pi_0(S)$ using (3), $\beta \circ ((f \times id_Y)^*(\sigma)) = (f \times id_Y)^*(\beta' \circ \sigma)$ is zero. Hence $(f \times id_D)^*(\beta' \circ \sigma_D)$ is zero. But this implies that $f \times id_D$ factors through M and hence f factors through N_0.

Therefore $p_0: N_0 \to T$ represents π_0 where p_0 is the restriction of $p: H \to T$. The universal section in $H^0(N_0 \times Y, (p_0 \times id_Y)^*(V))$ is σ_0 given by the pull back of σ by $N_0 \times Y \subsetneq H \times Y$. Let M_1 be the subscheme of $N_0 \times Y$ which is the pull back of the subscheme $(p_0 \times id_Y)^*(C) \subset (p_0 \times id_Y)^*(V)$ by σ_0. If C is a multicone in V the sheaf of ideals of M_1 in $N_0 \times Y$ is homogeneous with respect to the natural $\mathbb{N}^r$-gradation of the structure sheaf of $N_0 \times Y$. (Since the ideal sheaf of N_0 in H is homogeneous the structure sheaf of N_0 has a natural $\mathbb{N}^r$-gradation which gives the $\mathbb{N}^r$-gradation on the structure sheaf of $N_0 \times Y$.)

Again applying Lemma 3.20 we get a closed subscheme $N \subset N_0$ such that any morphism $f: S \to N_0$ factors through N_1 if and only if $f \times id_Y: S \times Y \to N_0 \times Y$ factors through M_1. It is then easy to see that N represents the functor π. By Lemma 3.19 if C is a multicone the ideal sheaf of N in N_0 (and hence in H) is homogeneous. This completes the proof of the lemma.

To prove the corollary note that if C is a multi-cone in V we have proved that

$$N = \mathrm{Spec}(S(\mathscr{H}_1^*) \otimes \cdots \otimes S(\mathscr{H}_r^*)/J)$$

where J is an $\mathbb{N}^r$-graded ideal of $S(\mathscr{H}_1^*) \otimes \cdots \otimes S(\mathscr{H}_r^*)$. Such an ideal defines a closed subscheme Q of $\mathrm{Proj}(S(\mathscr{H}_1^*)) \times \cdots \times \mathrm{Proj}(S(\mathscr{H}_r^*))$ and clearly the set defined in the corollary is $\tau(Q)$ where $\tau: \mathrm{Proj}(S(\mathscr{H}_1^*)) \times \cdots \times \mathrm{Proj}(S(\mathscr{H}_r^*)) \to T$ is the projection. Since τ is a proper morphism $\tau(Q)$ is closed in T.

3.22. *Lemma. Let $\xi \to S \times X$ be a family of semistable G-bundles. Let P be a maximal parabolic subgroup of G. Let χ be the dominant character on P which generates the group*

of characters of P which are trivial on Z_0. Let L be a line bundle of degree zero on X. Then the set

$$S_{P,L} = \left\{ s \in S \,\middle|\, \begin{array}{l} \xi_s \text{ has an admissible reduction } \sigma \text{ of structure} \\ \text{group to } P \text{ such that } \chi_* \sigma^* E \approx L \end{array} \right\}$$

is a closed subset of S.

Proof. Let $Y = S \times X$ and $\xi_1 = S \times L$. Then $\xi_1 \underset{Y}{\times} \xi$ is a $\mathbb{C}^* \times G$-bundle on Y. Let $\bar{P}$ be the subgroup $\{(\chi(p), p) | p \in P\}$ of $\mathbb{C}^* \times G$. It is a closed algebraic subgroup of $\mathbb{C}^* \times G$ since it is the image of the homomorphism $P \to \mathbb{C}^* \times G$, $p \mapsto (\chi(p), p)$ ([2], (1.4), p. 88). Under the projection $\mathbb{C}^* \times G \to G$, $\bar{P}$ maps onto P. Therefore, the projection $\xi_1 \underset{Y}{\times} \xi \to \xi$ induces $\xi_1 \underset{Y}{\times} \xi/\bar{P} \to \xi/P$. A reduction of structure group of $\xi_1 \underset{Y}{\times} \xi$ to $\bar{P}$ gives, by composing with $\xi_1 \underset{Y}{\times} \xi/\bar{P} \to \xi/P$, a reduction of structure group of ξ to P and it is easily checked that $\chi_*(\xi[P])$ is isomorphic to ξ_1. Conversely one can check that given a reduction of structure group σ of ξ to P such that $\chi_* \sigma^* \xi$ is isomorphic to ξ_1 then there is a reduction of structure group of $\xi_1 \underset{Y}{\times} \xi$ to $\bar{P}$ which when composed with $\xi_1 \underset{Y}{\times} \xi/\bar{P} \to \xi/P$ gives back σ. By Lemma 3.16 there is an irreducible representation $\rho: G \to SL(V)$ with a line $\{v\} \subset V$ such that $P = \{g \in G / \rho(g)\{v\} = \{v\}\}$ and $\rho(p)v = \chi(p)v$ for $p \in P$. Let $\mathbb{C}^* \times G$ act on $\mathrm{Hom}(\mathbb{C}, V)$ by $(z, g)f = \rho(g) \circ f \circ z^{-1}$ where $z \in \mathbb{C}^*$, $g \in G$, $f \in \mathrm{Hom}(\mathbb{C}, V)$ and we have denoted by z^{-1} the multiplication by the scalar $z^{-1} \in \mathbb{C}^*$. Let $f_0 \in \mathrm{Hom}(\mathbb{C}, V)$ be defined by $f_0(z) = z \cdot v$. Then the stabilizer of f_0 in $\mathbb{C}^* \times G$ is $\bar{P}$ and hence $\mathbb{C}^* \times G \to \mathrm{Hom}(\mathbb{C}, V)$ is given by $(z, g) \mapsto (z, g)\ f_0$ which induces an isomorphism of schemes of $\mathbb{C}^* \times G/\bar{P}$ with the orbit C of f_0 under $\mathbb{C}^* \times G$ (C is locally closed in $\mathrm{Hom}(\mathbb{C}, V)$ and endows it with the canonical reduced subscheme structure, (cf. [2], (1.8), p. 98)). Let $\bar{C}$ be the closure of C in $\mathrm{Hom}(\mathbb{C}, V)$. Since C is invariant under scalar multiplication in $\mathrm{Hom}(\mathbb{C}, V)$, $\bar{C}$ taken with the canonical reduced scheme structure is a cone in $\mathrm{Hom}(\mathbb{C}, V)$. We claim that $\bar{C} = C \cup \{0\}$. Clearly $0 \in \bar{C}$. Suppose $0 \neq f \in \bar{C}$. Since C is locally closed, $\bar{C}$ is also the closure of C with respect to the strong (Hausdorff) topology. Therefore, there is a sequence $\rho(g_n) \circ f_0 \circ z_n^{-1}$ tending to f, with $g_n \in G$, $z_n \in \mathbb{C}^*$. Let K be a maximal compact subgroup of G. Then $G = K \cdot P$ ([14], proof of Proposition 2.1, p. 130). Write $g_n = k_n p_n$ with $k_n \in K$, $p_n \in P$. Since K is compact we can assume that $\lim_{n \to \infty} k_n = k$. Then $\lim_{n \to \infty} \rho(p_n) \circ f_0 \circ z_n^{-1} = \rho(k)^{-1} \circ f$ so that $(\rho(p_n) \circ f_0 \circ z_n^{-1})(z) = (\chi(p_n) z_n^{-1} z)v$ tends to $\rho(k)^{-1} f(z)$. This implies $\lim_{n \to \infty} \chi(p_n) z^{-1} = z_0$. Therefore $\rho(k)^{-1} f(z) = z_0 z \cdot v = (f_0 \circ z_0)(z)$, since $f \neq 0$, $z_0 \neq 0$ and we have $f = (z_0^{-1}, k) f_0$. Therefore $f \in C$. Thus we have proved $\bar{C} = C \cup \{0\}$.

A reduction of structure group of $\xi_1 \underset{Y}{\times} \xi$ to $\bar{P}$ is given by a section of $\xi_1 \underset{Y}{\times} \xi/\bar{P} = (\xi_1 \underset{Y}{\times} \xi)(\mathbb{C}^* \times G/\bar{P}) = (\xi_1 \underset{Y}{\times} \xi)(C)$. The inclusions $C \subset \bar{C} \subset \mathrm{Hom}(\mathbb{C}, V)$ induce the inclusions $(\xi_1 \underset{Y}{\times} \xi)(C) \subset (\xi_1 \underset{Y}{\times} \xi)(\bar{C}) \subset (\xi_1 \underset{Y}{\times} \xi)(\mathrm{Hom}(\mathbb{C}, V))$ (See § 2.4).

By Lemma 3.21, the set $S'_{P,L}$ of points $s \in S$ such that $(\xi_1 \underset{Y}{\times} \xi)_s(\bar{C}) = (L \times \xi_s)(\bar{C})$ has a non-zero section is a closed subset of S. We shall now show that $S'_{P,L} = S_{P,L}$. Clearly $S_{P,L} \subset S'_{P,L}$. Suppose $s \in S'_{P,L}$. Then we have a non-zero homomorphism $\sigma: L \to \xi_s(V)$. Since $\deg L = 0$ and $\xi_s(V)$ is a semistable vector bundle of degree zero (Proposition 3.17) it follows that σ is an injection ([19], Proposition 3.1, p. 306). This implies that σ factors

through the open immersion $(L \times \xi_s)(C) \subsetneq (L \times \xi_s)(\bar{C})$. Therefore $L \times \xi_s$ has a reduction to $\bar{P}$ and hence $s \in S_{P,L}$.

3.23. *Lemma. Let $\xi \to S \times X$ be a family of semistable G-bundles. Suppose there exist*

(i) *a dense open subset T of S*
(ii) *a reductive subgroup M_0 of G of maximal rank and*
(iii) *a stable M_0-bundle E_0*

with the following property: For every $t \in T$ the G-bundle $\xi_t \to X$ has an admissible reduction of structure group σ to a parabolic subgroup Q having M_0 as a Levi component such that the M_0-bundles $\xi_t[Q, M_0]$ and E_0 are isomorphic. Then for all $s \in S$ $\operatorname{gr} \xi_s$ *is isomorphic to $E_0(G)$ and hence the G-bundles ξ_s for any $s \in S$ are all equivalent.*

Proof. We can assume without loss of generality that S is reduced, irreducible and affine. We shall prove the lemma by induction on the semisimple rank of the structure group.

Suppose $M_0 = G$. Let $S \times E_0 \to S \times X$ be the trivial family of G-bundles given by E_0. Let $\rho: G \subsetneq GL(V)$ be a faithful representation and $V = V_1 \oplus \cdots \oplus V_r$ be a decomposition into irreducible subspaces. Let $\rho_i: G \to GL(V_i)$ be the representation of G on V_i induced by ρ. Define

$$C = \{(\rho_1 \lambda_1(g), \ldots, \rho_r \lambda_r(g)) \in GL(V) \mid \lambda_i \in \mathbb{C}^*, g \in G\}.$$

Let $\bar{C}$ be the closure of C in $\oplus_{i=1}^r \operatorname{End} V_i$. We consider $\bar{C}$ as a closed subscheme of $\oplus \operatorname{End} V_i$ with the reduced structure. Note that C is an open subscheme in $\bar{C}$ and $\bar{C}$ is a multicone in $\oplus \operatorname{End} V_i$. By the hypothesis of the lemma and assumption $M_0 = G$, for $t \in T$ we have an isomorphism $\varphi_t: E_0 \to \xi_t$. We can interpret φ_t as a section of the fiber bundle $\left(E_0 \underset{X}{\times} \xi_t\right)(G)$ with fiber G associated to the $G \times G$ bundle $E_0 \underset{X}{\times} \xi_t$ for the action of $G \times G$ on G given by $(g_1, g_2)(h) = g_2 h g_1^{-1}$, $g_1, g_2, h \in G$ (cf. [17], §3.5, Example, p. 1–19). Since ρ is faithful one can identify G with the locally closed subscheme (taken with reduced structure) $\rho(G)$ of $\bar{C} \subset \oplus \operatorname{End} V_i$. We extend the action of $G \times G$ to $\oplus \operatorname{End} V_i$ by setting $(g_1, g_2)f = \rho(g_2) \circ f \circ \rho(g_1)^{-1}, g_1, g_2 \in G$ and $f \in \oplus \operatorname{End} V_i$. Then φ_t gives a section of $(E_0 \times \xi_t)(\bar{C}) = \left((S \times E_0) \underset{S \times X}{\times} \xi\right)_t (\bar{C})$. By Corollary 3.21.1 it follows that $\left((S \times E_0) \underset{S \times X}{\times} \xi\right)_s (\bar{C})$ has a section for any $s \in S$ which induces a non-zero endomorphism of $E_0(V_i)$ for every i. Then by ([14], Proposition 3.1) such a section gives rise to a G-bundle isomorphism of E_0 with ξ_s. This proves the lemma when $M_0 = G$. In particular if G is of semisimple rank zero then $G \approx \mathbb{C}^{*n}$ and any reductive subgroup of maximal rank has to be $\mathbb{C}^{*n}$. Therefore we have proved the lemma in the case of semisimple rank zero. (When $G = \mathbb{C}^{*n}$, alternatively, the lemma follows by observing that the $\mathbb{C}^{*n}$-bundle ξ is merely an n-tuple of line bundles $\mathscr{L}_i \to S \times X$ and by the hypothesis the morphism from S into the Jacobian (of suitable degree) of X determined by $\mathscr{L}_i$ is constant on T and hence on S.)

Now assume M_0 is a proper subgroup of G. Let $P_1, \ldots, P_r$ be a set of representatives of conjugacy classes of parabolic subgroups of G containing M_0 as a Levi component. Let $P'_1, \ldots, P'_r$ be maximal parabolic subgroups such that $P_i \subset P'_i$, $i = 1, \ldots, r$. Let χ_i be the

dominant character of P'_i and let $L_i = \chi_{i*}(E_0)$. Since a reduction of structure group to P_i can be considered in a natural way as a reduction of structure group to $P'_i \supset P_i$, the hypothesis of the lemma implies that $T \subset \cup_{i=1}^r S_{P'_i, L_i}$ where $S_{P'_i, L_i}$ is the set of points $s \in S$ such that ξ_s has a reduction of structure group to P'_i such that $\chi_{i*}\sigma^* \xi_s \approx L_i$. Since T is dense in S and each $S_{P'_i, L_i}$ is closed (Lemma 3.22) we have $S \subset \cup_{i=1}^r S_{P'_i, L_i}$. But we have assumed that S is irreducible. Therefore $S = S_{P,L}$ for $P = P'_i$, $L = L_i$ for some i. Let $P = M \cdot U$ be a Levi decomposition chosen such that $M_0 \subset M$.

For this parabolic subgroup P and its dominant character χ let the cone $\bar{C} \subset \operatorname{Hom}(\mathbb{C}, V)$ be defined as in the proof of Lemma 3.22. Then $\left(\xi_1 \underset{Y}{\times} \xi\right)(\bar{C})$ is a cone in the vector bundle $\left(\xi_1 \underset{Y}{\times} \xi\right)(\operatorname{Hom}(\mathbb{C}, V)) \to S \times X$, where $Y = S \times X$. By Lemma 3.21, we have an S-scheme $p: H \to S$ representing the functor π defined by $\pi(S' \to S) = \operatorname{Hom}_{S \times X}\left(S' \times X, \left(\xi_1 \underset{Y}{\times} \xi\right)(\bar{C})\right)$. Moreover H is of the form $\operatorname{Spec}(S(\mathscr{F})/J)$ for a locally free sheaf $\mathscr{F}$ on S and a homogeneous ideal J of $S(\mathscr{F})$. Let $\varepsilon: S \to H$ be the zero section ([EGA II], § 8·3) and $H' = H - \varepsilon(S)$. Let $p': H' \to S$ be the restriction of p to H'. Let σ be the universal element in $\pi(H)$. Let $\xi'_1 = (p' \times id_X)^*(\xi_1)$ and $\xi' = (p' \times id_X)^*(\xi)$. Then σ gives a section of $\left(\xi'_1 \underset{Y}{\times} \xi'\right)(\bar{C}) \to H' \times X$. As in the proof of Lemma 3.22, this section factors through $\left(\xi'_1 \underset{Y}{\times} \xi'\right)(C) \subsetneq \left(\xi'_1 \underset{Y}{\times} \xi\right)(\bar{C})$. It follows that the G-bundle $\xi' \to H' \times X$ has an admissible reduction of structure group $\mathscr{K}'$ to $P = M \cdot U$. Let $\xi'' = \xi'[P, M]$. Since $\xi'_h \approx \xi_{p(h)}$ for $h \in H$, the reduction of structure group $\mathscr{K}'_h$ of ξ'_h induced by $\mathscr{K}'$ gives canonically a reduction of structure group of $\xi_{p(h)}$ which we denote by $\mathscr{K}_h$.

Let $\pi': H' \to \operatorname{Proj}(S(\mathscr{F})/J)$ and $\mathscr{F}: \operatorname{Proj}(S(\mathscr{F})/J) \to S$ be the natural morphisms ([EGA, II], § 8.3). Since $S = S_{P,L}$, we have that p' is surjective. Since $p' = \tau \circ \pi'$, τ is surjective. Therefore, τ being proper and surjective and S being irreducible there is an irreducible component of $\operatorname{Proj}(S(\mathscr{F})/J)$ which maps onto S. It follows from ([EGA, II], Corollary 8.3.6, p. 165) that the irreducible components of $\operatorname{Proj}(S(\mathscr{F})/J)$ are the images under π' of the irreducible components of H'. Therefore, there is an irreducible component H'' of H' which maps onto S. Let p'' be the restriction of p to H''. Let us denote by the same letter ξ'' the restriction of the M-bundle $\xi'' = \xi'[P, M]$ to $H'' \times X$. The open subset $T'' = p''^{-1}(T)$ of H'' is dense since H'' is irreducible. Let $t'' = p''^{-1}(t)$ for $t \in T$. Then $\xi''_{t''} \approx q_* \mathscr{K}^*_{t''} \xi_t$ where $q: P \to M$ is the projection. Since $t \in T$ there is a reduction of structure group w of ξ_t to a parabolic subgroup Q with M_0 as a Levi component such that $j_* q_{0*} w^* \xi_t \approx j_* E_0$ where $q_0: Q \to M_0$ is a projection and $j: M_0 \subsetneq G$ is the inclusion. We can assume that (P, Q) is compatible with $(\mathscr{K}_{t''}, w)$. Using Lemma 3.11 we see that we have reduction of structure group $\mathscr{K}_{t''} \cap w$ of the M-bundle $q_* \mathscr{K}^*_{t''} \xi_t \approx \xi_{t''}$ to the parabolic subgroup $M \cap Q$ of M which has M_0 as a Levi component. Also for this reduction $\xi''_{t*}[M \cap Q, M_0] \approx E_0$ by 3.11 (iv). This shows that for the M-bundle $\xi'' \to H'' \times X$ the conditions (i), (ii) and (iii) of the hypothesis of the lemma are satisfied (with 'T' $= T''$, 'M_0' $= M_0$ and 'E_0' $= E_0$). Since the semisimple rank of M is one less than that of G we can apply the induction hypothesis to conclude that $\operatorname{gr} \xi''_h \approx E_0(M)$ for all $h \in H''$. But this implies that $\operatorname{gr}(q_* \mathscr{K}^*_h \xi_{p(h)}) \approx E_0(M) \approx \operatorname{gr}(E_0(M))$. Therefore by 3.13(ii) $\operatorname{gr} \xi_s \approx E_0(G)$ for all $s \in S$.

3.24. PROPOSITION

Let E be a semistable G-bundle. Then

(i) *if in a family of semistable G-bundles* $\xi \to S \times X$ *we have* $\xi_t \approx E$ *for t varying in a dense open subset of S then* ξ_s *is equivalent to E for all* $s \in S$.
(ii) *there exists a family of semistable G-bundles* $\xi \to \mathbb{C} \times X$ *such that for* $0 \neq z \in \mathbb{C}$, $\xi_z \approx E$ *and* $\xi_0 \approx \mathrm{gr}\, E$.

Proof. (i) follows from the preceding lemma and (ii) follows from Propositions 3.5 and 3.12.

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Normal subgroups of $SL_{1,D}$ and the classification of finite simple groups

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Abstract. Let D be a division algebra of degree three over an algebraic number field K and let $\mathbb{G} = S\mathbb{L}_D$. We prove that the normal subgroup structure of $\mathbb{G}(K)$ is given by the Platonov–Margulis conjecture. The proof uses the classification of finite simple groups.

Keywords. Division algebra; finite simple group.

1. Introduction

Let G be a simple simply connected algebraic group over an algebraic number field K. To formulate the conjecture, due to Platonov and Margulis (cf. [11], ch. IX), describing the normal subgroup structure of $G(K)$, we let T be the (finite) set of all nonarchimedean places v of K such that G is K_v-anisotropic, and define $G(K, T)$ to be $\Pi_{v\in T} G(K_v)$ with the topology of the direct product if $T \neq \emptyset$, and let $G(K, T) = \{e\}$ if $T = \emptyset$ (which is always the case if G is not of type A_n). Let δ: $G(K) \to G(K, T)$ be the diagonal embedding in the first case, and the trivial homomorphism in the second case. Then the Platonov–Margulis conjecture is stated as follows:

> for any noncentral normal subgroup $N \subset G(K)$ there should be an open normal subgroup $W \subset G(K, T)$ such that $N = \delta^{-1}(W)$; in particular, if $T = \emptyset$, the group $G(K)$ should not have any proper noncentral normal subgroups (i.e. should be projectively simple) [PM].

(the topology on $G(K)$ induced from $G(K, T)$ is usually referred to as T-adic, and in this terminology (PM) asserts that every noncentral normal subgroup of $G(K)$ should be T-adically open (or equivalently, T-adically closed (cf. [11], 9.1))).

By now, this conjecture has been proved for almost all K-isotropic groups and most K-anisotropic groups of type different from A_n (cf. *loc. cit*). So, major attention is currently focussed on the main remaining case of anisotropic groups of type A_n. The first result dealing with (PM) for this case was obtained in [9]; it was shown that for $G = \mathbf{SL}_{1,D}$, D a quaternion division algebra such that $T = \emptyset$, the group $G(K)$ is perfect (i.e. $G(K) = [G(K), G(K)]$). Note that for $G = \mathbf{SL}_{1,D}$, the set T introduced in the statement of (PM) coincides with the set of non-archimedean places v of K such that $D_v = D \otimes_K K_v$ is a division algebra. Using some methods from [9], Margulis [8] proved (PM) for $G = \mathbf{SL}_{1,D}$ where D is an arbitrary quaternion algebra, in full. The technique of [9] was extended in [10] to division algebras of arbitrary degree to prove (under some minor restriction on the dyadic ramification places of D) that the commutator subgroup $[G(K), G(K)]$ is always T-adically open in $G(K)$. Subsequently, Raghunathan [12] lifted the restriction imposed on D in [10] and proved that, more generally, the

commutator subgroup $[U, U]$ of any T-adically open subgroup $U \subset G(K)$ is again T-adically open. Finally, Tomanov [17] gave reduction of (PM) for $G = \mathbf{SL}_{1,D}$ to division algebras of odd degree; in particular he proved (PM) for all algebras of degree 2^d. In the present paper we prove (PM) for $G = \mathbf{SL}_{1,D}$ where D is a division algebra of degree 3. This is the first result on (PM) for division algebras of odd degree, and by Tomanov's theorem it implies (PM) also for all division algebras of degree $3 \cdot 2^d$.

Theorem 1.1. *Let D be a central division algebra over K of degree 3, $G = \mathbf{SL}_{1,D}$. Then the normal subgroup structure of $G(K)$ is given by (PM).*
The proof of this theorem is based on methods which are essentially different from those employed in the previous works on (PM). Its main feature is the usage of (CSG), the classification of finite simple groups. (By (CSG) we mean the statement that any finite non-abelian simple group is either a group of Lie type or an alternating group, or else is one of the 26 sporadic groups, cf. [1].) To be more precise, (CSG) is used at two places in the proof of Theorem 1.1. One of them is so-called push-up theorem (cf. Theorem 2.1) which claims that if for $G = \mathbf{SL}_{1,D}$, D a division algebra of arbitrary degree, the group $G(K)$ fails to satisfy (PM), then there exists a normal subgroup $N \subset D^*$ such that the quotient D^*/N is a finite non-abelian simple group. Actually, the proof of this assertion uses not (CSG) itself but its following important consequence (Theorem 2.4): any class-preserving automorphism of a finite simple group F is inner.

This result suggests the following conjecture which can be regarded as a generalization of (PM) to division algebras over arbitrary field: let D be a finite dimensional algebra over an arbitrary field; then D^* does not have any normal subgroup N such that D^*/N is a non-abelian finite simple group.

In the present paper, we prove this conjecture for algebras of degree 2 and 3, which yields Theorem 1.1 and gives an alternate proof of Margulis' theorem [8]. The argument for algebras of degree 2 actually imitates the proof of Theorem 3 in [9] and does not use (CSG), while consideration of algebras of degree 3 is based on a new approach.

First, we show that all quotients of D^*, D a division algebra of degree 3, satisfy a certain abstract property (called here property (C), cf. § 3), and then prove that, on the contrary, none of the (known) finite non-abelian simple groups has this property (cf. Theorem 3.5). The proof of Theorem 3.5 takes the major part of this paper, and it is the other spot in the proof of Theorem 1.1 which depends on (CSG) (note that here the dependence on (CSG) is stronger than in the proof of the push-up theorem, viz. to handle the sporadic groups, we use detailed information about centralizers of elements from [1]).

We hope to apply the technique of this paper to other division algebras too. Apparently, this might be done via a more thorough analysis of "additive" properties of multiplicative normal subgroups in D^* of finite degree. (Some important results in this direction were obtained in [3], [18].) So, in § 7 we describe some additive properties that might enable us to prove (PM).

2. A push-up theorem

Theorem 2.1. *Let D be a finite dimensional division algebra over an algebraic number field K. If $\mathbf{SL}_{1,D}$ fails to satisfy (PM) then there exists a normal subgroup $N \subset D^*$ such that D^*/N is a non-abelian finite simple group.*

Proof. It is easy to show (cf. [11], Lemma 9.2) that the assertion of (PM) is true for a noncentral normal subgroup $N \subset SL_1(D)$ if and only if N is T-adically closed (in $SL_1(D)$). Therefore, our assumption entails the existence of a normal subgroup $M \subset SL_1(D)$ such that $M \neq \bar{M}$ where the bar denotes closure in $SL_1(D)$ with respect to the T-adic topology. Obviously, $\bar{M}$ contains a T-adically open subgroup $V \subset G(K)$ which is normalized by D^*. Then $\overline{M \cap V} = \bar{M} \cap V = V$ and

$$(\overline{M \cap V})/(M \cap V) \simeq \bar{M}/M \neq \{e\},$$

so we may (and we will) assume that $\bar{M}$ is normalized by D^*. Furthermore, by Margulis's finiteness theorem (cf. [7]), M is of finite index in $SL_1(D)$; in particular, the group $\bar{M}/M$ is finite. Pick a subgroup B, with $M \subset B \subset \bar{M}$, which is normal in $\bar{M}$ and such that $F = \bar{M}/B$ is a nontrivial simple group. As we mentioned in the introduction, by Raghunathan's theorem (cf. [12] or [11], Theorem 9.3) the commutator subgroup $[\bar{M}, \bar{M}]$ is T-adically open; therefore, F cannot be abelian, as $\bar{B} = \bar{M}$. If $m = [SL_1(D){:}M]$ then M contains the subgroup $C = \langle x^m \rangle$ generated by m-th powers of all elements $x \in SL_1(D)$. Then C is normalized by D^* and, again by Margulis's theorem, has finite index in $G(K)$. Since

$$C \subset R = \bigcap_{g \in D^*} (g^{-1}Bg),$$

R has finite index in $SL_1(D)$ as well. For any $g \in D^*$, we have $g^{-1}Bg \subset \bar{M}$ and

$$\bar{M}/(g^{-1}Bg) \simeq \bar{M}/B = F$$

implying that $\bar{M}/R$ is a direct product of a certain number of copies of F; in particular, $\bar{M}/R$ is a perfect group. To show that in fact $R = B$, or equivalently, B is normalized by D^*, we need the following statement (cf. [12], Proposition 3, or [11], Proposition 9.3).

PROPOSITION 2.2

Assume that $N \subset SL_1(D)$ is a non-central subgroup normalized by D^*, and for $x \in D^*$ let $C(x) = \{y \in D^* \mid [x, y] \in N\}$ (where, as usually, $[x, y] = xyx^{-1}y^{-1}$). Then for any $x \in \bar{N}$ we have $D^* = C(x)\bar{N}$.

An automorphism of a group is called class-preserving if it preserves all conjugacy classes.

COROLLARY 2.3

Inner automorphisms of D^* induce class-preserving automorphisms of $\bar{N}/N$. In particular, any intermediate subgroup $P, N \subset P \subset \bar{N}$, normalized by $\bar{N}$, is a normal subgroup in D^*.

To apply this fact in our situation, notice that $\bar{R} = \bar{M}$. Indeed, being of finite index in $SL_1(D)$, the subgroup R cannot be central, and then, it follows from the structure of local division algebras (cf. [14]) that $\bar{M}/\bar{R}$ is soluble; on the other hand, as we have seen above, in our case $\bar{M}/\bar{R}$ is perfect. Now, we obtain from the corollary above that B is a normal subgroup in D^*, as required. Using this corollary once more, we conclude that the action of D^* on $F = \bar{M}/B$ induced by inner automorphisms is class-preserving, and we may apply the following fact from the theory of finite simple groups (W. Feit, private communication).

Theorem 2.4. *Let F be a finite simple group. Then every class-preserving automorphism of F is inner.*

It follows that the image of the homomorphism $\alpha: D^* \to \mathrm{Aut}(F)$ defined by the above action, coincides with $\mathrm{Int}(F)$. So, letting $N = \mathrm{Ker}(\alpha)$ we have $D^*/N \simeq \mathrm{Int}(F) \simeq F$. Theorem 2.1 is proved. ■

COROLLARY 2.5

(PM) for $\mathbf{SL}_{1,D}$ is equivalent to the non-existence of normal subgroups $N \subset D^*$ such that D^*/N is a non-abelian finite simple group.

Indeed, as we already mentioned in the proof of Theorem 2.1, it follows from [14], that for any T-adically open normal subgroup $M \subset SL_1(D)$ the quotient $SL_1(D)/M$ is soluble. This fact combined with (PM) implies that for any normal subgroup $N \subset D^*$ of finite index, the image of $SL_1(D)$ in D^*/N is soluble, and therefore the group D^*/N is itself soluble since $SL_1(D) = [D^*, D^*]$.

At present, there does not seem to be a reasonable way to control all possible quotients of the multiplicative group of a division algebra; however, the following conjecture, which can be regarded as a generalization of (PM) to division algebras over arbitrary fields, appears to be plausible.

Conjecture (FSQ). Let D be a finite dimensional division algebra over an arbitrary field. Then D^* does not have any normal subgroup N such that D^*/N is a non-abelian finite simple group.

The rest of the paper is devoted to proving this fact for two particular cases.

Theorem 2.6. *Conjecture (FSQ) holds true for division algebras of degree two and three.*

3. Proof of theorem 2.6

First, we consider the case of D a quaternion division algebra. The argument here is based on the following statement.

Lemma 3.1 (cf. Lemma 4 in [9]). *Fix some maximal subfield $L \subset D$. Then any element $z \in SL_1(D)$ can be written in the form $z = [x, y]$ for suitable $x \in D^*$, $y \in L^*$ where $[x, y] = xyx^{-1}y^{-1}$ is the commutator of x and y.*

Proof. Fix $z \in SL_1(D)$ and consider the following linear homogeneous equation for y:

$$\mathrm{Trd}_{D/K}(y) = \mathrm{Trd}_{D/K}(zy). \tag{3.1}$$

Since $\dim_K(L) = 2$, (3.1) has a non-zero solution $y \in L$. Then the elements y and zy have the same characteristic polynomial and therefore, they are conjugate in D by the Skölem–Noether theorem. Thus, $zy = xyx^{-1}$ for some $x \in D^*$, and $z = [x, y]$ as required. ■

Let now $N \subset D^*$ be a normal subgroup such that $F = D^*/N$ is a finite non-abelian simple group. Pick a maximal subfield $L \subset D$ and let A denote the image of L^* in F. Then by Lemma 3.1, the set of commutators $\{[x, y] \mid x \in F, y \in A\}$ coincides with F and the proof is completed by the following elementary.

Lemma 3.2. *Let F be a finite simple group. If for some abelian subgroup $A \subset F$ the set of commutators $Y(A) = \{[x, y] \mid x \in F, y \in A\}$ coincides with F, then $F = \{e\}$.*

Proof. We have

$$Y(A) = \bigcup_{a \in A} Y(a)$$

where $Y(a) = \{[x, a] \mid x \in F\}$. Since A is abelian, any two elements in the same right coset modulo A define the same commutator implying that

$$\# Y(a) \leqslant [F:A]$$

for any $a \in A$. Besides, for the unity element $e \in A$ we have $Y(e) = \{e\}$. Therefore,

$$\# F = \# Y(A) \leqslant \sum_{a \in A \setminus \{e\}} Y(a) + 1 \leqslant (\# A - 1)[F:A] + 1 = \# F + (1 - [F:A]),$$

forcing $F = A$ and eventually $F = \{e\}$. ■

The given proof of Theorem 2.6 for quaternion division algebras is close to the proof of Theorem 3 in [9]. Note, that Theorem 3 in [9] combined with Theorem 2.4 above (which, as we learned later, was already known at that time) implies that if for a quaternion algebra D over an algebraic number field, the set T introduced in the statement of (PM), is empty then $SL_1(D)$ is projectively simple (Kneser's conjecture [6] generalized later in the form of (PM), and proved in that form for quaternions by Margulis [8]).

The proof of Theorem 2.6 for division algebras of degree 3 is much more complicated (and longer). First of all, we need to introduce one group-theoretic property. Let F be an abstract group. For any $x \in F$ let $C_F(x)$ denote the centralizer of x in F, and for $x, y \in F$ let

$$C_F(x, y) = \bigcup_{z \in y C_F(x)} C_F(z),$$

$$E_F(x, y) = C_F(x, y) C_F(x) C_F(x, y).$$

DEFINITION 3.3

We say that F has property (C) if $E_F(x, y) = F$ for any $x, y \in F$.
This property looks rather odd and, to the best of our knowledge, has never appeared in the literature before. However due to the following statement, it plays an important role in the proof of Theorem 2.6.

PROPOSITION 3.4

Let D be a division algebra of degree 3. Then its multiplicative group D^* satisfies property (C), and so does any of its quotients.

Proof. Clearly, it suffices to prove (C) for D^*. Fix $x, y \in D^*$ and put $L = K(x)$. Then obviously $L^* \subset C(x)$ (we shall omit the subscript D^*), and there is nothing to prove if either $x \in K$ or $y \in L$. Otherwise, $V = L + yL$ is a 6-dimensional K-subspace in D, and since $\dim_K D = 9$, we have $V \cap (aV) \neq (0)$ for any $a \in D^*$; in other words, $a \in (V \backslash (0))(V \backslash (0))^{-1}$. To prove that $E(x, y) = D^*$, it remains to notice that $V \backslash (0) \subset C(x, y) C(x)$. Indeed, let $t = c + yd$ where $c, d \in L^*$. If $c = 0$ then $t = yd \in C(y) C(x) \subset C(x, y) C(x)$. Otherwise,

$$t = (1 + y(dc^{-1}))c \in C(y(dc^{-1}))C(x) \subset C(x, y) C(x).$$

Proposition 3.4 is proved. ■

Now, Theorem 2.6 is a consequence of the following result which seems to be of independent interest.

Theorem 3.5. *None of the (known) finite simple groups has property* (C).
(Note, that by a "known" finite simple group we mean one of the following: an alternating group, a finite group of Lie type or one of 26 sporadic groups, and (CSG) is the statement that all finite simple groups are known).

4. Proof of Theorem 3.5 for groups of Lie type: Generic case

For most groups of Lie type and for the alternating groups, it is possible to prove a stronger result than Theorem 3.5. Namely, with notations as in the previous section, for elements x, y of a group F we let $R_F(x, y)$ denote the subgroup in F generated by $C_F(x)$ and $C_F(x, y)$. Then we have the following theorem.

Theorem 4.1. *Let F be either an alternating group A_n ($n \geqslant 5$, $n \neq 8$) or one of simple (adjoint) Chevalley groups (normal or twisted) not in the following list:*

(i) $PSL_n(q)$, *where either* $q = 2$, *or* $n = 3$, $q = 4$, *or else* $n = 4$, $q = 3$;
(ii) $PSU_n(q^2)$, *where* $q = 2$ or $q = 3$, $n = 4$;
(iii) $PSp_{2n}(q)$, $P\Omega_n(f, q)$, $E_7(q)$, $E_8(q)$, *and* $F_4(q)$ *where* $q = 2$ *or* 3;
(iv) ${}^3D_4(q)$, $E_6(q)$ *and* ${}^2E_6(q^2)$, $q = 2$;
(v) $G_2(q)$, $q = 3$.

Then there exist elements $x, y \in F$ such that $R_F(x, y) \neq F$; in particular F does not have property (C).

We begin the proof with the alternating groups which naturally require a sort of special analysis, while all other groups of Lie type are treated in a more or less uniform way.

Alternating groups. Let $F = A_n$. First, consider the case of even $n \geqslant 10$. Letting $l = (n-4)/2$, we introduce the following subsets $I_j (j = 1, 2, 3)$ of the set $I = \{1, 2, \ldots, n\}$:

$$I_1 = \{1, 2\}, I_2 = \{3, \ldots, 2 + l\}, I_3 = \{3 + l, \ldots, 3 + 2l = n - 1\}.$$

Obviously, $\#I_2 = l$, $\#I_3 = l + 1$, and since $n \geqslant 10$, we have $l > 2$. Now, let $x = x_1 x_2 x_3$, where x_j is a cyclic permutation of I_j. Then each of I_j's is invariant under the centralizer $C_F(x)$. Pick $y \in F$ such that $y(n) = n$ and $y(I_j) \cap I_j = \emptyset$ for each $j = 1, 2, 3$; for example, one can take

$$y = (1, 3 + l, 2, 4 + l, \ldots, 3 + 2l, 2 + l).$$

For any $z \in C_F(x)$, and $t = yz$ we have the following: $t(n) = n$ and $t(I_j) \cap I_j = \emptyset$ for each $j = 1, 2, 3$, in particular the fixed point set I^t is $\{n\}$. So, n is fixed by every centralizer $C_F(t)$, $t \in yC_F(x)$, and therefore also by $R_F(x, y)$, implying that $R_F(x, y) \neq F$.

Next, we consider the case of $n \equiv 1 \bmod (4)$, $n \geqslant 17$. Here we let $l = (n-7)/2$ and define the subsets $I_j \subset I = \{1, 2, \ldots, n\}$ ($j = 1, 2, 3, 4$) as follows:

$$I_1 = \{1, 2\}, I_2 = \{3, 4, 5\}, I_3 = \{6, \ldots, 5 + l\},$$

$$I_4 = \{6 + l, \ldots, 6 + 2l = n - 1\}.$$

Since $\#I_3 = l$, $\#I_4 = l+1$, and $l > 3$, the element $x = x_1x_2x_3x_4$ where x_j is a cyclic permutation of I_j, has the property that each of I_j's is invariant under $C_F(x)$. Then letting

$$y = (1,3,6,6+l)(2,4,7,7+l)(5,n-1)\cdot \prod_{j=8}^{5+l} (j,j+l),$$

we see just as above that the group $R_F(x,y)$ fixes n, and therefore is a proper subgroup of F.

For $n \equiv 3 \bmod (4)$, $n \geqslant 7$, we set $l = (n-1)/2$,

$$I_1 = \{1,\ldots,l\}, I_2 = \{l+1,\ldots,2l\},$$

and let $x = x_1x_2$ where x_j is a cyclic permutation of I_j. Since l is odd, the centralizer $C_F(x)$ is generated by x_1 and x_2; and therefore leaves each of I_j's invariant. Then one can take

$$y = (1,l+1)(2,l+2,\ldots,l,2l)$$

and argue as above.

Remaining small n's require special treatment.

Assume that $n = 5$ and let $x = (1,2)(3,4)$ and $y = (1,2,3)$. Then the centralizer $C_F(x)$ is the Klein group

$$\{e,(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\},$$

so any $t = yz$, where $z \in C_F(x)$, has order 3. This implies that $C_F(t) = \langle t \rangle$, so $R_F(x,y)$ fixes 5, and $R_F(x,y) \neq F$.

Since $A_6 \simeq PSL_2(9)$, and the group $PSL_2(9)$ will be considered later, we don't write out here x and y for this case.

If $n = 9$, let $x = (1,2,3,4,5,6,7)$ and $y = (2,4,3,7,5,6)(8,9)$. One can readily verify that $C_F(x) = \langle x \rangle$, $C_F(y) = \langle y \rangle$ and $y^{-1}xy = x^3$. The last relation implies that the coset $yC_F(x)$ coincides with the conjugacy class $t^{-1}yt, t \in \langle x \rangle$. So $R_F(x,y)$ is generated by $\{x,y\}$, and therefore $R_F(x,y) \neq F$.

If $n = 10$, let $x = (1,2)(3,4,5)(6,7,8,9)$ and $y = (1,3,6)(2,7)(4,8,5,9)$. Then $C_F(x) = \langle (1,2)(6,7,8,9),(3,4,5) \rangle$ leaves invariant each of the following sets

$$I_1 = \{1,2\}, I_2 = \{3,4,5\}, I_3 = \{6,7,8,9\},$$

and y has the property that $y(I_j) \cap I_j = \emptyset$ for each $j = 1,2,3$; so arguing as above, we obtain that $R_F(x,y)$ fixes 10.

For $n = 13$ we argue exactly as in the previous case, with

$$x = (1,2,3)(4,5)(6,7)(8,9,10,11,12)$$

and

$$y = (1,4)(2,8)(3,9)(5,10)(6,11)(7,12).$$

We will describe now the general strategy of the proof of Theorem 4.1 for the groups of Lie type (excluding those listed above). So, let G be a finite simple (adjoint) Chevalley group, $\tilde{G}$ be the corresponding simply connected Chevalley group, and $\varphi : \tilde{G} \to G$ be the natural epimorphism. Then $\tilde{G}$ can be obtained as the group of fixed points of a suitable algebraic endomorphism $\sigma : \mathbf{G} \to \mathbf{G}$ of the corresponding simply connected algebraic group $\mathbf{G}$ over the algebraic closure K of a finite field. To recall the description of endomorphisms which occur in this set-up, we fix a maximal split torus $\mathbf{T} \subset \mathbf{G}$ and

consider the associated root system $R = R(\mathbf{T}, \mathbf{G})$. Let Π be a system of simple roots in R, R_+ be the corresponding system of positive roots, $\mathbf{B}$ be the associated Borel subgroup, and $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Then the endomorphism σ required to define $\tilde{G}$ can be written in the form

$$\sigma = \theta \circ \tau, \tag{4.1}$$

where θ is the automorphism of $\mathbf{G}$ corresponding to a power of the Frobenius automorphism $\mathbf{p}$ of K where $p = \mathrm{char} K$ (in the sequel, θ will also be used to denote the underlying field automorphism), and τ is either an algebraic automorphism of $\mathbf{G}$ corresponding to a symmetry of Π (in this case the order of τ can be 1, 2 or 3) or an algebraic endomorphism switching positive multiples of short and long roots (this occurs only for the types $B_2 = C_2$ and F_4 if $p = 2$, and for the type G_2 if $p = 3$). Let k be the field of elements fixed under θ, and let l be the fixed field of θ^d if τ is an automorphism of order d, and the fixed field of $\mathbf{p}\theta^2$ if τ is an endomorphism but not an automorphism.

With these notations in mind, we proceed with our agrument. Let U be the group of σ-fixed elements in $\mathbf{U}$ (i.e. $U = \mathbf{U} \cap \tilde{G}$). Lemma 4.2 below shows that U always contains a regular element $\tilde{x}$. Suppose there exists an element $\tilde{y} \in T = \mathbf{T} \cap \tilde{G}$ with the following properties:

(A) the centralizer $C_{\mathbf{G}}(\tilde{y})$ is contained in a proper standard σ-invariant parabolic subgroup $\mathbf{P} \subset \mathbf{G}$;
(B) $C_G(\varphi(\tilde{y})) = \varphi(C_{\tilde{G}}(\tilde{y}))$.

Then the elements $x = \varphi(\tilde{x})$ and $y = \varphi(\tilde{y})$ are as required. Indeed, since $\tilde{x}$ is regular unipotent,

$$C_{\tilde{G}}(\tilde{x}) = Z(\tilde{G}) \cdot C_U(\tilde{x})$$

and

$$C_G(x) = \varphi(C_{\tilde{G}}(\tilde{x}))$$

(in the above $Z(\tilde{G})$ denotes the center of $\tilde{G}$). We will show that $R_G(x, y)$ is contained in $\varphi(P)$ where $P = \mathbf{P} \cap \tilde{G}$. Since P is a proper subgroup of $\tilde{G}$ (Lemma 4.2(ii)) and $\tilde{G}$ is its own commutator subgroup, we obtain that $\varphi(P) \neq G$ yielding the required result. Now, we pick an arbitrary $u \in C_G(x)$, and writing it as $u = \varphi(v)$ for a suitable $v \in C_U(\tilde{x})$ let $z = yu$ and $\tilde{z} = \tilde{y}v$. Letting p be the characteristic of k as before, we can put $\tilde{z}$ in the form $\tilde{z} = z_1 z_2$ where both z_1 and z_2 belong to the cyclic subgroup generated by $\tilde{z}$, and the order of z_1 (resp. z_2) is prime to p (resp. is a power of p). Clearly T is a maximal subgroup of the soluble group $B = \mathbf{B} \cap \tilde{G}$ of order prime to p (recall that $B = TU$, a semi-direct product) so it follows from the Hall theorem (cf. [5], §9.3) that $t = g z_1 g^{-1} \in T$ for a suitable $g \in U$. We have

$$t = z_1(z_1^{-1} g z_1 g^{-1}) = \tilde{z} z_2^{-1}(z_1^{-1} g z_1 g^{-1}) = \tilde{y}[v z_2^{-1}(z_1^{-1} g z_1 g^{-1})].$$

Since U is a normal Sylow p-subgroup of B, the expression in the square brackets belongs to U, and we conclude that $t = \tilde{y}$. Furthermore, as $\varphi(z_1)$ is a power of z, the centralizer of the latter is contained in the centralizer of the former, so we obtain the inclusion

$$C_G(z) \subset C_G(\varphi(g^{-1}\tilde{y}g)) = \varphi(g)^{-1}\varphi(C_{\tilde{G}}(\tilde{y}))\varphi(g) \subset \varphi(P).$$

Eventually $R_G(x, y) \subset \varphi(P)$, as we claimed.

Lemma 4.2. (i) *U contains a regular element*; (ii) *$\tilde{G}$ is not contained in any proper σ-invariant standard parabolic subgroup* $\mathbf{P} \supset \mathbf{B}$.

Proof. For $\alpha \in R$, let $x_\alpha : K^+ \to \mathbf{U}_\alpha \subset \mathbf{G}$ be the root one-parameter subgroup. Then any element $u \in \mathbf{U}$ can be uniquely written in the form

$$u = \prod_{\alpha \in R_+} x_\alpha(u_\alpha) \tag{4.2}$$

(with arbitrary but fixed order of positive roots), and regular elements are determined by the condition

$$u_\alpha \neq 0 \text{ for all } \alpha \in \Pi$$

(cf. [15], ch. III, 1.13). The endomorphism τ involved in the above description of σ induces an automorphism τ^* of the vector space $V = \mathbf{X}(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ where $\mathbf{X}(\mathbf{T})$ is the character group of $\mathbf{T}$. Let us consider the orthogonal projection $\pi_\tau : V \to V_\tau$ on the space $V_\tau \subset V$ of τ^*-fixed elements and define the equivalence relation on R as follows: $\alpha \sim \beta$ iff $\pi_\tau(\alpha) = c\pi_\tau(\beta)$ for some $c > 0$ (details can be found in [16], § 11). Let E denote the set of equivalence classes with respect to $\sim$, and let E_+ be the set of classes having a representative in R_+. Then for the proof of the lemma it is convenient to combine the factors in (4.2) as follows:

$$u = \Pi_{\mathfrak{a} \in E_+} (\Pi_{\alpha \in \mathfrak{a}} x_\alpha(u_\alpha)).$$

We will show that given any $\mathfrak{a} \in E_+$, there exists a $u(\mathfrak{a}) \in U$ such that

$$u(\mathfrak{a}) = \prod_{\alpha \in \mathfrak{a}} x_\alpha(u_\alpha) \tag{4.3}$$

and

$$u_\alpha \neq 0 \text{ for all } \alpha \in \mathfrak{a} \cap \Pi. \tag{4.4}$$

It is known that the roots belonging to $\mathfrak{a} \in E_+$ form a system of positive roots in a certain root system $R(\mathfrak{a})$, and for each case there is an explicit description of the set $U_\mathfrak{a}$ of elements of the form (4.3) which are fixed by σ, i.e. belong to U (cf. [16], § 11, Lemma 63). To be more precise, the full list of possibilities looks as follows (the fields l and k and the field automorphism θ are as introduced above):

(a) $R(\mathfrak{a}) = A_1$ and $\mathfrak{a} = \{\alpha\}$; then $U_\mathfrak{a} = \{x_\alpha(t) \mid t \in k\}$;
(b) $R(\mathfrak{a}) = (A_1)^n$ where $n = 2$ or 3; then $U_\mathfrak{a} = \{x \cdot \sigma(x) \cdots \mid x = x_\alpha(t),\ \alpha \in \mathfrak{a},\ t \in l\}$;
(c) $R(\mathfrak{a}) = A_2$ and $\mathfrak{a} = \{\alpha, \beta, \alpha + \beta\}$; then $U_\mathfrak{a} = \{x_\alpha(t) x_\beta(t^\theta) x_{\alpha+\beta}(u) \mid t, u \in l$ and $tt^\theta + u + u^\theta = 0\}$ and $\theta^2 = 1$;
(d) $R(\mathfrak{a}) = C_2$ and $\mathfrak{a} = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}$; then $U_\mathfrak{a} = \{x_\alpha(t) x_\beta(t^\theta) x_{\alpha+2\beta}(u) x_{\alpha+\beta} \cdot (t^{1+\theta} + u^\theta) \mid t, u \in l\}$ and $2\theta^2 = 1$.
(e) $R(\mathfrak{a}) = G_2$ and $\mathfrak{a} = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}$; then $U_\mathfrak{a} = \{x_\alpha(t) x_\beta(t^\theta) x_{\alpha+3\beta}(u) \cdot x_{\alpha+\beta}(u^\theta - t^{1+\theta}) x_{2\alpha+3\beta}(v) x_{\alpha+2\beta}(v^\theta - t^{1+2\theta}) \mid t, u, v \in l\}$.

From this description, the existence of $u \in U_\mathfrak{a}$ satisfying (4.4) follows immediately for all cases except (c), and to handle (c) one can take $t = 1$ and then notice that since the trace map $\mathrm{Tr}_{l/k}$ is surjective, one can find $u \in l$ to satisfy the required equation, proving (i).

To prove (ii) suppose that our parabolic corresponds to a (proper) subset $\Delta \subset \Pi$. Let $\alpha \in \Pi \backslash \Delta$ and $\mathfrak{a} \in E_+$ be the class containing α. Consider the corresponding root system $R(\mathfrak{a})$ and the group $\mathbf{G}_\mathfrak{a}$ generated by root subgroups $\mathbf{U}_\alpha$ for $\alpha \in R(\mathfrak{a})$. Then $\mathbf{G}_\mathfrak{a}$ is stable

under σ and $\mathbf{P}\cap\mathbf{G}_{\mathfrak{a}}=\mathbf{B}\cap\mathbf{G}_{\mathfrak{a}}$. On the other hand, arguing as above, one can show that $\mathbf{G}_{\mathfrak{a}}$ always contains a non-identity σ-fixed element of the form

$$u(-\mathfrak{a})=\prod_{\alpha\in(-\mathfrak{a})} x_\alpha(u_\alpha),$$

which belongs to $\tilde{G}$ but not to $\mathbf{P}$. The lemma is proved. ■

For convenience of reference, we formulate here one simple group-theoretic statement.

Lemma 4.3. *Let $\phi: G_1\to G_2$ be a surjective homomorphism of finite groups such that $N=\mathrm{Ker}(\phi)$ is contained in the center of G_1, and let $x\in G_1$. Then*

(i) $\#C_{G_2}(\phi(x))\leqslant\#C_{G_1}(x)$;
(ii) *if n is the exponent of N, then $C_{G_2}(\phi(x))\subset\phi(C_{G_1}(x^n))$.*

Proof. Let $C_1=C_{G_1}(x)$, $C_2=C_{G_2}(\phi(x))$. For $g\in C_2$ we pick a lift $\tilde{g}\in\phi^{-1}(g)$ and consider the commutator $[\tilde{g},x]$. Since N is central in G_1, this commutator depends only on g, but not on the choice of $\tilde{g}$ and therefore can be denoted simply as $c(g)$. Furthermore, one easily verifies that for $g\in C_2$, $c(g)\in N$, and the map

$$\theta: C_2\to N,\quad g\mapsto c(g)$$

is a homomorphism of groups the kernel of which is exactly $\phi(C_1)$. It follows that

$$\#C_2\leqslant(\#\phi(C_1))\cdot(\#N).$$

On the other hand, $N\subset C_1$ and therefore

$$\#\phi(C_1)=\frac{\#C_1}{\#N},$$

and we obtain (i). To prove (ii), it remains to observe that with notations as above, we have

$$[\tilde{g},x^n]=[\tilde{g},x]\cdot(x[\tilde{g},x]x^{-1})\ldots(x^{n-1}[\tilde{g},x]x^{-(n-1)})=[\tilde{g},x]^n,$$

implying that $[\tilde{g},x^n]=1$, and $\tilde{g}\in C_{G_1}(x^n)$. ■

While Lemma 4.2 and the preceding remarks provide a uniform construction of the element $\tilde{x}$, the choice of $\tilde{y}$ requires a case-by-case consideration of all types.

Groups $PSL_n(q)$. Here $\tilde{y}$ can be presented explicitly in the matrix form, without appealing to the root notations. For the Borel subgroup $\mathbf{B}$ (resp. the maximal torus $\mathbf{T}$) we take the group of the upper triangular (resp. diagonal) matrices in $\mathbf{G}=\mathbf{SL}_n$. Suppose $k=\mathbf{F}_q$, the field of q elements. We will show that except for the cases listed in Theorem 4.1 and also the group $PSL_2(5)$ one can take $\tilde{y}=\mathrm{diag}(s,s^{-1},1,\ldots,1)$ where s is a generator of $\mathbf{F}_q^*$. Since $q\neq 2$, the centralizer $C_{\tilde{G}}(\tilde{y})$ is always contained in the parabolic subgroup $\mathbf{P}\subset\mathbf{G}$ consisting of matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ 0 & 0 & a_{33} & \cdots & a_{3n}\\ \vdots & \vdots & & \ddots & \\ 0 & 0 & a_{n3} & \cdots & a_{nn}\end{pmatrix}.$$

Now, let $\varphi:\tilde{G}=SL_n(q)\to PSL_n(q)=G$ be the canonical homomorphism, and $y=\varphi(\tilde{y})$.

Lemma 4.4. *If $q \neq 2$ then except for the following cases:*

1) $n=2$, $q=3,5$;
2) $n=3$, $q=4$;
3) $n=4$, $q=3$,

we have $C_G(y)=\varphi(C_{\tilde{G}}(\tilde{y}))$.

Proof. Let $u=\varphi(v)\in C_G(y)\setminus\varphi(C_{\tilde{G}}(\tilde{y}))$ if possible. Then $v\tilde{y}v^{-1}=z\tilde{y}$ where $z=\mathrm{diag}(\xi,\dots,\xi)$, and $1\neq\xi\in\mathbf{F}_q^*$ satisfies $\xi^n=1$. If $n>4$ then $\tilde{y}$ has eigenvalue 1 with multiplicity $n-2>2$, and since the only eigenvalue with this multiplicity for $z\tilde{y}$ is ξ we obtain $\xi=1$, a contradiction. If $n=4$ and $q>3$, then again $\tilde{y}$ has eigenvalue 1 with multiplicity 2, and the only eigenvalue with this multiplicity for $z\tilde{y}$ is ξ. For $n=3$, v belongs to $C_{\tilde{G}}(\tilde{y}^3)$ (cf. the proof of Lemma 4.3) which coincides with $C_{\tilde{G}}(\tilde{y})$ unless $s^3=\pm 1$, i.e. if $q\neq 4$ or 7. However, for $q=7$, $\tilde{y}$ has eigenvalue 1 while $z\tilde{y}$ does not. In the last remaining case $n=2$, we have $v\in C_{\tilde{G}}(\tilde{y}^2)$ which equals $C_{\tilde{G}}(\tilde{y})$ provided $s^2\neq\pm 1$, i.e. if $q\neq 3,5$. ■

Now, to complete consideration of $G=PSL_n(q)$, it remains to observe that for $n=2$, $q=3$ the group G is not simple, and therefore is automatically excluded from our consideration, and for $n=2$, $q=5$, G is isomorphic to the alternating group A_5 which has been considered earlier.

Groups $PSU_n(q^2)$. The group $SU_n(q^2)$ is obtained as the subgroup of $\mathbf{G}=\mathbf{SL}_n$ consisting of elements fixed by the automorphism σ of $\mathbf{G}$ given by the formula

$$\sigma(x)=F^{-1}(x^*)^{-1}F,$$

where $x^*_{ij}=(\theta(x_{ji}))$, θ is the field automorphism $x\mapsto x^q$, and

$$F=\begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}.$$

Obviously, in this case $k=\mathbf{F}_q$, $l=\mathbf{F}_{q^2}$. Besides, since $SU_2(q^2)$ is isomorphic to $SL_2(q)$, we may assume that $n\geqslant 3$. As in the previous case, one can take for $\mathbf{B}$ (resp. $\mathbf{T}$) the group of upper triangular (resp. diagonal) matrices in $\mathbf{G}$.

If $n\geqslant 4$, we let $\tilde{y}=\mathrm{diag}(s,s^{-1},1,\dots,1,\theta(s),\theta(s)^{-1})$ where s is a generator of l^*. Since $q>2$, all the elements in the set $\{s,s^{-1},\theta(s),\theta(s)^{-1},1\}$ are distinct, and therefore $C_{\mathbf{G}}(\tilde{y})$ is contained in a proper parabolic $\mathbf{P}\supset\mathbf{B}$. As in the previous case, to verify (B) it suffices to show that the matrices $\tilde{y}$ and $z\tilde{y}$ where $z=\mathrm{diag}\{\xi,\dots,\xi\}$ with ξ satisfying $\xi^n=1$ and $\xi\theta(\xi)=1$, are not conjugate unless $\xi=1$. For $n\geqslant 6$ this follows from the fact that $\tilde{y}$ has eigenvalue 1 with multiplicity $n-4\geqslant 2$, and the only eigenvalue of this multiplicity for $z\tilde{y}$ is ξ. For $n=5$, one should observe that 1 is an eigenvalue for $\tilde{y}$ but not for $z\tilde{y}$ unless $\xi=1$, since the order of s is $q^2-1\geqslant 8$.

Suppose now that $n=4$. If $\xi\neq 1$, then for s to be contained in the set of eigenvalues of $z\tilde{y}$, ξ should equal one of the elements $s^2, s\theta(s^{-1})$ or $s\theta(s)$. The orders of these elements are respectively $(q^2-1)/2, q+1, q-1$, and the conditions on ξ imply that $q=3$ which is excluded in Theorem 4.1.

In the case $n=3$ we let $\tilde{y}=\mathrm{diag}(s,s^{-1}\theta(s),\theta(s)^{-1})$ where s is as above. Since $q>2$, all the elements $s, s^{-1}\theta(s)$ and $\theta(s)^{-1}$ are distinct, and therefore $C_{\mathbf{G}}(\tilde{y})=\mathbf{T}$. In verifying (B) we may assume that $q>3$. If $q\geqslant 5$ then all eigenvalues of $\tilde{y}^3$ are also distinct, so

$C_G(\tilde{y}) = C_G(\tilde{y}^3)$ and $C_G(y) \subset \varphi(C_{\tilde{G}}(\tilde{y}^3)) = \varphi(C_{\tilde{G}}(\tilde{y}))$, as required. In the last remaining case $q = 4$, if s is contained in the set of eigenvalues of $z\tilde{y}$ and $\xi \neq 1$, then ξ can be only either s^{-2} or s^5. Using the properties of ξ, we easily check that none of these is possible.

Groups $PSp_{2n}(q), n \geqslant 2$. Here $\mathbf{G} = \mathbf{Sp}_{2n}(f)$ where f is an alternating form with the matrix

$$\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}, \quad J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

For $\mathbf{B}$ (resp. $\mathbf{T}$) one can take the group of upper triangular (resp. diagonal) matrices in $\mathbf{G}$. Let $\tilde{y} = \mathrm{diag}(s, 1, \ldots, 1, s^{-1})$ where s is a generator of the multiplicative group of $k = \mathbf{F}_q$. Since the cases $q = 2$ or 3 are excluded in Theorem 4.1, $s \neq s^{-1}$ implies that $C_G(\tilde{y})$ is contained in a proper standard parabolic, hence (A). If $p = 2$, then the center of $\mathbf{G}$ is trivial, and (B) is satisfied automatically. Otherwise, for $q > 3$, $\tilde{y}$ is not conjugate to $-\tilde{y}$, and again (B) follows.

Groups ${}^2C_2(2^{2m+1})$. Since the group ${}^2C_2(2)$ is soluble, we may assume that $m \geqslant 1$. In this case $k = \mathbf{F}_{2^m}, l = \mathbf{F}_{2^{2m+1}}$ and σ acts on the root elements as follows:

$$\sigma: \begin{cases} x_\alpha(t) \to x_\beta(t^{2^m}), \\ x_\beta(t) \to x_\alpha(t^{2^{m+1}}), \end{cases} \qquad \overset{\alpha \quad \beta}{\circ\!\Longrightarrow\!\circ}$$

(In the realization of $R = C_2$ as $\{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm \varepsilon_1 \pm \varepsilon_2\}$ given in ([2], table III) we have $\alpha = 2\varepsilon_2, \beta = \varepsilon_1 - \varepsilon_2$.) Then σ acts on the elements $h_\alpha(t), h_\beta(t)$ by similar formulas, implying that T consists of elements of the form $r(s) = h_\alpha(s)h_\beta(s^{2^m}), s \in l^*$. Since the center of $\tilde{G}$ is trivial, it suffices to show that for a generator s of l^*, the element $r = r(s)$ is regular. Obviously, for any root γ, the value $\gamma(r)$ is a certain power of s, to be denoted as $|\gamma(r)|$. For positive γ's, the value $|\gamma(r)|$ are given in table 1, which shows that there is no γ such that $\gamma(r) = 1$, hence regularity of r.

Groups $P\Omega_n(f, q), n \geqslant 5, q > 3$. For a non-degenerate quadratic form f over k in n variables we let $\Omega_n(f)$ denote the commutator subgroup of the special orthogonal group $SO_n(f)$, and then $P\Omega_n(f) = \Omega_n(f)$ if n is odd or else if $p = 2$, and $P\Omega_n(f) = \Omega_n(f)/\{\pm 1\}$ otherwise. (Recall that if f is isotropic, which is always the case over a finite field, then $\Omega_n(f)$ coincides with the kernel of the spinor norm for $p \neq 2$ and $\Omega_n(f) = SO_n(f)$ for $p = 2$.)

For the groups $P\Omega_n(f)$ the above method requires some minor modifications. Let f_0 be the split form in n variables, i.e. $f_0 = x_1x_n + \cdots + x_rx_{r-1}$ if $n = 2r$, and

Table 1.

γ	$\lvert\gamma(r)\rvert$
$\varepsilon_1 - \varepsilon_2$	$-1 + 2^{m+1}$
$\varepsilon_1 + \varepsilon_2$	1
$2\varepsilon_1$	2^{m+1}
$2\varepsilon_2$	$2 - 2^{m+1}$

$f_0 = x_1x_n + \cdots + x_rx_{r+2} + x_{r+1}^2$ if $n = 2r+1$. It is more convenient to think of the occurring groups as obtained not from the simply connected spinor group $\mathrm{Spin}_n(f_0)$, but rather from the special orthogonal group $\mathbf{G} = \mathbf{SO}_n(f_0)$. We fix the maximal torus $\mathbf{T} \subset \mathbf{G}$ consisting of matrices of the form $\mathrm{diag}(t_1, \ldots, t_r, t_r^{-1}, \ldots, t_1^{-1})$ (resp. $\mathrm{diag}(t_1, \ldots, t_r, 1, t_r^{-1}, \ldots, t_1^{-1})$) for $n = 2r$ (resp., $n = 2r+1$), and also the Borel subgroup $\mathbf{B} \subset \mathbf{G}$ consisting of the upper triangular matrices in $\mathbf{G}$ whose diagonal part is in $\mathbf{T}$. Then the split orthogonal group $SO_n(f_0)$ over $k = \mathbf{F}_q$ is obtained as the group of σ-fixed points for $\sigma = \theta$ where θ corresponds to the field automorphism $x \mapsto x^q$, while non-split (but quasi-split) group $SO_n(f)$ where $n = 2r, f = x_1x_n + \cdots + x_{r-1}x_{r+2} + x_r^2 + dx_{r+1}^2$, $d \notin (-1)^r k^{*2}$ if $p \neq 2$, and $f = x_1x_n + \cdots + x_{r-1}x_{r+2} + dx_r^2 + x_rx_{r+1} + dx_{r+1}^2, d \in k^*$ is such that the polynomial $dt^2 + t + d$ is irreducible over k if $p = 2$ is isomorphic to the group of σ-fixed points for $\sigma = \tau \circ \theta$ where θ is as above, and $\tau = \mathrm{Int}(s)$,

$$s = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Let $\tilde{G} = \Omega_n(f)$, $G = P\Omega_n(f)$ and $\varphi: SO_n(f) \to PSO_n(f)$ be the canonical projection. The same argument as before shows that it suffices to find $\tilde{y}$ in $T = \tilde{G} \cap \mathbf{T}$ with the following properties.

(A′) $C_{\mathbf{G}}(\tilde{y})$ is contained in a proper parabolic $\mathbf{P} \subset \mathbf{G}$ containing $\mathbf{B}$;
(B′) $C_G(\varphi(\tilde{y})) = \varphi(C_{\tilde{G}}(\tilde{y}))$.

(Obviously (B′) is satisfied automatically if either n is odd or $p = 2$.) Now, suppose that $q > 3$, and let s be the generator of k^* and

$$\tilde{y} = \mathrm{diag}(s, s, 1, \ldots, 1, s^{-1}, s^{-1}).$$

One easily verifies that $\tilde{y}$ is fixed by σ and has the spinor norm one in both split and non-split cases, so $\tilde{y} \in T$. Since $s \neq s^{-1}$, $C_{\mathbf{G}}(\tilde{y})$ is contained in a proper parabolic containing $\mathbf{B}$, hence (A′). Finally, if $u = \varphi(v)$ is in $C_G(\varphi(\tilde{y}))$ but not in $\varphi(C_{\tilde{G}}(\tilde{y}))$ then $v^{-1}\tilde{y}v = -\tilde{y}$ which is impossible since the eigenvalue 1 or $\tilde{y}$ is not an eigenvalue for $-\tilde{y}$.

Verification of (A) and (B) for groups of other types is based on the root technique, so let us recall some facts and fix notations. For a subset $\Delta \subset \Pi$, the standard parabolic subgroup $\mathbf{P} = \mathbf{P}_\Delta$ corresponding to Δ is generated by $\mathbf{B}$ and $\mathbf{U}_\alpha$, for all $\alpha \in -[\Delta]$ where $[\Delta]$ denotes the set of roots which are linear combinations of roots from Δ with non-negative coefficients. On the other hand, for $\tilde{y} \in \mathbf{T}$, the centralizer $C_{\mathbf{G}}(\tilde{y})$ is generated by $\mathbf{T}$ and $\mathbf{U}_\alpha$ for $\alpha \in \Phi(\tilde{y}) = \{\alpha \in R_+ \mid \alpha(\tilde{y}) = 1\}$; this is a consequence of the Bruhat decomposition and the following fact proved in [15], ch. II, § 4: in the simply connected case, the centralizer $C_W(\tilde{y})$ in the Weyl group $W = W(\mathbf{T}, \mathbf{G})$ is generated by reflections w_α for $\alpha \in \Phi(\tilde{y})$. So, to ensure (A) we need to find $\tilde{y} \in T$ such that

$$\Phi(\tilde{y}) \subset [\Delta] \tag{4.5}$$

for a suitable proper subset $\Delta \subset \Pi$. Such an element $\tilde{y}$ will be presented in the form $\tilde{y} = \chi_1(t_1) \ldots \chi_m(t_m)$ where $\chi_1, \ldots, \chi_m$ are (multiplicative) one-parameter subgroups in $\mathbf{T}$.

The group $\mathbf{X}_*(\mathbf{T})$ of one-parameter subgroups which is identifiable with the dual group $\mathrm{Hom}(\mathbf{X}(\mathbf{T}), \mathbb{Z})$ of the character group, coincides in the simply connected case with the $\mathbb{Z}$-lattice $Q(R^\vee)$ generated by the dual roots $\alpha^\vee = 2\alpha/(\alpha|\alpha)$, and the one-parameter subgroup corresponding to $\alpha^\vee$ is H_α in the notations of Steinberg [16]. Also, let $P(R^\vee)$ be the dual group to the subgroup $Q(R) \subset \mathbf{X}(\mathbf{T})$ generated by the roots $\alpha \in R$. Then $Q(R^\vee)$ is embeddable into $P(R^\vee)$ as a subgroup of finite index in such a way that the pairing between $P(R^\vee)$ and $Q(R)$ is compatible with that between $\mathbf{X}(\mathbf{T})$ and $\mathbf{X}_*(\mathbf{T})$, and the index $e = [P(R^\vee):Q(R^\vee)]$ is called the *connectedness index* (l'indice de connexion) of R.

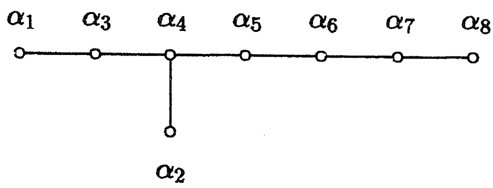
If $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, then $P(R^\vee)$ is a free $\mathbb{Z}$-module with the basis $\omega_1, \ldots, \omega_r$ such that $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$. Then $\chi_i = e\omega_i \in \mathbf{X}_*(\mathbf{T})$, and in some cases for $\tilde{y}$ one can take an element of the form $\tilde{y} = \chi_i(s)$ for some i, where s is a generator of k^*. To describe the precise conditions when this is possible, let us denote the coefficient of $\beta \in Q(R)$ at α_i as $n_i(\beta)$. Also, let $\tilde{\alpha} = n_1\alpha_1 + \cdots + n_r\alpha_r$ be the highest root. We claim that if

$$s^{e(n_i!)} \neq 1, \tag{4.6}$$

then the element $\tilde{y} = \chi_i(s)$ satisfies (4.5) with $\Delta = \Pi \backslash \{\alpha_i\}$, thereby yielding (A). Indeed, let $\alpha \in \Phi(\tilde{y})$ and $\alpha \notin [\Delta]$; then $n_i(\alpha) \neq 0$ and $n_i(\alpha) \leqslant n_i$. This means that $s^{e(n_i!)}$ is a power of $\alpha(\tilde{y}) = s^{en_i(\alpha)}$, and then (4.6) gives a contradiction.

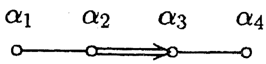
For the groups of types E_8, F_4, G_2 the corresponding n_i can be given explicitly (for these types $e = 1$, implying that the center of $\widetilde{G}$ is trivial (cf. [16], § 3) and (B) is satisfied automatically).

Groups $E_8(q), q > 3$. Labeling simple roots as in [2], Table VII



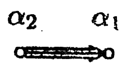
we have $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$, and (4.6) is satisfied for $i = 1$.

Groups $F_4(q), q > 3$. We have the following Dynkin diagram (cf. [2], table VIII):



and the highest root $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Obviously (4.6) is satisfied for $i = 1$.

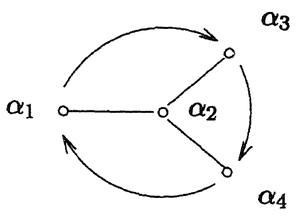
Groups $G_2(q), q > 3$. The Dynkin diagram is



and the highest root is $\tilde{\alpha} = 3\alpha_1 + 2\alpha_2$ (cf. [2] table IX). Then (4.6) is satisfied for $i = 2$.

Other groups require more detailed treatment.

Groups ${}^3D_4(q), q \geqslant 3$. Here $\widetilde{G}$ is the group of σ-fixed points for the automorphism σ of the group $\mathbf{G}$ of type D_4 of the form $\sigma = \tau \circ \theta$ where τ is the algebraic automorphism of $\mathbf{G}$ corresponding to the following symmetry of the Dynkin diagram:



and θ is induced by the field automorphism $x \to x^q$ (in this case $k = \mathbf{F}_q, l = \mathbf{F}_{q^3}$). Then R can be realized as $\{\pm \varepsilon_i \pm \varepsilon_j | 1 \leqslant i < j \leqslant 4\}$ with the following system of simple roots:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_3 + \varepsilon_4$$

(cf. [2], table IV). The center of $\widetilde{G}$ is trivial, and we need only to find $\tilde{y} \in T$ satisfying (A), i.e. such that $\Phi(\tilde{y}) \subset [\Delta]$ for some proper subset $\Delta \subset \Pi$. Let t be a generator of l^*. Assume now that $q > 3$, and so we may pick an element $s \in k^*$ different from 1 and $N_{l/k}(t)$. Then the element

$$\tilde{y} = h_{\alpha_1}(t) h_{\alpha_3}(t^q) h_{\alpha_4}(t^{q^2}) h_{\alpha_2}(s)$$

is fixed by σ, and the values at $\tilde{y}$ of positive roots containing α_1 are given in table 2. Going through the table, one easily ascertains that all values are different from 1, i.e. $\Phi(\tilde{y}) \subset [\Delta]$ for $\Delta = \{\alpha_2, \alpha_3, \alpha_4\}$. In fact, one can say even more: since $\tilde{y}^\sigma = \tilde{y}$, then using the action of τ on the Dynkin diagram, we conclude that $\Phi(\tilde{y}) \subset [\Delta]$ for $\Delta = \{\alpha_2\}$. On the other hand, $\alpha_2(\tilde{y}) = s^2 \cdot N_{l/k}(t)^{-1}$; so picking $s \in k^*$ such that in addition $s^2 \neq N_{l/k}(t)$ (this is always possible if $q \geqslant 5$; for $p \neq 2$ one can take $s = -1$), we obtain a regular $\tilde{y}$. For $q = 3$, we take

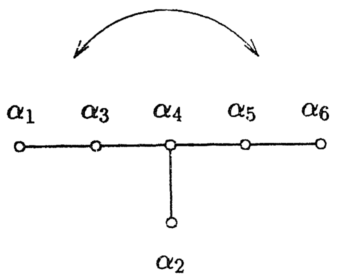
$$\tilde{y} = h_{\alpha_1}(t^2) h_{\alpha_3}(t^{2q}) h_{\alpha_4}(t^{2q^2}) h_{\alpha_2}(-1).$$

Then again with the help of table 2 we verify that $\Phi(\tilde{y}) \subset [\Delta]$ for $\Delta = \{\alpha_2, \alpha_3, \alpha_4\}$.

Table 2.

γ	$\lvert\gamma(\tilde{y})\rvert$
$\varepsilon_1 - \varepsilon_2$	$t^2 \cdot s^{-1}$
$\varepsilon_1 + \varepsilon_2$	s
$\varepsilon_1 - \varepsilon_3$	$t^{1-q-q^2} s$
$\varepsilon_1 + \varepsilon_3$	$N_{l/k}(t) \cdot s^{-1}$
$\varepsilon_1 - \varepsilon_4$	t^{1+q-q^2}
$\varepsilon_1 + \varepsilon_4$	t^{1-q+q^2}

Groups $E_6(q)$ *and* ${}^2E_6(q^2)$, $q>2$. The Dynkin diagram looks as follows:



and ${}^2E_6$ is the group of σ-fixed points for $\sigma=\tau\circ\theta$ where θ is associated with the field automorphism $x\mapsto x^q$, and τ is the algebraic automorphism corresponding to the symmetry indicated above (in this case $k=\mathbf{F}_q, l=\mathbf{F}_{q^2}$).

First, let $q>3$ and let $\omega_2=\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$ be the highest root. Then $\omega_2 \in Q(R^\vee)=\mathbf{X}_*(\mathbf{T})$ (cf. [2], table V); let $\tilde{y}=\omega_2(s)$ where s is a generator of k^*. Since $n_2=2$ and $s^2\neq 1$, we obtain $\Phi(\tilde{y})\subset[\Delta]$ for $\Delta\subset\Pi\backslash\{\alpha_2\}$, proving (A). To verify (B), pick a $u=\varphi(v)$ which is in $C_G(\varphi(\tilde{y}))$ but not in $\varphi(C_{\tilde{G}}(\tilde{y}))$ if (B) is not true. Then $v^{-1}\tilde{y}v=z\tilde{y}$ for some non-trivial element z of the center $Z(\tilde{G})$, and there exists $w\in W=W(\mathbf{T},\mathbf{G})$ such that

$$w\tilde{y}w^{-1}=z\tilde{y}. \tag{4.7}$$

It is easy to check that $Z(\mathbf{G})$ is generated by the element

$$z_0=h_{\alpha_1}(\xi)h_{\alpha_3}(\xi^2)h_{\alpha_5}(\xi)h_{\alpha_6}(\xi^2)$$

where ξ is a primitive 3rd root of unity (or $\xi=1$ is case of characteristic 3), and therefore (4.7) implies that for $\beta=\omega_2-w(\omega_2)$ we have

$$n_i(\beta)\equiv 0 \bmod (q-1),\ i=2,4. \tag{4.8}$$

Obviously $w\in C_G(\tilde{y})$; since for $q=5$ the order of $\tilde{y}$ is four, we obtain $C_G(\tilde{y}^3)=C_G(\tilde{y})$ implying that $z=1$. On the other hand, for $q\geqslant 7$ we can never have (4.8) for $i=2$, since $n_2(\omega_2)=2$ and $|n_2(w(\omega_2))|\leqslant 2$. So, it remains to consider the case $q=4$. Since $n_2(\omega_2)=2$ and $n_4(\omega_2)=3$, we can have (4.8) only if

$$n_2(w(\omega_2))=2 \text{ or } -1,\ n_4(w(\omega_2))=0 \text{ or } \pm 3. \tag{4.9}$$

Using the table of roots in [2], one can see that the possibilities for $w(\omega_2)$ are the following:

$$\omega_2,\ -\alpha_2,\ -(\omega_2-\alpha_2).$$

However, in the first case $z=1$, and in the last two cases we respectively have

$$\beta=\alpha_1+3\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$$

and

$$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6,$$

and therefore

$$w^{-1}\tilde{y}w\tilde{y}^{-1}=h_{\alpha_1}(s^{-1})h_{\alpha_3}(s^{-2})h_{\alpha_5}(s^{-2})h_{\alpha_6}(s^{-1})$$

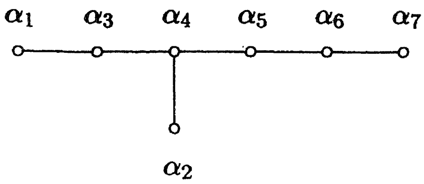
or

$$h_{\alpha_1}(s^{-2})h_{\alpha_3}(s^{-4})h_{\alpha_5}(s^{-4})h_{\alpha_6}(s^{-2}),$$

but neither of these elements belongs to $Z(\mathbf{G})$, and (B) is established.

Finally, if $q=3$, then both groups $E_6(3)$ and ${}^2E_6(9)$ have trivial centers, so we need only to check (A). Since $n_1=n_6=1$, in the non-twisted case one can take $\tilde{y}=\chi_1(-1)\in T$ where $\chi_1=3\omega_1\in \mathbf{X}_*(\mathbf{T})$ to ensure that $\Phi(\tilde{y})\subset[\Delta]$ for $\Delta=\Pi\backslash\{\alpha_1\}$. In the twisted case we let $\tilde{y}=\chi_1(s)\chi_6(\theta(s))$ where s is a generator of the multiplicative group of $l=\mathbf{F}_9$, θ is the field automorphism $x\mapsto x^3$, and $\chi_6=3\omega_6$. Then for $\alpha\notin[\Delta]$ where $\Delta=\Pi\backslash\{\alpha_1,\alpha_2\}$, the value $\alpha(\tilde{y})$ is one of the following: s, $\theta(s)$ or $s\theta(s)$, and neither of these is 1.

Groups $E_7(q), q>3$. Labeling the simple roots as in [2]:



we have $\omega_1\in Q(R^\vee)=\mathbf{X}_*(\mathbf{T})$ (cf. *loc. cit.*, table VI); let $\tilde{y}=\omega_1(s)$ where s is a generator of the multiplicative group of $k=\mathbf{F}_q$. Then, since $n_1=2$ and $s^2\neq 1$, we have $\Phi(\tilde{y})\subset[\Delta]$ for $\Delta=\Pi\backslash\{\alpha_1\}$, hence (A). If (B) fails then, as above, there exists $w\in W=W(\mathbf{T},\mathbf{G})$ such that $w\tilde{y}w^{-1}=z\tilde{y}$. Then w is contained in $C_W(\tilde{y}^2)$, which, as we mentioned above, is generated by w_α for $\alpha\in\Phi(\tilde{y}^2)$. For any such α we have

$$\alpha(\tilde{y}^2)=s^{2n_1(\alpha)}=1, \tag{4.10}$$

and since $q>3$, we conclude that $n_1(\alpha)>1$. However it follows from the table that there exists the only positive root in the root system of type E_7 with $n_1(\alpha)>1$, namely $\tilde{\alpha}=\omega_1$, the highest root; besides in this case $n_1(\tilde{\alpha})=2$, and therefore (4.10) holds only if $q=5$. So, $C_W(\tilde{y}^2)$ is generated by w_α for α not containing α_1, and also by $w_{\tilde{\alpha}}$ for $q=5$. So, if the element $x=w\tilde{y}w^{-1}\tilde{y}^{-1}$ is non-trivial, then it equals $\omega_1(-1)$. But $\omega_1(-1)=h_{\alpha_3}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1)$ while the non-trivial element in the center is $h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1)$. Thus $z\notin Z(\mathbf{G})$, a contradiction which proves (B).

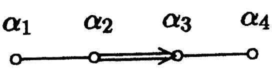
Groups ${}^2F_4(2^{m+1})$. Since the group ${}^2F_4(2)$ is not simple (however, its derived subgroup ${}^2F_4(2)'$ known as Tits' sporadic group, is simple and is considered in§ 5), we may assume that $m\geqslant 1$. We will be using the realization of the root system of type F_4 given in [2], table VIII:

$$R=\{\pm\varepsilon_i,\pm\varepsilon_i\pm\varepsilon_j|1\leqslant i<j\leqslant 4\}\cup\{\tfrac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)\}.$$

Then the system of simple roots consists of the following elements:

$$\alpha_1=\varepsilon_2-\varepsilon_3,\ \alpha_2=\varepsilon_3-\varepsilon_4,\ \alpha_3=\varepsilon_4,\ \alpha_4=\tfrac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$$

and the corresponding Dynkin diagram looks as follows:



Then $\tilde{G}$ is obtained as the group of σ-fixed elements of the endomorphism σ of the group

G of type F_4 acting on the root elements as follows:

$$\sigma = \begin{cases} x_{\alpha_1}(t) \to x_{\alpha_4}(t^{2^m}), \\ x_{\alpha_2}(t) \to x_{\alpha_3}(t^{2^m}), \\ x_{\alpha_3}(t) \to x_{\alpha_2}(t^{2^{m+1}}), \\ x_{\alpha_4}(t) \to x_{\alpha_1}(t^{2^{m+1}}). \end{cases} \tag{4.11}$$

In this case $k = \mathbf{F}_{2^m}, l = \mathbf{F}_{2^{2m+1}}$. It follows from (4.11) that elements of the form

$$r(s) = h_{\alpha_1}(s) h_{\alpha_2}(s^2) h_{\alpha_3}(s^{2^{m+1}}) h_{\alpha_4}(s^{2^m}),\ s \in l^* \tag{4.12}$$

are fixed by σ. Recall that the center of $\tilde{G}$ is trivial, and therefore all we need to do is to find $\tilde{y}$ of the form (4.12) such that $\Phi(\tilde{y}) \subset [\Delta]$ for some proper subset $\Delta \subset \Pi$. Let $s \in l^*$ be a generator. We will show that

$$\Phi(r(s)) \subset [\Delta] \text{ for } \Delta = \{\alpha_1, \alpha_4\}. \tag{4.13}$$

For a root $\gamma \in R$, let $|\gamma(r)|$ be the integer such that $\gamma(r) = s^{|\gamma(r)|}$ where $r = r(s)$. The values of $|\gamma(r)|$ for all positive roots are shown in table 3. For $m = 1$, s has order 7, but no number

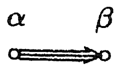
Table 3.

γ	$\lvert\gamma(r)\rvert$	$\lvert\gamma(r)\rvert$ for $m = 1$
ε_1	2^m	2
ε_2	$1-2^m$	-1
ε_3	$1-2^m$	-1
ε_4	$-2+3\cdot 2^m$	4
$\varepsilon_1-\varepsilon_2$	$-1+2^{m+1}$	3
$\varepsilon_1+\varepsilon_2$	1	1
$\varepsilon_1-\varepsilon_3$	$-1+2^{m+1}$	3
$\varepsilon_1+\varepsilon_3$	1	1
$\varepsilon_1-\varepsilon_4$	$2-2^{m+1}$	-2
$\varepsilon_1+\varepsilon_4$	$-2+2^{m+2}$	6
$\varepsilon_2-\varepsilon_3$	0	0
$\varepsilon_2+\varepsilon_3$	$2-2^{m+1}$	-2
$\varepsilon_2-\varepsilon_4$	$3-2^{m+2}$	-5
$\varepsilon_2+\varepsilon_4$	$-1+2^{m+1}$	3
$\varepsilon_3-\varepsilon_4$	$3-2^{m+2}$	-5
$\varepsilon_3+\varepsilon_4$	$-1+2^{m+1}$	3
$\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$	2^m	2
$\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)$	$2-2^{m+1}$	-2
$\frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)$	$-1+2^{m+1}$	3
$\frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)$	$1-2^m$	-1
$\frac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)$	$-1+2^{m+1}$	3
$\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4)$	$-2+3\cdot 2^m$	4
$\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$	0	0
$\frac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4)$	$1-2^m$	-1

in the third column (except two zeroes) is divisible by 7, and (4.13) follows. For $m > 1$, (4.13) is easily obtained via elementary estimations.

Groups ${}^2G_2(3^{2m+1})$. In this case, $\tilde{G}$ is obtained as the group of σ-fixed points of the automorphism σ of the group $\mathbf{G}$ of type G_2 acting on the root elements as follows:

$$\sigma = \begin{cases} x_\alpha(t) \to x_\beta(t^{3^m}), \\ x_\beta(t) \to x_\alpha(t^{3^{m+1}}). \end{cases}$$



Then $k = \mathbf{F}_{3^m}, l = \mathbf{F}_{3^{2m+1}}$. Since the group ${}^2G_2(3)$ is not simple, we may assume that $m \geqslant 1$. One easily verifies that σ-fixed elements of $\mathbf{T}$ are parameterized as follows:

$$r(s) = h_\alpha(s) h_\beta(s^{3^m}),\ s \in l^*.$$

Since the center of $\tilde{G}$ is trivial, it suffices to show that for a generator s of l^*, the element $r = r(s)$ is regular. The values of $|\gamma(r)|$ for all positive roots are given in table 4, and simple analysis shows that there is no root γ such that $\gamma(r) = 1$, hence the required results.

5. Proof of Theorem 3.5 for groups of Lie type: Exceptional cases

First, we outline the basic idea of constructing the required elements in the Chevalley groups which are not covered by Theorem 4.1 and then present these for each type (we will see in the next section that the method used here is also applicable to the sporadic groups).

Lemma 5.1. *Let G be a finite group, $x, y \in G$. Then*

$$\#E(x, y) \leqslant c^3 d^2, \tag{5.1}$$

where c is the order of $C = C_G(x)$ and $d = \max(\#C_G(z))$, z runs through the coset yC. If y normalizes C and $C \cap C_G(y) = (e)$, then $d = \#C_G(y)$ and (5.1) *can be sharpened to*

$$\#E(x, y) \leqslant \frac{c^3 d^2}{\operatorname{ord}(y)}. \tag{5.2}$$

Proof. Obviously, the number of elements in

$$C_G(x, y) = \bigcup_{z \in yC} C_G(z)$$

Table 4.

γ	$\lvert\gamma(r)\rvert$
α_2	$2 - 3^{m+1}$
α_1	$-1 + 2\cdot 3^m$
$\alpha_2 + \alpha_1$	$1 - 3^m$
$\alpha_2 + 2\alpha_1$	3^m
$\alpha_2 + 3\alpha_1$	$-1 + 3^{m+1}$
$2\alpha_2 + 3\alpha_1$	1

cannot exceed cd, and (5.1) is immediate. Furthermore, if y is as in the second statement, the map

$$\mu: C \to C, \ c \mapsto y^{-1}c^{-1}yc,$$

is a bijection. This implies that any element $z \in yC$ can be written as $c^{-1}yc$ for a suitable $c \in C$; in particular, $\#C_G(z) = \#C_G(y)$. Besides, in this case

$$E(x, y) = CC_G(y)CC_G(y)C, \tag{5.3}$$

and writing $C_G(y)$ as $C_G(y) = \cup_{i=1}^{t} z_i D$ where $D = \langle y \rangle$, $t = [C_G(y):D]$, and using the fact that y normalizes C, we obtain

$$\#E(x, y) \leqslant c^3 \cdot d \cdot t = \frac{c^3 \cdot d^2}{\operatorname{ord}(y)}.$$

The lemma is proved. ∎

The basic construction for producing the required elements x, y in most types of Chevalley groups is as follows. Let $k = \mathbf{F}_q$, the field of q elements, l/k be the extension of degree n, and $\rho_{l/k}: l \to M_n(k)$ be the regular representation. Then $T_0 = \rho(l^*)$ and $T_1 = \rho(l^{(1)})$ are cyclic subgroups of $G_0 = GL_n(k)$; let x_0 (resp. x_1) be a generator of T_0 (resp. T_1). Obviously, x_0 is a regular element in $\mathbf{GL}_n$ implying that its centralizer C_0 in G_0 coincides with the cyclic subgroup $\langle x_0 \rangle$. We claim that x_1 is also regular, and therefore its centralizer in G_0 (resp. $G_1 = SL_n(k)$) coincides with C_0 (resp. $C_1 = \langle x_1 \rangle$). To see the regularity it suffices to observe that if $s \in l^{(1)}$ is a generator, then all the elements $s, s^q, \ldots, s^{q^{n-1}}$, obtained by applying the consecutive powers of the Frobenius automorphism, are pairwise distinct (since for any i, j between 0 and $n-1$, the difference $q^i - q^j$ is strictly less than the order of s which is equal to $(q^n-1)/(q-1)$).

Next, let σ be a generator of $\operatorname{Gal}(l/k)$. Since $N_{l/k}(l^*) = k^*$, the Skolem–Noether theorem easily implies the existence of an element $y_1 \in G_1$ which normalizes T_0 and is such that the inner automorphism $\operatorname{Int}(y_1)$ induces on $\rho_{l/k}(l)$ the automorphism $\rho_{l/k} \circ \sigma \circ \rho_{l/k}^{-1}$. Then any element $y \in y_1 C_0$ enjoys the latter two properties as well, and obviously the intersection $C_0 \cap C_{G_0}(y)$ coincides with $\rho_{l/k}(k^*)$ which is the center of G_0. Another property of y to be used in the sequel is given by the following statement.

Lemma 5.2. *Any $y \in y_1 C_0$ is conjugate in $GL_n(k)$ to a matrix of the form*

$$w(\alpha) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ \alpha & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \end{pmatrix}$$

for some $\alpha \in k^*$. In particular, y is a regular element in $\mathbf{GL}_n$.

Proof. Since $y^n \in C_0 \cap C_{G_0}(y_1)$, we have $y^n = \alpha E_n$ for some $\alpha \in k^*$ (where E_n is the identity matrix). Let us show that y is conjugate to the corresponding matrix $w = w(\alpha)$, it being sufficient to establish the conjugacy in $\mathbf{GL}_n$. Obviously, y normalizes the torus $\mathbf{T}_0 = \mathbf{R}_{l/k}(\mathbb{G}_m)$, and therefore the conjugation which brings $\mathbf{T}_0$ to the diagonal form takes y to a certain monomial matrix μ. To every monomial matrix there corresponds

in an obvious way a permutation of indices $\{1,\ldots,n\}$, and the permutation corresponding to μ is by our construction a cycle of length n. This means that conjugating μ by a suitable permutation matrix, if necessary, we may assume that μ looks as follows

$$\mu(\alpha_1,\ldots,\alpha_n)=\begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_n \\ \alpha_1 & 0 & \cdots & 0 & 0 \\ & \alpha_2 & & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & & & \alpha_{n-1} & 0 \end{pmatrix}.$$

The conjugation by the diagonal matrix

$$\operatorname{diag}(1,\alpha_2\ldots\alpha_n,\ldots,\alpha_{n-1}\alpha_n,\alpha_n)$$

brings $\mu(\alpha_1,\ldots,\alpha_n)$ to the form $w(\beta), \beta=\alpha_1,\ldots,\alpha_n$. However, since $w(\beta)^n=\beta E_n$, we conclude that $\beta=\alpha$.

Recall now the following criterion for regularity (cf. [13], § 1): $g\in\mathbf{GL}_n$ is regular if and only if for any λ in some algebraically closed field containing the entries of g, the rank of the matrix $(g-\lambda E_n)$ is $\geqslant n-1$. To verify this criterion for y, notice that deleting the first row and the last column in the matrix $w(\alpha)-\lambda E_n$, we obtain the matrix

$$\begin{pmatrix} \alpha & -\lambda & & & \\ 0 & 1 & -\lambda & & \\ & & 1 & & \\ & & & \ddots & -\lambda \\ & & & & 1 \end{pmatrix}$$

implying that $\operatorname{rank}(y-\lambda E_n)=\operatorname{rank}(w(\alpha)-\lambda E_n)\geqslant n-1$. ■

Using this information, we are already in a position to consider the following.

Groups $PSL_n(q)$. First, let $q=2$. Then $G=PSL_n(q)$ is isomorphic to $SL_n(q)=GL_n(q)$. Since the group $SL_2(2)$ is not simple and $SL_3(2)$ is isomorphic to $PSL_2(7)$, we may assume that $n\geqslant 4$. We will prove that for $x=x_1$ and $y=y_1$

$$E(x,y)\neq G. \tag{5.4}$$

This will be done by estimating the number of elements in $E(x,y)$ and comparing it to the order of G. Since in our setting $C_G(x)\cap C_G(y)=(e)$, $\#C_G(x)\leqslant 2^n-1$ and $\#C_G(y)\leqslant 2^n-1$ (the last inequality follows from the fact that the centralizer of a regular element in M_n is a vector space of dimension n), we obtain from Lemma 5.1 that

$$\#E(x,y)\leqslant\frac{2^{5n}}{n}.$$

Let $\alpha_n=2^{5n}/n$, $\beta_n=\#SL_n(\mathbf{F}_2)$. We will show that for any $n\geqslant 5$

$$\alpha_n<\beta_n, \tag{5.5}$$

proving (5.4) for all $n\neq 4$. To establish (5.5), observe that for $n\geqslant 3$ we have

$$\frac{\alpha_{n+1}}{\alpha_n}\leqslant\frac{\beta_{n+1}}{\beta_n}. \tag{5.6}$$

Indeed, obviously

$$\frac{\alpha_{n+1}}{\alpha_n} = 2^5 \cdot \frac{n}{n+1} < 2^5,$$

while, using the formula for β_n (cf. [16]), we obtain

$$\frac{\beta_{n+1}}{\beta_n} = \frac{(2^{n+1}-1)(2^{n+1}-2)\dots(2^{n+1}-2^n)}{(2^n-1)(2^n-2)\dots(2^n-2^{n-1})} = 2^n \cdot (2^{n+1}-1) > 2^5.$$

By virtue of (5.6), the inequality $\alpha_n < \beta_n$ implies $\alpha_{n+1} < \beta_{n+1}$, so it suffices to prove (5.5) for $n = 5$, which is done by direct computations

$$\alpha_5 = \frac{33,554,432}{5}, \beta_5 = 9,999,360.$$

The case $n = 4$ requires slightly more delicate estimations. Since $y^4 = E_4$, y is a regular unipotent element, and therefore $\#C_G(y) = 8$. Thus, $C_G(y) = \{1, z\} \cdot \langle y \rangle$, and in view of (5.3) we obtain

$$E(x, y) = (C\{1, z\} C\{1, z\} C) \cdot \langle y \rangle.$$

Then

$$\#E(x, y) \leqslant (15^3 + 2 \cdot 15^2 + 15) \cdot 4 = 15,360$$

while $\#SL_4(\mathbf{F}_2) = 20,160$.

Now, consider one of the two remaining cases: $G = PSL_4(\mathbf{F}_3)$. Let $\tilde{G} = SL_4(\mathbf{F}_3)$ and $\varphi: \tilde{G} \to G$ be the canonical homomorphism. Let $x = \varphi(x_1)$ and $y = \varphi(y_1)$, and show that for these elements we have (5.4). Indeed, it is easy to see that if s is a generator of $l^{(1)}$ where l is the extension of $k = \mathbf{F}_3$ of degree 4, then $k(s^2) = l$, implying that $C_G(x) = \varphi(C_{\tilde{G}}(x_1))$, and therefore $\#C_G(x) = 20$. Next the characteristic polynomial of any $z_1 \in y_1 T_1$ has the form $\lambda^4 + 1$, implying that the algebra $\mathbf{F}_3[z_1]$ is isomorphic to $\mathbf{F}_9 \oplus \mathbf{F}_9$, and therefore $C_{\tilde{G}}(z_1)$ is of order 32. There exists an element $t \in T_1$ such that $y_1 t y_1^{-1} = -t$, and then $z_1 t z_1^{-1} = -t$ for any $z_1 \in y_1 T_1$. Then for $z = \varphi(z_1)$ we have $C_G(z) = \{1, \varphi(t)\} \cdot D$, where $D = \varphi(C_{\tilde{G}}(z_1))$ is of order 16. Using (5.3), we obtain the following estimation:

$$\#E(x, y) \leqslant 20^3 \cdot 16^2 = 2,048,000$$

while $\#G = 6,065,280$, and (5.4) is proved.

The last case $G = PSL_3(4)$ could be treated similarly, however we prefer to bring out at this point one technical statement which is applicable in a number of cases (in particular, to all sporadic groups).

Lemma 5.3. *Let G be a finite group, and let $x \in G$ be an element of a prime order p, such that the centralizer C of x in G coincides with the cyclic group $\langle x \rangle$. Suppose that p^2 does not divide the order of G, and that G has $c(p)$ conjugacy classes of elements of order p. Then*

(i) *$N_G(C)$ contains an element y of order $d(p) = (p-1)/c(p)$;*
(ii) *if m is the maximum of orders of the centralizers of elements of order $d(p)$ in G, then*

$$\#E(x, y) \leqslant f(x, y) = \frac{m^2 p^3}{d(p)}.$$

(As was pointed out by Fischer, the condition that p^2 does not divide the order of G is superfluous).

Proof. Consider the homomorphism $\mu: N_G(C) \to \mathrm{Aut}(C)$ sending $g \in N_G(C)$ to the restriction of the inner automorphism $\mathrm{Int}(g)$. It follows from our assumptions, that any element of order p in G is conjugate to an element of C, and therefore $c(p)$ is equal, in fact, to the number of orbits of $\mathrm{Im}(\mu)$ on the set $X = \{x, \ldots, x^{p-1}\}$. Since $\mathrm{Aut}(C)$ is a cyclic group acting simply transitively on X, we obtain that $c(p) = [\mathrm{Aut}(C):\mathrm{Im}(\mu)]$, implying that $\mathrm{Im}(\mu)$, and consequently also $N_G(C)$, contain an element of order $d(p)$.

Part (ii) is immediate from Lemma 5.1. ∎

So, to prove that $E(x, y) \neq G$ it suffices to verify that $f(x, y) < \#G$. This is done by using information from [1] about the centralizers of elements. For example, if $G = PSL_3(4)$, one finds in [1], p. 24, that an element $x \in G$ of order 7 satisfies the assumptions of Lemma 5.3. Besides, $c(7) = 2$ and for y we take an element of order $d(7) = 3$. Again from the table in [1] we conclude that $m = 9$. Thus $f(x, y) = 9{,}261$ while $\#G = 20{,}160$.

Alternating groups. The only case left out in Theorem 4.1 is $G = A_8$. This group, however, is known to be isomorphic to $SL_4(2)$ which has already been considered above.

To exhibit the required elements x, y in the groups of other classical types, we embed into these a suitable group $GL_n(q)$ or $SL_n(q)$ and take the images of x_0 (or x_1) and y_1 under this embedding. The calculation of the centralizer of the element x thus obtained is based on the following statement.

Lemma 5.4. *Assume that $n \geqslant 3$. If x is one of the matrices x_0 or x_1 constructed above, then x does not have any common eigenvalues with $-x$ (if* char $\neq 2$) *and* $\pm x^{-1}$. *In particular, the matrix* $\mathrm{diag}(x, {}^t x^{-1})$ *is regular in* GL_{2n}.

Proof. If λ is one eigenvalue of x, then all others are $\lambda^q, \ldots, \lambda^{q^{n-1}}$. It follows that if λ is common to x and $-x$, then $\lambda = -\lambda^{q^i}$ for some $1 \leqslant i \leqslant n-1$. Since the order of λ is either $(q^n - 1)$ (if $x = x_0$) or $l(n, q) = (q^n - 1)/(q - 1)$ (if $x = x_1$), we obtain

$$2(q^i - 1) \equiv 0 \bmod l(n, q)$$

which is impossible for $n \geqslant 3$, as is easily seen. Similarly, if λ is common to x and $\pm x^{-1}$, there should be $i, 1 \leqslant i \leqslant n-1$ such that

$$2(q^i + 1) \equiv 0 \bmod l(n, q)$$

which is not possible either. The lemma is proved. ∎

Groups $PSp_{2m}(q), m > 1, q = 2$ or 3. The group $PSp_4(3)$ requires special treatment and will be considered at the end of this subsection; therefore, since the group $PSp_4(2)$ is not simple, we may assume that $m \geqslant 3$. Now it is convenient to use a presentation of $Sp_{2m}(q)$ other than the one used in § 4. Namely, the corresponding alternating matrix I will be taken in the following form

$$I = \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix},$$

and then $Sp_{2m}(q) = \{x \in M_{2m}(q) \mid {}^t x I x = I\}$. In this realization we have the embedding $\varepsilon: GL_m(q) \to Sp_{2m}(q)$ given by the formula

$$\varepsilon: a \to \begin{pmatrix} a & 0 \\ 0 & ({}^t a)^{-1} \end{pmatrix}.$$

Also, let $\varphi:\tilde{G}=Sp_{2m}(q)\to PSp_{2m}(q)=G$ be the natural projection (note that $\mathrm{Ker}(\varphi)=1$ for $q=2$), $\tilde{x}=\varepsilon(x_0)$, $\tilde{y}=\varepsilon(y_0)$, and $x=\varphi(\tilde{x})$, $y=\varphi(\tilde{y})$.

It follows from Lemma 5.4 that $C_{\tilde{G}}(\tilde{x})$ is the cyclic group $\langle\tilde{x}\rangle$. Besides, the matrices $\tilde{x}$ and $-\tilde{x}$ are not conjugate if char $\neq 2$, implying that $C_G(x)=\varphi(C_{\tilde{G}}(\tilde{x}))$. Thus,

$$\#C_G(x)=\begin{cases}2^m-1 & \text{if } q=2,\\ 1/2\cdot(3^m-1), & \text{if } q=3.\end{cases}$$

Furthermore, it follows from Lemma 5.2 that any $\tilde{z}\in\tilde{y}C_{\tilde{G}}(\tilde{x})$ is conjugate in $\tilde{G}$ to a matrix of the form $s=\mathrm{diag}(w,{}^tw^{-1})$ where $w=w(\alpha)$ for some $\alpha\in\mathbf{F}_q^*$. However, since $q=2$ or 3, w is orthogonal (i.e. ${}^tww=E_m$), implying that $C_{\tilde{G}}(\tilde{z})$ is identifiable with the unitary group $U_2(R)$ over the ring $R=\mathbf{F}_q[w]$ defined by the equation

$$a^*Ja=J,$$

where $J=\begin{pmatrix}0 & e\\ -e & 0\end{pmatrix}$ ($e\in R$ is the multiplicative identity), $(a_{ij})^*=(\tau(a_{ji}))$, τ being the involution of R defined by taking the matrix transpose. To estimate the order of $U_2(R)$, consider the subgroup $H\subset GL_2(R)$ consisting of matrices of the form $\mathrm{diag}(r,e)$, $r\in R^*$. Obviously, H has trivial intersection with $U_2(R)$, hence

$$\#(H\cdot U_2(R))=(\#H)\cdot(\#U_2(R))\leqslant\#GL_2(R)$$

and

$$\#U_2(R)\leqslant\frac{\#GL_2(R)}{\#R^*}=\#SL_2(R).$$

However, as is well-known, $\#SL_2(R)\leqslant(\#R)^3$, and we obtain the estimation

$$\#C_{\tilde{G}}(\tilde{y})\leqslant q^{3m}.$$

Also, recall that the order of G is given by the formula (cf. [16]):

$$\#G=\gamma\cdot q^{m^2}(q^2-1)(q^4-1)\dots(q^{2m}-1)\tag{5.7}$$

where $\gamma=1$ if char $=2$, and $\gamma=1/2$ otherwise. Now, let us verify (5.4). First, let $q=2$. It follows from the above that $C_G(x)\cap C_G(y)=(e)$, and therefore by Lemma 5.1

$$\#E(x,y)\leqslant\frac{(\#C_G(x))^3\cdot(\#C_G(y))^2}{\mathrm{ord}(y)}\leqslant\frac{2^{9m}}{m}.\tag{5.8}$$

Then, using (5.7) one easily verifies that $\#E(x,y)<\#G$ for any $m\geqslant4$. For $q=3$, we obtain the following estimation

$$\#E(x,y)\leqslant(\#C_G(x))^3\cdot(\#C_G(y))^2\leqslant\tfrac{1}{8}\cdot3^{9m},\tag{5.9}$$

and again it is easy to check that $\#E(x,y)<\#G$ for any $m\geqslant4$.

The case $m=3$ requires slightly sharper estimations. With the notations as above, let W be the image of the (R)-determinant on $U_2(R)$; obviously W is intermediate between

$$U_1(R)=\{a\in R^*\,|\,a\cdot\tau(a)=1\}\text{ and }(R^*)^{1-\tau}=\{a/\tau(a)\,|\,a\in R^*\}.$$

Then

$$\#U_2(R)=(\#W)\cdot(\#SU_2(R)).$$

On the other hand, direct computation show that $SU_2(R)=SL_2(R_0)$ where $R_0=R^\tau$ is the subring of τ-symmetric elements. If $m=3$, $q=2$, then $R=\mathbf{F}_2[t]/(f(t))$ where

$f(t) = t^3 - 1$, i.e. $R = \mathbf{F}_2 \oplus \mathbf{F}_4$. Besides, τ acts trivially on $\mathbf{F}_2$ and non-trivially on $\mathbf{F}_4$. Obviously, $U_1(R) = (R^*)^{1-\tau}$ has order 3 and $R_0 = \mathbf{F}_2 \oplus \mathbf{F}_2$, implying

$$\#U_2(R) = (\#U_1(R)) \cdot (\#SL_2(\mathbf{F}_2))^2 = 108.$$

For $m = 3$, $q = 3$ we have $R = \mathbf{F}_3[t]/(f(t))$ where $f(t) = t^3 \pm 1$. So, R is identifiable with the ring $\mathbf{F}_3[\delta]$, $\delta^3 = 0$, with τ sending δ to $2\delta \pm \delta^2$. Obviously, under the homomorphism $R \to \mathbf{F}_3$ sending δ to zero, $U_2(R)$ is mapped onto $Sp_2(\mathbf{F}_3)$, and therefore W is contained in $(1 + \delta\mathbf{F}_3[\delta])$. Using this fact, it is easy to show that W has order 3. Also, R_0 coincides with the subring generated by δ^2, and consequently

$$\#U_2(R) = (\#W) \cdot (SL_2(R_0)) = 1944.$$

Thus, we have shown that if $m = 3$, the order of $C_{\tilde{G}}(\tilde{y})$ is 108 for $q = 2$, and is 1944 for $q = 3$.

Using the figures in (5.8), we obtain for $m = 3$, $q = 2$ that

$$\#E(x, y) \leqslant \tfrac{1}{3} \cdot 7^3 \cdot 108^2 = 1{,}333{,}584,$$

while the order of G equals 1,451,520. For $m = 3$, $q = 3$, one should observe that $C_G(x) \cap C_G(y) = (e)$ (this is the case for any odd m) and therefore (5.9) can be sharpened to

$$\#E(x, y) \leqslant \tfrac{1}{24} \cdot (\#C_G(x))^3 \cdot (\#C_G(y))^2,$$

using which it is easy to verify that again $\#E(x, y) < \#G$.

The last remaining case is $G = PSp_4(3)$. It follows from [1] that G has a single conjugacy class of elements of order 5, and if x is one of them, then C, the centralizer of x in G, is $\langle x \rangle$. By Lemma 5.3, $N_G(C)$ contains an element y of order 4, and then

$$\#E(x, y) \leqslant \frac{5^3 \cdot (\#C_G(y))^2}{4}. \tag{5.10}$$

Reference [1] provides two possibilities for the order of $C_G(y)$: 8 or 48. We shall show that in our case $\#C_G(y) = 8$, and then (5.10) easily yields the required result (observe, that $\#G = 25{,}920$). Let $\tilde{x} \in Sp_4(3)$ be an element of order 10 in $\varphi^{-1}(x)$, and $\tilde{y} \in \varphi^{-1}(y)$. It is easy to see that $\tilde{x}$ has 4 distinct eigenvalues in the natural action on the 4-dimensional space, and the conjugation by $\tilde{y}$ induces a cyclic permutation of the lines spanned by the corresponding eigenvectors. Now, it follows from the proof of Lemma 5.2, that $\tilde{y}$ is a regular element in GL_4. Thus, the centralizer of $\tilde{y}$ in $Sp_4(3)$ is a 2-dimensional torus splitting over the quadratic extension of $\mathbf{F}_3$, easily implying that the order of $C_{\tilde{G}}(\tilde{y})$ is $\leqslant 16$, hence $\#C_G(y) = 8$.

Groups $PSU_n(q^2)$. As is well-known, the group $G = PSU_n(q^2)$ for $n = 3$, $q = 2$, is not simple, and for $n = 4$, $q = 2$ is isomorphic to $PSp_4(3)$ which has already been considered. Besides, the following cases are easily considered with the help of Lemma 5.1, using elements of the indicated orders:

$$\begin{aligned} &n = 4, \quad q = 3, \quad \operatorname{ord}(x) = 5,\\ &n = 5, \quad q = 2, \quad \operatorname{ord}(x) = 11,\\ &n = 6, \quad q = 2, \quad \operatorname{ord}(x) = 7. \end{aligned}$$

So, throughout this subsection $n \geqslant 7$ and $q = 2$. We let $k = \mathbf{F}_2$; $k_0 = \mathbf{F}_4$, and let $x \to \bar{x}$ denote the natural automorphism of k_0 over k. Also, for $x = (x_{ij})$ in $GL_m(k_0)$ we let $x^* = (\overline{x_{ji}})$.

Lemma 5.5. Let $x \in GL_m(k_0)$ be one of the matrices x_0 or x_1. Then for any $\mu \in k_0$, x does not have common eigenvalues with $\mu x (\mu \neq 1)$ and $\mu (x^*)^{-1}$.

Proof. As in the proof of Lemma 5.4, if λ is one of the eigenvalues of x, then all others are: $\lambda^4, \ldots, \lambda^{4^{m-1}}$. Thus, if λ is common to x and $\mu x (\mu \neq 1)$, it satisfies a relation of the form: $\lambda = \mu \lambda^{4^i}$, for some $i = 1, \ldots, m-1$. This implies the congruence $9(4^i - 1) \equiv 0 \bmod (4^m - 1)$, which never holds true. Similarly, if λ is common to x and $\mu (x^*)^{-1}$, then for some i we have $\lambda = \mu \lambda^{-2 \cdot 4^i}$, implying $9 \cdot (2 \cdot 4^i + 1) \equiv 0 \bmod (4^m - 1)$, which is again impossible for $m \geqslant 3$. The lemma is proved. ∎

First, we consider the case $n = 2m, m \geqslant 4$. It is convenient to use the following realization of the unitary group:

$$U_n = \{x \in GL_n(\mathbf{F}_4) | x^* I x = I\}, \tag{5.11}$$

where

$$I = \begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix}$$

and $(x_{ij})^* = (\overline{x_{ji}})$. Let $\varepsilon: GL_m(\mathbf{F}_4) \to U_n$ be the embedding given by the formula

$$\varepsilon: a \mapsto \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}.$$

Obviously, for $\tilde{G} = SU_n$ one has $\varepsilon^{-1}(\tilde{G}) = SL_m(\mathbf{F}_4)$. Let $\tilde{x} = \varepsilon(x_1)$, $\tilde{y} = \varepsilon(y_1)$, and $x = \varphi(\tilde{x})$, $y = \varphi(\tilde{y})$. It follows from Lemma 5.5 that the centralizer $C_G(x)$ is equal to $(\varphi \circ \varepsilon)(T_1)$, and therefore its order is $\leqslant \frac{1}{3}(4^m - 1)$. Any $z \in y C_G(x)$ is of the form $z = \varphi(\tilde{z})$ where $\tilde{z} \in \tilde{y} \varepsilon(T_1)$, and it suffices to estimate the order of $C_{\tilde{G}}(\tilde{z})$ (which is $\geqslant \# C_G(z)$). Actually, for the case of n odd we will need the estimation of the order of the centralizer in U_n (for n even) of any $z \in \tilde{y} \varepsilon(T_0)$. For this reason, we produce here the latter estimation, which is also the upper bound for $C_{\tilde{G}}(\tilde{z})$. It follows from Lemma 5.2 that $\tilde{z}$ is conjugate in U_n to the matrix $w = \mathrm{diag}(w(\alpha), (w(\alpha)^*)^{-1})$, for some $\alpha \in k_0^*$. Since $\alpha \bar{\alpha} = 1$, the matrix $w(\alpha)$ is unitary, and therefore the centralizer of w in U_n is isomorphic to the unitary group $U_2(R)$ over the ring $R = k_0[w(\alpha)]$ (which is the centralizer of $w(\alpha)$ in $M_m(k_0)$), provided with the involution τ, acting as $*$ on $w(\alpha)$ and as $^-$ on k_0, and with respect to the matrix

$$J = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}.$$

To estimate the order of $U_2(R)$, observe that $SU_2(R) = SL_2(R_0)$ where $R_0 = R^\tau$ (since char $= 2$), and therefore

$$\# SU_2(R) \leqslant (\# R_0)^3 = 2^{3m}.$$

Then

$$\# U_2(R) \leqslant 2^{3m} \cdot d, \tag{5.12}$$

where d is the order of the subgroup $U_1(R)$ of elements $r \in R^*$ satisfying $\tau(r) r = e$. In particular, since the order of the multiplicative group of R^* is $< 2^{2m}$, we obtain that

$$\# U_2(R) \leqslant 2^{5m}.$$

Then

$$\# E(x, y) \leqslant \frac{2^{16m}}{27},$$

and using the formula for the order of G,

$$\#G=\gamma\cdot 2^{(2m-1)m}(2^2-1)(2^3+1)\ldots(2^{2m-1}+1)(2^{2m}-1),$$

where γ is 1 or 1/3, we easily obtain that $\#E(x,y)<\#G$ for any $m\geqslant 4$.

Now, let $n=2m+1, m\geqslant 3$. Then U_n can again be presented in the form (5.11) where

$$I=\begin{pmatrix} 0 & E_m & 0 \\ E_m & 0 & \vdots \\ & & 0 \\ 0 \quad \cdots & 0 & 1 \end{pmatrix},$$

and we may consider the embedding $\varepsilon: GL_m(k_0)\to SU_n$ given by the formula

$$\varepsilon: a\mapsto\begin{pmatrix} a & 0 & 0 \\ 0 & (a^*)^{-1} & \vdots \\ & & 0 \\ 0 \quad \cdots & 0 & \chi(a) \end{pmatrix},$$

where $\chi(a)=(\det(a))^{-1}\overline{(\det(a))}$. Let $\tilde{x}=\varepsilon(x_0)$, $\tilde{y}=\varepsilon(y_0)$ and $x=\varphi(\tilde{x})$, $y=\varphi(\tilde{y})$.

It easily follows from Lemma 5.5 that the centralizer $C_G(x)$ is equal to $(\varphi\circ\varepsilon)(T_0)$, hence its order is $\leqslant 4^m-1$. To estimate the order of $C_{\tilde{G}}(\tilde{z})$ for $\tilde{z}\in\tilde{y}\varepsilon(T_0)$, observe that if $\tilde{z}(e_n)=\lambda e_n$, then the λ-eigenspace V of $\tilde{z}$ in the natural action on $\mathbf{F}_4^n$ is at most 3-dimensional (since any element in $y_1 T_0$ is regular). Obviously, V is invariant under $C_{\tilde{G}}(\tilde{z})$, and therefore the index in $C_{\tilde{G}}(\tilde{z})$ of the subgroup B fixing e_n, does not exceed the number of elements in V, i.e. 2^6. On the other hand, B is isomorphic to the centralizer in U_{2m} of an element of the form similar to that of $\tilde{z}$, and we obtain that $\#C_{\tilde{G}}(\tilde{z})\leqslant 2^{5m+6}$. The corresponding estimation for $\#E(x,y)$ looks as follows:

$$\#E(x,y)\leqslant 2^{16m+12},$$

using which and the formula for the order of G:

$$\#G=\gamma\cdot 2^{(2m+1)m}\cdot(2^2-1)(2^3+1)\ldots(2^{2m}-1)(2^{2m+1}+1),$$

we easily obtain that $\#E(x,y)<\#G$ for all $m\geqslant 4$.

The case $n=7$ requires slightly more careful estimations. Assume that $\lambda=1$, i.e. $\tilde{z}(e_n)=e_n$. Observe now that the index $[C_{\tilde{G}}(\tilde{z}):B]$ cannot exceed, in fact, the number of elements in the subspace V defined above, of length 1. Using Witt's theorem, and the fact that V is non-degenerate in this case, it is easy to calculate that this number is 36. On the other hand, we *a priori* know that the centralizer of the corresponding element in SU_6 cannot exceed 2^{15}, and from the information in [1], we conclude that this bound, in fact, can be reduced to 1944. Thus, $\#C_{\tilde{G}}(\tilde{z})\leqslant 36\cdot 1944$, using which, it is easy to compute that $\#E(x,y)<\#G$ also in this case.

Now, assume that $\lambda\neq 1$. Then one can easily check that the corresponding λ-eigenspace V is 1-dimensional and the algebra R introduced above is the field $\mathbf{F}_{4^3}$. It follows that $U_1(R)$ is of order 9, and therefore according to (5.12),

$$\#U_2(R)\leqslant 9\cdot 2^9.$$

Eventually, we obtain that $\#C_{\tilde{G}}(\tilde{z})\leqslant 9\cdot 2^9$, using which one can easily check again that $\#E(x,y)<\#G$.

Groups $P\Omega_n(f), q=2$ or 3. First, let $n=2m, m\geqslant 4$ and f has Witt index m, i.e. $f(x_1,\dots,x_{2m})=x_1x_{m+1}+\cdots+x_mx_{2m}$. Then we have the embedding $\varepsilon: GL_m(q)\to SO_n(f)$ given by the formula

$$\varepsilon: a\mapsto\begin{pmatrix} a & 0\\ 0 & ({}^t a)^{-1}\end{pmatrix}.$$

Note that for $q=2$ we have $GL_m(q)=SL_m(q)$ and $SO_n(f)=\Omega_n(f)$, while for $q=3$ we have $\varepsilon^{-1}(\Omega_n(f))=SL_m(q)$. Also, let

$$\varphi:\tilde{G}=\Omega_n(f)\to P\Omega_n(f)=G$$

be the canonical projection; obviously $\mathrm{Ker}(\varphi)=1$ if either $q=2$ or $q=3$ and m is odd. Let $\tilde{x}=\varepsilon(x_1)$, $\tilde{y}=\varepsilon(y_1)$, and $x=\varphi(\tilde{x})$, $y=\varphi(\tilde{y})$. It follows from Lemma 5.5 that the centralizer $C_G(x)$ coincides with $(\varphi\circ\varepsilon)(T_1)$, and therefore has order $(q^m-1)/(q-1)$.

To estimate the order of $C_{\tilde{G}}(\tilde{z}), \tilde{z}\in\tilde{y}\varepsilon(T_1)$ observe that $\tilde{z}$ is conjugate in $SO_n(f)$ to a matrix of the form $\mathrm{diag}(w,w)$ where $w=w(\alpha), \alpha=\pm 1$. Then $C_{\tilde{G}}(\tilde{z})$ embeds into the unitary group $U_2(R)$ of the ring $R=\mathbf{F}_q[w]$ with the involution induced by taking the matrix transpose and with respect to the matrix

$$J=\begin{pmatrix} 0 & e\\ e & 0\end{pmatrix}.$$

One easily verifies that the subgroup $H\subset GL_2(R)$ consisting of matrices of the form $\mathrm{diag}(a,e), a\in R^*$, has trivial intersection with $U_2(R)$, implying that

$$\#(U_2(R)\cdot H)=(\#U_2(R))\cdot(\#H)\leqslant\#GL_2(R)$$

and, consequently,

$$\#U_2(R)\leqslant\frac{\#GL_2(R)}{\#H}=\#SL_2(R)\leqslant q^{3m}. \tag{5.13}$$

For $q=2$, we have $C_G(x)\cap C_G(y)=(1)$, so by Lemma 5.1

$$\#E(x,y)\leqslant\frac{(2^m)^3\cdot(2^{3m})^2}{m}=\frac{2^{9m}}{m}. \tag{5.14}$$

Similarly, for $q=3$

$$\#E(x,y)\leqslant\frac{3^{9m}}{8}. \tag{5.15}$$

On the other hand, the order of G is given by the formula

$$\#G=\gamma\cdot q^{m(m+1)}(q^2-1)(q^4-1)\dots(q^{2m-2}-1)(q^m-1), \tag{5.16}$$

where $\gamma=1$ for $q=2$, and $\gamma=1/2$ or $1/4$ for $q=3$. Using (5.14), (5.15) and (5.16), it is easy to show that $\#E(x,y)<\#G$ for any $m\geqslant 5$. The case $m=4$ is considered using Lemma 5.1 and elements of the following orders:

$$m=4,\quad q=2,\quad \mathrm{ord}(x)=7,$$
$$m=4,\quad q=3,\quad \mathrm{ord}(x)=13.$$

Next, let $n=2m+1$ and $f(x_1,\dots,x_{2m+1})=x_1x_{m+1}+\cdots+x_mx_{2m}+x_{2m+1}^2$. In view of the isomorphism $B_n(2)\simeq C_n(2)$, we need to consider here only the case $q=3$. Take the

following embedding $\varepsilon: GL_m(q) \to SO_n(f)$,

$$\varepsilon: a \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & ({}^t a)^{-1} & \vdots \\ & & 0 \\ 0 \quad \cdots & 0 & 1 \end{pmatrix}.$$

Obviously, $G = P\Omega_n(f)$ coincides with $\Omega_n(f)$ and $\varepsilon^{-1}(\Omega_n) = SL_m(q)$. Let $x = \varepsilon(x_1)$, $y = \varepsilon(y_1)$. As usual, it follows from Lemma 5.5 that $C_G(x) = \varepsilon(T_1)$, and therefore has order $1/2 \cdot (3^m - 1)$. To estimate $C_G(z)$ for $z \in y\varepsilon(T_1)$, we let V be the 1-eigenspace for z. Obviously, V is invariant under $C_G(z)$, contains the last basic vector e_n and is of dimension $\leqslant 3$. It follows that the subgroup D of $C_G(z)$ fixing e_n is of index at most $\# V = 3^3$, and from (5.13) we derive the estimation

$$\# C_G(z) \leqslant 3^{3m+3}.$$

Then

$$\# E(x, y) \leqslant (\tfrac{1}{2} \cdot 3^m)^3 \cdot (3^{3m+3})^2 = \frac{3^{9m+6}}{8}.$$

Since

$$\# G = \tfrac{1}{2} 3m^2 \cdot (3^2 - 1)(3^4 - 1) \ldots (3^{2m} - 1),$$

we obtain that $\# E(x, y) < \# G$ for any $m \geqslant 5$. To consider the case $m = 4$, notice that here $y_1^4 = -E_4$, in particular, y_1 does not have eigenvalue 1; therefore V is 1-dimensional and $[C_G(z):D] \leqslant 3$. Using [1] we see that the order of the centralizer of an element of order 4 in $P\Omega_8$ is $\leqslant 14 \cdot 10^3$ which implies that $\# C_G(z) \leqslant 6 \cdot 10^4$, and eventually that $\# E(x, y) < \# G$.

The case $m = 3$, $q = 3$ requires special consideration. It follows from [1] that G has a single conjugacy class of elements of order 15; let x be one of them. Then $C = C_G(x)$ coincides with $\langle x \rangle$, and the quotient-group $N_G(\langle x \rangle)/\langle x \rangle$ is isomorphic to $(\mathbb{Z}/15\mathbb{Z})^* \simeq (\mathbb{Z}/3\mathbb{Z})^* \times (\mathbb{Z}/5\mathbb{Z})^*$. Let S be a Sylow 2-subgroup in $N_G(\langle x \rangle)$. Then S projects isomorphically onto $(\mathbb{Z}/15\mathbb{Z})^*$; let $y \in S$ be the element corresponding to the pair $(-1 \bmod (3), 2 \bmod (5))$. Obviously, $\mathrm{Int}(y)$ acts on $\langle x \rangle$ without fixed points, and therefore

$$E(x, y) = C C_G(y) C C_G(y) C.$$

Now, writing $C_G(y)$ as a union of cosets modulo S and using the fact that S normalizes C, we conclude that

$$\# E(x, y) \leqslant \frac{(\# C)^3 \cdot (\# G_G(y))^2}{8} \leqslant \frac{15^3 \cdot (2880)^2}{8} < 35 \cdot 10^9$$

while $\# G > 4{\cdot}5 \times 10^9$.

Finally we consider the case: $n = 2m, m \geqslant 4$ and $f(x_1, \ldots, x_{2m}) = x_1 x_m + \cdots + x_{m-1} x_{2m-2} + g(x_{2m-1}, x_{2m})$ where g is an anisotropic binary form over $\mathbf{F}_q$. Consider the embedding $\varepsilon: GL_{m-1}(q) \to SO_n(f)$ given by the formula

$$\varepsilon: a \to \begin{pmatrix} a & 0 & 0 \\ 0 & ({}^t a)^{-1} & \vdots \\ & & 0 \\ 0 \quad \cdots & 0 & E_2 \end{pmatrix}.$$

The commutator subgroup $\Omega_n(f)$ coincides with $SO_n(f)$ if $q=2$, and is a subgroup of index 2 if $q=3$; moreover, in the latter case $\varepsilon^{-1}(\Omega_n(f)) = SL_{m-1}(q)$. As before, let $\varphi: \tilde{G} = \Omega_n(f) \to P\Omega_n(f) = G$ be the natural homomorphism (note that $\mathrm{Ker}(\varphi) = 1$ if $q=2$), and let $\tilde{x} = \varepsilon(x_1)$, $\tilde{y} = \varepsilon(y_1)$, and $x = \varphi(\tilde{x})$, $y = \varphi(\tilde{y})$. As usual, it follows from Lemma 5.5 that $C_G(x) = \varphi(C)$, where $C = (\varepsilon(T_0) \times SO_2(g)) \cap \Omega_n(f)$. Taking into account that in the case $q=3$, $-E_n \in \Omega_n(f)$ if and only if m is odd, we see that the order of $C_G(x)$ is as follows:

$$\#C_G(x) = \begin{cases} 3(2^{m-1}-1); & q=2, \\ 2(3^{m-1}-1); & q=3 \text{ and } m \text{ even}, \\ 3^{m-1}-1; & q=3 \text{ and } m \text{ odd}. \end{cases} \tag{5.17}$$

The estimation of the order of $C_{\tilde{G}}(\tilde{z})$ for $\tilde{z} \in \tilde{y}C$ is based on the reduction to the case of split form in $2(m-1)$ variables. Namely, if $B \subset C_{\tilde{G}}(\tilde{z})$ is the subgroup fixing the last two basic vectors e_{n-1}, e_n, then we already know that $\#B \leqslant q^{3(m-1)}$, and it suffices to estimate the index $d = [C_{\tilde{G}}(\tilde{z}):B]$. Let $\tilde{z} = \tilde{y}c$ where $c = \mathrm{diag}(a,b), a = \varepsilon(t)$, $t \in T_0$ and $b \in SO_2(g)$.

Case 1: $b = \pm 1$. Let V be respectively $(+1)$ or (-1)-eigenspace for $\tilde{z}$. Then V is invariant under $C_{\tilde{G}}(\tilde{z})$ and has dimension 2 or 4. In the first case V coincides with $V_1 = \langle e_{n-1}, e_n \rangle$ implying that d is bounded by the order of the orthogonal group $O(V_1)$ which is 3 for $q=2$ and 8 for $q=3$. Now, assume that $\dim V = 4$, and let V_2 be the orthogonal complement of V_1 in V. Then there are two possibilities: either V_1 is non-degenerate of Witt index 1 (this happens if, respectively, $+1$ or -1 is a simple eigenvalue for $y_1 t$) or the restriction of f to V_2 is zero. Obviously

$$d \leqslant \frac{\#O(V)}{\#O(V_2)},$$

and in the first case

$$\frac{\#O(V)}{\#O(V_2)} = \frac{q^2(q^4-1)}{q-1} = q^2(q^3+q^2+q+1),$$

this number being equal to 60 if $q=2$, and 360 if $q=3$, while in the second case

$$\frac{\#O(V)}{\#O(V_2)} = \frac{(\#GL_2(q)) \cdot q^4 \cdot (\#O(V_1))}{\#GL_2(q)} = q^4 \cdot (\#O(V_1))$$

which is 48 for $q=2$ and 648 for $q=3$.

Case 2: $b \neq \pm 1$. Then b has order 3 if $q=2$, and order 4 if $q=3$. If $y_1 t$ and b have no common eigenvalues, then $V_1 = \langle e_{n-1}, e_n \rangle$ is invariant under $C_{\tilde{G}}(\tilde{z})$ and $d \leqslant \#O(V_1)$. Now, let λ be a common eigenvalue for $y_1 t$ and b; λ belongs to $\mathbf{F}_{q^2}$, but not to $\mathbf{F}_q$. Obviously, the λ-eigenspace $V(\lambda)$ for $\tilde{z}$ in $\mathbf{F}_{q^2}^n$ is 3-dimensional and $C_{\tilde{G}}(\tilde{z})$-invariant. If v is the λ-eigenvector for b in V_1 and for $h \in C_{\tilde{G}}(\tilde{z})$ we have $h(v) = v$, then at the same time $h(\bar{v}) = \bar{v}$ where the bar denotes the nontrivial automorphism of $\mathbf{F}_{q^2}/\mathbf{F}_q$, and therefore h acts on V_1 identically, i.e. $h \in B$. This implies that

$$d \leqslant \#(V(\lambda) - \{0\})$$

which is 63 if $q=2$, and is 728 if $q=3$.

Putting the above information together, we obtain that

$$C_G(z) \leqslant \begin{cases} 63 \cdot 2^{3(m-1)}, & \text{if } q=2, \\ 728 \cdot 3^{3(m-1)}, & \text{if } q=3. \end{cases} \tag{5.18}$$

Using (5.17), (5.18) and the formula for the order of G:

$$\#G = \gamma \cdot q^{m(m-1)}(q^2-1)(q^4-1)\cdots(q^{2m-2}-1)(q^m+1),$$

where $\gamma = 1$ for $q=2$ and $\gamma = 1/2$ or $1/4$ for $q=3$, one easily shows that $\#E(x,y) < \#G$ for any $m \geqslant 6$.

The case $q=3$, $m=5$ requires slightly more precise estimations. First of all, observe that if in the above argument $b = \pm 1$, then necessarily $t \in T_1$ and therefore $(y_1 t)^4 = -1$. This implies that $y_1 t$ and b have no common eigenvalues and

$$\#C_{\tilde G}(\tilde z) \leqslant 8 \cdot 3^{12} < 4 \cdot 3 \cdot 10^6.$$

On the contrary, if b has order 4, then $t \in T_0 \setminus T_1$, implying that $y_1 t$ is conjugate to the matrix $w = w(1)$ (notations as in Lemma 5.2). To estimate the order of the centralizer of $s = \operatorname{diag}(w, w)$ in $\Omega_8(g)$ where g has Witt index 4, write $\mathbf{F}_3^8$ as

$$\mathbf{F}_3^8 = V_+ \perp V_- \perp V_0,$$

where V_+ (resp. V_-) is $(+1)$ (resp. (-1)) eigenspace of s, and V_0 is the orthogonal complement of $V_+ \perp V_-$. Obviously, each of these spaces is invariant under $C_{\tilde G}(\tilde z)$, and we obtain the estimation

$$\#C_{\tilde G}(\tilde z) \leqslant 728 \cdot \tfrac{1}{4}(\#O(V_+)) \cdot (\#O(V_-)) \cdot (\#O(V_0)) = 728 \cdot 4 \cdot 2 \cdot 24^2 < 4 \cdot 3 \cdot 10^6.$$

Using the value $4 \cdot 3 \cdot 10^6$ as the upper bound for the order of $C_{\tilde G}(\tilde z)$, we obtain by direct computation that $\#E(x,y) < \#G$.

To consider the following cases, one applies Lemma 5.1 to the elements of the indicated order:

$$\begin{aligned} &m=4, \quad q=2, \quad \operatorname{ord}(x)=7, \\ &m=5, \quad q=2, \quad \operatorname{ord}(x)=17. \end{aligned}$$

The case ${}^2D_4(3)$ requires special treatment. Here $G = P\Omega_8(f)$ where f is a quadratic form in 8 variables over $\mathbf{F}_3$ of Witt index 3. Obviously, in this case G is identifiable with $\tilde G = \Omega_8(f)$. Let F denote the matrix of f, and τ be the involution of $M_8(\mathbf{F}_3)$ given by the formula $\tau(a) = F^{-1}({}^t a)F$. Then $O_8(f)$ is defined by the equation $\tau(g)g = E_8$. The order computation shows that $\tilde G = \Omega_8(f)$ contains an element x of order 41. Then $L = \mathbf{F}_3[x]$, the subalgebra generated by x, is τ-invariant and is, in fact, a field extension of $\mathbf{F}_3$ of degree 8. One easily finds that $T = \{l \in L \mid \tau(l) \cdot l = 1\}$ has order 82, and therefore $T = \pm\langle x \rangle$ and $T \cap \Omega_8(f) = \langle x \rangle$. This allows to conclude that the centralizer $C = C_G(x)$ is $\langle x \rangle$.

Let $g \in GL_8(3)$ be the element such that $\operatorname{Int}(g)$ induces on L an automorphism of order 8. Since the restrictions of τ and $\operatorname{Int}(g)$ on L commute, we obtain that $g\tau(g) \in L^\tau$, and then for a suitable $l \in L^*$ we have $g\tau(g) = l\tau(l)$, and therefore $s = l^{-1}g$ satisfies $s\tau(s) = 1$. If we assume that $\det(s) = 1$, the characteristic polynomial of s would be $\lambda^8 + 1$, and therefore $-E_8 = s^8 \in \Omega_8(f)$ – a contradiction; hence $\det(s) = -1$, and the characteristic polynomial of s is $\lambda^2 - 1$, i.e. every 8th root of unity is eigenvalue of s with multiplicity 1. Let y be one of the elements s^2 or $-s^2$ which belongs to $\Omega_8(f)$. Then $\operatorname{Int}(y)$ induces on L an

automorphism of order 4,

$$E(x,y)=CC_G(y)CC_G(y)C,$$

and

$$\#E(x,y)\leqslant\frac{(\#C)^3\cdot(\#C_G(y))^2}{4}.$$

To estimate $\#C_G(y)$, observe that y has the following eigenvalues: $1, -1, i, -i$, (where $i^2=-1$), each with multiplicity 2. We have the following decomposition:

$$\mathbf{F}_3^8=V_+\perp V_-\perp V_0$$

where V_+ and V_- are $(+1)$ and (-1)-eigenspaces for y, and V_0 is the orthogonal complement of $V_+\perp V_-$. Each of these spaces is invariant under y, and the restriction of y to V_0 is not scalar. It follows that

$$\#C_G(y)<(O(V_+)\times O(V_-)\times H)\cap\Omega_8(f)$$

where H is *a proper* subgroup of $O(V_0)$, and therefore

$$\#C_G(y)\leqslant\frac{8\cdot 8\cdot 720}{4}=11{,}520.$$

Thus,

$$\#E(x,y)\leqslant 2{\cdot}29\cdot 10^{12}$$

while $\#G>10^{13}$.

The group ${}^3D_4(2)$. It follows from [1] that the group G of this type has a single conjugacy class of elements of order 7 having the centralizer of order 49; let x be one of them, and $C=C_G(x)$. Then $N_G(\langle x\rangle)/C$ is the cyclic group of order 6, and therefore there exists an element $y\in N_G(\langle x\rangle)$ of order 6. Again using [1] we see that the order of $C_G(y)$ can be either 72 or 24. Since neither of these numbers is divisible by 7, we conclude that the action of $\mathrm{Int}(y)$ on C is fixed-point free implying the fact that

$$E(x,y)=CC(y)CC(y)C.$$

Therefore

$$\#E(x,y)\leqslant\frac{49^2\cdot 72^2}{6}<1{\cdot}20\cdot 10^8$$

which is less than $\#G(>2{\cdot}1\cdot 10^8)$.

The groups ${}^2E_6(4), F_4(2)$ *and* $G_2(3)$. These are considered by applying Lemma 5.1 to the elements of the following orders:

$$\begin{array}{ll} {}^2E_6 & \mathrm{ord}\ (x)=13,\\ F_4(2) & \mathrm{ord}\ (x)=13,\\ G_2(3) & \mathrm{ord}\ (x)=7. \end{array}$$

Tits group ${}^2F_4(2)'$. This is considered using Lemma 5.1, if one takes for x an element of order 13.

For consideration of the remaining types we need the following proposition.

PROPOSITION 5.6

1) *Let* $\mathbf{D}$ *be a semisimple group of rank r over the field* $k=\mathbf{F}_q$. *Assume that* $\mathbf{D}$ *does not contain any factors of type* G_2. *Then for* $r\leqslant 5$ *we have the estimations for the order of* $\mathbf{D}(k)$ *given in table 5.*

Table 5.

r	$\#D(k)$
1	$< q^3$
2	$< 2 \cdot q^{11}$
3	$< 2 \cdot q^{21}$
4	$< q^{36}$
5	$< q^{55}$

2) $\#E_7(q) > \frac{1}{2^8} \cdot q^{133}$ *and* $\#E_8(q) < \frac{1}{2^8} \cdot q^{248}$. *Besides,* $\#E_6(2) > 2 \cdot 10^{23}$ *and* $\#F_4(3) > 5 \cdot 10^{24}$.

Proof. The proof easily follows from the formulas for the order of finite Chevalley groups (cf. [16]). ■

The group $E_6(2)$. Let $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) be the subgroup of the simple simply connected group $\mathbf{G}$ of type E_6 over $k = \mathbf{F}_2$ generated by the root subgroups $\mathbf{G}_\alpha$ for $\alpha \in \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ (resp. for $\alpha = \tilde{\alpha}$), simple roots being labeled as in the previous section. Then $\mathbf{G}_1 = \mathbf{SL}_6$, $\mathbf{G}_2 = \mathbf{SL}_2$, and besides, $\mathbf{G}_1$ and $\mathbf{G}_2$ commute. Let $\mathbf{T}_1$ be the following subtorus in $\mathbf{G}_1$:

$$\mathbf{T}_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbf{R}^{(1)}_{l/k}(\mathbb{G}_m) \right\},$$

where l/k is the extension of degree five, and $\mathbf{T} = (\mathbf{R}_{l/k}(\mathbb{G}_m) \times \mathbb{G}_m) \cap \mathbf{SL}_6$ be the maximal torus of $\mathbf{SL}_6$ which contains $\mathbf{T}_1$. A simple computation with roots shows that $C_\mathbf{G}(\mathbf{T}_1) = \mathbf{T} \times \mathbf{G}_2$. (Indeed, $\mathbf{T}_1$ is conjugate in $\mathbf{G}_1$ to the subgroup generated by h_α, $\alpha \in \{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, and one easily checks that the roots orthogonal to these α_i's are $\pm\tilde{\alpha}$.) Now, let x_1 be a generator of $\mathbf{T}_1(k)$; $\mathrm{ord}(x_1) = 31$. We claim that again $C_\mathbf{G}(x_1) = \mathbf{T} \cdot \mathbf{G}_2$. Assume the contrary. Since $\mathbf{T} = \mathbf{T}_1 \cdot \mathbb{G}_m$ and $\mathbf{T}_1$ does not have any proper k-defined subtori, the maximal central subtorus in $C_\mathbf{G}(x_1)$ would reduce to $\mathbb{G}_m$. But the group of the form $\mathbb{G}_m \cdot \mathbf{Z}$ where $\mathbf{Z}$ is the center of some semisimple group of rank $\leqslant 5$ cannot contain an element of order 31. Let $x = x_1 x_2$ where x_2 is some non-trivial unipotent element in $\mathbf{G}_2(k)$. Then the above argument shows that $C_G(x) = \mathbf{T}(k) \times \langle x_2 \rangle$, and therefore has order 62.

Next, let $y = \mathrm{diag}(y_0, 1) \in \mathbf{G}_1$ where $y_0 \in \mathbf{SL}_5(k)$ is such that $\mathrm{Int}(y_0)$ induces on l an automorphism of order 5, and let $z \in y C_G(x)$. Then $z = ytx_2^\varepsilon$ ($\varepsilon = 0$ or 1), and $C_G(z) \subset C_G(yt)$, so in estimating the order of $C_G(z)$ we may assume that $z = yt$ is an element of order 5 in $\mathbf{G}_1$. Obviously, z is contained in a k-defined subtorus $\mathbf{S} \subset \mathbf{G}_1$ of the form $\mathbf{R}_{m/k}(\mathbb{G}_m)$ where m is the extension of degree 4, and the maximal central torus in $C_G(\mathbf{S}_1)$ is $\mathbf{S} = \mathbf{R}_{m/k}(\mathbb{G}_m)$. Suppose $C_G(z)$ is different from $C_G(\mathbf{S}_1) = C_G(\mathbf{S})$. Then z is a k-element in a group of the form $\mathbf{S}' \cdot \mathbf{Z}$ where $\mathbf{S}' \subset \mathbf{S}$ is a proper k-subtorus different from $\mathbf{S}_1$ and $\mathbf{Z}$ is the center of a certain group of rank $\leqslant 5$. Since $\mathbf{Z}(k)$ does not contain elements of order 5, we conclude that $\mathbf{S}'$ must contain such an element, implying that $\mathbf{S}' \supset \mathbf{S}_1$ – a contradiction.

It follows that $C_G(z) = \mathbf{S} \cdot \mathbf{D}$ where $\mathbf{D}$ is some semisimple group of rank $\leqslant 2$. Using table 5, we conclude that

$$\#C_G(z) \leqslant 5 \cdot 2^{12}.$$

Therefore

$$\#E(x,y) \leqslant 62^3 \cdot 5^2 \cdot 2^{24} < 5 \cdot 10^9$$

while $\#G > 2 \cdot 10^{23}$.

The group $E_7(2)$. Let $\mathbf{G}_1$ be the subgroup of the group $\mathbf{G}$ of type E_7 over $k = \mathbf{F}_2$ generated by $\mathbf{G}_\alpha$ for $\alpha \in \{\tilde{\alpha}, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$; $\mathbf{G}_1 \simeq \mathbf{SL}_8$. Let $\mathbf{T} \subset \mathbf{G}_1$ be the maximal torus of the form

$$\mathbf{T} = (\mathbf{R}_{l_1/k}(\mathbb{G}_m) \times \mathbf{R}_{l_2/k}(\mathbb{G}_m)) \cap \mathbf{SL}_8,$$

where l_1 and l_2 are extensions of degree 5 and 3, respectively. Put $\mathbf{T}_i = \mathbf{R}^{(1)}_{l_i/k}(\mathbb{G}_m)$ $(i = 1, 2)$ and $\mathbf{T}_3 = \mathbf{T}_1 \times \mathbf{T}_2$. A simple computation with roots shows that $C_\mathbf{G}(\mathbf{T}_3) = \mathbf{T}$. Picking generators $x_i \in \mathbf{T}_i(k)$ for $i = 1, 2$, we let $x = x_1 x_2$ (note that x_1 and x_2 have orders 31 and 7, respectively, so the order of x is 217). We claim that $C_\mathbf{G}(x) = \mathbf{T}$. Indeed, if $C_\mathbf{G}(x) = \mathbf{T}' \cdot \mathbf{D}$ where $\mathbf{T}'$ is the central torus, $\mathbf{D}$ is semisimple, then $x \in \mathbf{T}' \cdot Z(\mathbf{D})$. Obviously, it is impossible that $\dim \mathbf{T}' = 1$. However, if $\dim \mathbf{T}' > 1$ then $\dim \mathbf{T}_3 \cap \mathbf{T}' \geqslant 1$, and since $\mathbf{T}_3$ does not have any k-defined subtori except for $\mathbf{T}_1$ and $\mathbf{T}_2$, we conclude that $\mathbf{T}'$ contains at least one of them. On the other hand, $\mathbf{T}'(k)$ contains an element of order 217. Suppose, for example, that $\mathbf{T}_1 \not\subset \mathbf{T}'$. We have $\mathbf{T} = \mathbf{T}_1 \times \mathbf{T}_2 \times \mathbb{G}_m$, and therefore $\mathbf{T}/(\mathbf{T}_1 \times \mathbb{G}_m) = \mathbf{T}_2$. Comparing the orders, we see that $\mathbf{T}' \not\subset \mathbf{T}_1 \times \mathbb{G}_m$, therefore $\mathbf{T}'$ projects surjectively onto $\mathbf{T}_2$. But this immediately implies that $\mathbf{T}' \supset \mathbf{T}_3$, so $C_\mathbf{G}(x) = \mathbf{T}$, and $\#C_G(x) = 217$.

Next, let $y_1 \in \mathbf{SL}_5(k)$ and $y_2 \in \mathbf{SL}_3(k)$ be the elements inducing on l_1 and l_2 the automorphisms of orders 5 and 3, respectively; let $y = y_1 y_2 \in \mathbf{G}_1(k)$. Clearly, any $z \in y\mathbf{T}(k)$ has order 15.

Lemma 5.7. $\#C_G(z) \leqslant 45 \cdot 2^{36}$.

Proof. Obviously, z is contained in a maximal k-torus $\mathbf{S} \subset \mathbf{G}_1$ of the form $\mathbf{S} = ((\mathbb{G}_m)^2 \times \mathbf{R}_{m_1/k}(\mathbb{G}_m) \times \mathbf{R}_{m_2/k}(\mathbb{G}_m)) \cap \mathbf{SL}_8$ where m_1 and m_2 are extensions of degrees 4 and 2, respectively. Let $C_G(z) = \mathbf{S}' \cdot \mathbf{D}$. We claim that rank $\mathbf{D} \leqslant 4$. Indeed, $z = (y_1 t_1)(y_2 t_2)$ where $t_i \in \mathbf{T}_i(k)$ and $\mathbf{C}_\mathbf{G}(z) = \mathbf{C}_\mathbf{G}(y_1 t_1) \cap \mathbf{C}_\mathbf{G}(y_2 t_2)$. Since there is an unipotent element in $C_\mathbf{G}(y_1 t_1)$ which does not commute with $y_2 t_2$, the semisimple rank of $C_\mathbf{G}(z)$ cannot exceed that of $C_\mathbf{G}(y_1 t_1)$ minus one. So, it remains to be shown that for $z_1 = y_1 t_1$ the semisimple rank is $\leqslant 5$. Assume that $C_\mathbf{G}(z_1) = \mathbf{S}'_1 \cdot \mathbf{D}_1$ and rank $\mathbf{D}_1 = 6$, i.e. $\dim \mathbf{S}'_1 = 1$. But $z_1 \in \mathbf{S}'_1 \cdot Z(\mathbf{D}_1)$ and since $\#\mathbf{S}'_1(k) \leqslant 3$ and no k-defined quotient of $Z(\mathbf{D}_1)$ can contain elements of order 5, we obtain a contradiction. So, using table 5, we get

$$\#C_G(z) = (\#\mathbf{S}'(k))(\#\mathbf{D}(k)) \leqslant 45 \cdot 2^{36},$$

as required. ■

Now, we obtain that

$$\#E(x,y) \leqslant (217)^3 \cdot (45 \cdot 2^{36})^2 \leqslant 2 \cdot 1 \cdot 10^{10} \cdot 2^{72},$$

on the other hand, $\#G > 2^{125}$, and one easily checks that $\#E(x,y) < \#G$.

The group $E_7(3)$. We will use the same subgroup $\mathbf{G}_1 \subset \mathbf{S}$ as in the previous case and let

$$\mathbf{T} = (\mathbf{R}_{l_1/k}(\mathbb{G}_m) \times \mathbf{R}_{l_2/k}(\mathbb{G}_m)) \cap \mathbf{SL}_8,$$

where l_1 and l_2 are extensions of degrees 5 and 2, respectively. Let $\mathbf{T}_i = \mathbf{R}^{(1)}_{l_i/k}(\mathbb{G}_m)$, $(i=1,2)$ and $\mathbf{T}_3 = \mathbf{T}_1 \times \mathbf{T}_2$. Again, a simple computation shows that $C_G(\mathbf{T}_3) = \mathbf{T}$. Picking arbitrary generators $x_i \in \mathbf{T}_i(k)$, let $x_0 = x_1 x_2$; the order of x_0 is 484. We claim that $C_G(x_0) = \mathbf{T}$. Indeed, if $C_G(x) = \mathbf{T}' \cdot \mathbf{D}$ then $x \in \mathbf{T}' \cdot Z(\mathbf{D})$, and $\mathbf{T}'$ must contain an element of order 121. Comparing the orders, we see that the projection of $\mathbf{T}'$ on $\mathbf{R}_{l_1/k}(\mathbb{G}_m)$ must contain $\mathbf{T}_1$; in particular, $\dim \mathbf{T}' \geqslant 4$. It follows that $\dim(\mathbf{T}' \cap \mathbf{T}_3) \geqslant 2$, and since $\mathbf{T}_3$ does not have any k-defined subtorus other than $\mathbf{T}_1$ and $\mathbf{T}_2$, we get $\mathbf{T}' \supset \mathbf{T}_1$. Now, the centralizer $C_G(\mathbf{T}_1)$ has semisimple rank $\leqslant 3$, and on the other hand, contains a unipotent element which does not commute with x_2. It follows, that rank $\mathbf{D} \leqslant 2$. However, then the exponent of the 2-component of $Z(\mathbf{D})$ is at most 2, and therefore $\mathbf{T}'$ contains an element of order $4 \cdot 121$. Now, the image of the projection of $\mathbf{T}'$ on $\mathbf{R}_{l_2/k}(\mathbb{G}_m)$ has to contain $\mathbf{T}_2$ and eventually $\mathbf{T}' = \mathbf{T}_3$, i.e. $C_G(x_0) = \mathbf{T}$. Let x_2' be a generator of l_2^*; put $\tilde{x} = \operatorname{diag}(x_1, x_2', -1)$. Then the above argument shows that $C_G(\tilde{x}^2) = \mathbf{T}$ implying that if $\varphi : \tilde{G} = \mathbf{G}(k) \to G$ is the canonical projection then for $x = \varphi(\tilde{x})$ we have $C_G(x) = \varphi(\mathbf{T}(k))$, and therefore $\# C_G(x) = 968$.

Now, let $y_1 \in \mathbf{SL}_5(k)$ and $y_2 \in \mathbf{SL}_2(k)$ be the elements which induce on l_1 and l_2 the automorphisms of orders 5 and 2, respectively. Put $\tilde{y} = \operatorname{diag}(y_1, y_2, 1)$ and $y = \varphi(\tilde{y})$.

Lemma 5.8. *For any $z \in yC_G(x)$, $\# C_G(z) \leqslant 1280 \cdot 3^{36}$.*

Proof. If suffices to estimate the order of $C_{\tilde{G}}(\tilde{z})$ where

$$\tilde{z} = \tilde{y}t,\ t = \operatorname{diag}(t_1, t_2, t_3) \in \mathbf{T}(k).$$

First of all, we are going to show that the semisimple rank of $C_{\mathbf{G}}(\tilde{z})$ is $\leqslant 4$. Write t_1 in the form $t_1 = \varepsilon t_1'$ where $t_1' \in \mathbf{R}^{(1)}_{l_1/k}(\mathbb{G}_m)$, $\varepsilon = \pm 1$, and let $t_2' = \operatorname{diag}(\varepsilon, t_2, t_3)$. Then $\tilde{z} = z_1 z_2$ where $z_i = y_i t_i$ and

$$C_{\mathbf{G}}(\tilde{z}) = C_{\mathbf{G}}(z_1) \cap C_{\mathbf{G}}(z_2).$$

Obviously, $C_{\mathbf{G}}(z_1)$ contains a unipotent element which does not commute with z_2, therefore it suffices to show that the semisimple rank of $C_{\mathbf{G}}(z_1)$ is $\leqslant 5$. Write $C_{\mathbf{G}}(z_1) = \mathbf{S}' \cdot \mathbf{D}$; then $z_1 \in \mathbf{S}' \cdot Z(\mathbf{D})$. If we assume that rank $\mathbf{D} = 6$, then $\dim \mathbf{S}' = 1$. However, this is impossible since $\# \mathbf{S}'(k) \leqslant 4$ and none of the k-defined quotients of $Z(\mathbf{D})$ can contain a k-element of order 5. Obviously, $\mathbf{S}'$ is a subtorus in a maximal k-torus $\mathbf{S} \subset \mathbf{G}_1$ of the form $(\mathbb{G}_m \times \mathbf{R}_{m_1/k}(\mathbb{G}_m) \times \mathbf{B} \times \mathbb{G}_m) \cap \mathbf{SL}_8$ where m_1 is the extension of degree 4, and $\mathbf{B}$ is either $(\mathbb{G}_m)^2$ or $\mathbf{R}_{m_2/k}(\mathbb{G}_m)$, $[m_2 : k] = 2$. In either case

$$\# C_{\tilde{G}}(\tilde{z}) = (\# \mathbf{S}'(k)) \cdot (\# \mathbf{D}(k)) \leqslant (\# \mathbf{S}(k)) \cdot (\# \mathbf{D}(k)) \leqslant 1280 \cdot 3^{36}.$$

The lemma is proved. ∎

Now,

$$\# E(x, y) \leqslant (968)^3 \cdot (1280 \cdot 3^{36})^2 < 2 \cdot 10^{15} \cdot 3^{72}.$$

On the other hand, $\# G > (1/2^8) \cdot 3^{133}$, and one can easily see that $\# E(x, y) < \# G$.

The group $E_8(2)$. Let $\mathbf{G}_1 \subset \mathbf{G}$ be the subgroup generated by the root subgroups $\mathbf{G}_\alpha$ for $\alpha \in \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \tilde{\alpha}\}$; $\mathbf{G}_1 = \mathbf{SL}_9$. Let $\mathbf{T} = \mathbf{R}^{(1)}_{l/k}(\mathbb{G}_m)$ be the maximal torus in $\mathbf{G}_1$ corresponding to the extension l of $k = \mathbf{F}_2$ of degree 9, and $x \in \mathbf{T}(k)$ be a generator $(\operatorname{ord}(x) = 7 \cdot 73 = 511)$. We claim that x is a regular element. Assume the contrary. Then

$x \in \mathbf{T}' \cdot \mathbf{Z}$ where $\mathbf{T}' \subset \mathbf{T}$ is a proper subtorus and $\mathbf{Z}$ is the center of some semisimple group of rank $\leqslant 7$. It follows that $\mathbf{T}'(k)$ contains an element of order 73 which immediately implies that $\mathbf{T}' = \mathbf{T}$, a contradiction. Now, let $y \in \mathbf{G}_1(k)$ be an element such that $\mathrm{Int}(y)$ induces on l an automorphism of order 9.

Lemma 5.9. *For any* $z \in y\mathbf{T}(k)$, *we have* $\#C_G(z) \leqslant 2^{23} \cdot 10^2$.

Proof. Obviously, z has order 9 and therefore is contained in a maximal torus $\mathbf{S} \subset \mathbf{G}_1$ of the form $(\mathbf{R}_{m_1/k}(\mathbb{G}_m) \times \mathbf{R}_{m_2/k}(\mathbb{G}_m) \times \mathbf{R}_{m_3/k}(\mathbb{G}_m)) \cap \mathbf{SL}_9$ where m_1, m_2, m_3 are extensions of degrees 6, 2 and 1 respectively. Let $\mathbf{R} = C_G(z)$ and write $\mathbf{R}$ as $\mathbf{R} = \mathbf{S}' \cdot \mathbf{D}$, almost direct product of the central torus $\mathbf{S}'$ and a semisimple group $\mathbf{D}$. Then $z \in \mathbf{S}' \cdot Z(\mathbf{D})$. We claim that the three component $\mathbf{Z}_3$ of the center of any semisimple group of rank $\leqslant 7$ has the following property: for any k-defined subgroup $\mathbf{Z}'_3 \subset \mathbf{Z}_3$ the map $\pi(k):\mathbf{Z}'_3(k) \to (\mathbf{Z}_3/\mathbf{Z}'_3)(k)$ is surjective.

First of all, notice that it suffices to prove that

$$\pi(f):\mathbf{Z}_3(f) \to (\mathbf{Z}_3/\mathbf{Z}'_3)(f),$$

where $[f:k] = 3$ is surjective. Indeed, then for $t \in (\mathbf{Z}_3/\mathbf{Z}'_3)(k)$ the fiber $\pi(f)^{-1}(t)$ is non-empty and $\mathrm{Gal}(f/k)$ invariant, hence contain a $\mathrm{Gal}(f/k)$ stable point, since $\mathrm{Gal}(f/k)$ has order 3. But $\mathbf{Z}_3$ is f-isomorphic to direct product of factors isomorphic to μ_3, the group of the 3rd roots of unity, or $\mu_3^{(1)} = \mathbf{R}^{(1)}_{m/k}(\mu_3)$, $[m:k] = 2$, and our assertion can be easily checked.

Applying this fact to the morphism $Z(\mathbf{D}) \to Z(\mathbf{D})/Z(\mathbf{D}) \cap \mathbf{S}'$ we obtain that $\mathbf{S}'(k)$ contains an element of order 9, and therefore $\dim \mathbf{S}' \geqslant 5$, i.e. the rank of $\mathbf{D}$ is $\leqslant 3$. It follows from table 5 that $\#\mathbf{D}(k) \leqslant 2^{22}$. On the other hand, $\#\mathbf{S}'(k) \leqslant \#\mathbf{S}(k) = 189$, hence our estimation. ∎

Now,

$$\#E(x, y) \leqslant (511)^3 \cdot (2^{23} \cdot 10^2)^2 < 2 \cdot 10^{27},$$

while $\#G > 3 \cdot 10^{74}$.

The group $E_8(3)$. Let $\mathbf{G}_1 \subset \mathbf{S}$ be the same as in the previous case, and let $\mathbf{T} \subset \mathbf{G}_1$ be a maximal k-torus of the form

$$\mathbf{T} = (\mathbf{R}_{l_1/k}(\mathbb{G}_m) \times \mathbf{R}_{l_2/k}(\mathbb{G}_m)) \cap \mathbf{SL}_9,$$

where l_1 and l_2 are extensions of degrees 7 and 2, respectively. Put $\mathbf{T}_i = \mathbf{R}^{(1)}_{l_i/k}(\mathbb{G}_m)$ $(i = 1, 2)$ and $\mathbf{T}_3 = \mathbf{T}_1 \times \mathbf{T}_2$. A simple computation shows that $C_G(\mathbf{T}_3) = \mathbf{T}$. Pick generators $x_i \in \mathbf{T}_i(k)$ $(i = 1, 2)$, (these have orders 1093 (a prime) and 4 respectively), and let $x = x_1 x_2$. We claim that $C_G(x) = \mathbf{T}$. Indeed, let $C_G(x) = \mathbf{T}' \cdot \mathbf{D}$; then $x \in \mathbf{T}' \cdot Z(\mathbf{D})$. This implies that $\mathbf{T}'(k)$ contains an element of order 1093. Then the projection of $\mathbf{T}'$ on $\mathbf{R}_{l_1/k}(\mathbb{G}_m)$ must contain $\mathbf{T}_1$. It follows that the semisimple rank of $C_G(x_1)$ is $\leqslant 2$. On the other hand, $C_G(x_1)$ contains a unipotent element which does not commute with x_2, implying that rank $\mathbf{D} \leqslant 1$. Then the projection of $\mathbf{T}'$ on $\mathbf{R}_{l_2/k}(\mathbb{G}_m)$ must contain $\mathbf{T}_2$, and eventually, $\mathbf{T}' \supset \mathbf{T}_3$, i.e. $C_G(x) = \mathbf{T}$ and $\#C_G(x) = 8744$.

Next, let $y_1 \in \mathbf{SL}_7(k)$, $y_2 \in \mathbf{SL}_2(k)$ be the elements which induce on l_1 and l_2 the automorphisms of orders 7 and 2 respectively, and $y = \mathrm{diag}(y_1, y_2)$.

Lemma 5.10. *For any* $z \in yC_G(x)$, $\#C_G(z) \leqslant 2912 \cdot 3^{55}$.

Proof. Let $z = yt$, $t = \operatorname{diag}(t_1, t_2)$. Write t_1 as $t_1 = \varepsilon t'_1$ where $t'_1 \in \mathbf{T}_1$, $\varepsilon = \pm 1$, and put $t'_2 = \operatorname{diag}(\varepsilon, t_2)$. Then $z = z_1 z_2$ where $z_i = y_i t_i$, and

$$C_G(z) = C_G(z_1) \cap C_G(z_2).$$

We claim that the semisimple rank of $C_G(z)$ is $\leqslant 5$. Since $C_G(z_1)$ obviously contains a unipotent element with does not commute with z_2, it suffices to show that the semisimple rank of $C_G(z_1)$ is $\leqslant 6$. Assume that the latter is 7. Then $z_1 \in \mathbf{S}' \cdot \mathbf{Z}$ where $\mathbf{S}'$ is a one-dimensional torus and $\mathbf{Z}$ is the center of a semisimple group of rank 6. However, since $\#\mathbf{S}'(k) \leqslant 4$ and none of the possible k-defined factors of $\mathbf{Z}$ contains a k-element of order 7, this is impossible. Thus, $C_G(z) = \mathbf{S}'' \cdot \mathbf{D}$ where rank $\mathbf{D} \leqslant 5$. Clearly, z is contained in a maximal k-torus $\mathbf{S} \subset \mathbf{G}_1$ of the form $\mathbf{S} = (\mathbb{G}_m \times \mathbf{R}_{m_1/k}(\mathbb{G}_m) \times \mathbf{B}) \cap \mathbf{SL}_9$ where $\mathbf{B}$ is either $(\mathbb{G}_m)^2$ or $\mathbf{R}_{m_2/k}(\mathbb{G}_m)$, $[m_2 : k] = 2$. Then

$$\#C_G(z) = (\#\mathbf{S}''(k)) \cdot (\#\mathbf{D}(k)) \leqslant (\#\mathbf{S}(k)) \cdot (\#\mathbf{D}(k)) \leqslant 2912 \cdot 3^{55}.$$

The lemma is proved. ■

It follows that

$$\#E(x, y) \leqslant (8744)^3 \cdot (2912 \cdot 3^{55})^2 < 10^{19} \cdot 3^{110}.$$

On the other hand, $\#G > (1/2^8) \cdot 3^{248}$, and $\#E(x, y) < \#G$.

The group $F_4(3)$. Let $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) be the subgroup in $\mathbf{G}$ of type F_4 over $k = \mathbf{F}_3$ generated by the root subgroups $\mathbf{G}_\alpha$ for $\alpha \in \{\tilde{\alpha}, \alpha_1, \alpha_2\}$ where $\tilde{\alpha}$ is the highest root (resp. for $\alpha = \alpha_4$). Then $\mathbf{G}_1 \simeq \mathbf{SL}_4$, $\mathbf{G}_2 \simeq \mathbf{SL}_2$ and $\mathbf{G}_1$ and $\mathbf{G}_2$ commute. Let $\mathbf{T}_1 = \mathbf{R}^{(1)}_{l/k}(\mathbb{G}_m)$ be the maximal torus in $\mathbf{G}_1$ corresponding to the extension l/k of degree 4. Obviously, $\mathbf{T}_1$ is the maximal central torus in its centralizer $C_G(\mathbf{T}_1)$. Let x_1 be a generator of $\mathbf{T}_1(k)$; x_1 has order 40. We claim that $C_G(x_1) = C_G(\mathbf{T}_1)$. Indeed, otherwise x_1 would be a k-element in a group of the form $\mathbf{T}' \cdot \mathbf{Z}$ where $\mathbf{T}' \subset \mathbf{T}$ is a k-subtorus of co-dimension $r \geqslant 1$ and $\mathbf{Z}$ is the center of a certain semisimple group of rank $\leqslant r + 1$. If $r = 1$ then $\mathbf{Z}$ has exponent 2 or 3 implying that $\mathbf{T}'$ should contain an element of order at least 20, which is impossible. If $r = 2$ then $\#\mathbf{T}'(k) = 4$ and $\#\mathbf{Z} \leqslant 8$, and again x_1 cannot belong to $(\mathbf{T}' \cdot \mathbf{Z})(k)$. Now, let x_2 be a non-trivial unipotent element in $\mathbf{G}_2(k)$, and $x = x_1 x_2$. Then the above argument shows that $C_G(x) = \mathbf{T}_1(k) \times \langle x_2 \rangle$; therefore $\#C_G(x) = 120$. Next, let $y \in \mathbf{G}_1(k) = SL_4(3)$ be the element which induces on l an automorphism of order 4. The characteristic polynomial of y is $\lambda^4 + 1$, i.e. y has order 8.

Lemma 5.11. *For any* $z \in yC_G(x)$, *we have* $\#C_G(z) \leqslant 8 \cdot 3^{14}$.

Proof. We have $z = ytx^\varepsilon$ where $t \in \mathbf{T}_1(k)$, $\varepsilon = 0, 1$ or 2. Since $C_G(z) \subset C_G(yt)$, in estimating the order of $C_G(z)$ we may assume that $z = yt \in \mathbf{G}_1$. Obviously z also has order 8. Since $\mathbf{G}$ is simply connected, $\mathbf{R} = C_G(z)$ is connected, and therefore $\mathbf{R} = \mathbf{S}_1 \mathbf{D}$, almost direct product of the maximal central torus $\mathbf{S}_1$ and a semisimple group $\mathbf{D}$. First, we show that $\mathbf{D}$ cannot belong to either of types A_3, B_3 or C_3. Indeed, if $\mathbf{D}$ were of type B_3 or C_3, the center of $\mathbf{D}$ would not be bigger than $\{\pm 1\}$, and $\dim \mathbf{S}_1 = 1$. Then either $\mathbf{S}_1$ or $\mathbf{S}_1 \times \{\pm 1\}$ would contain an element of order 8, which is impossible. To show that $\mathbf{D}$ cannot be of type A_3, notice that the semi-simple part of the centralizer of z^2 in $\mathbf{G}_1$ is

of type $(A_1)^2$, implying that $C_G(z^2)$ should contain $(A_1)^3$. So, if **D** were of type A_3, there would be an embedding $(A_1)^3 \subsetneq A_3$ which is impossible. As follows from table 5, in all other cases of rank three groups $\#D(k) \leqslant 2 \cdot 3^{14}$ and since $\#\mathbf{S}_1(k) \leqslant 4$, we obtain our estimation. If D is of rank $\leqslant 2$ then $\#D(k) \leqslant 2 \cdot 3^{11}$, and since $\#\mathbf{S}_1(k) \leqslant 40$, we again have our estimation. The lemma is proved. ∎

Using our estimations, we get

$$\#E(x, y) \leqslant (120)^3 \cdot (8 \cdot 3^{14})^2 < 3 \cdot 10^{21}$$

while $\#G > 5 \cdot 10^{24}$.

6. Proof of Theorem 3.5 for sporadic groups

All the sporadic groups are considered with the help of Lemma 5.1 and information from [1]. The results of computations are arranged in table 6, which for every sporadic group G contains: the orders of x and y, the maximum of orders of centralizers of

Table 6.

G	ord(x)	ord(y)	m	$f(x,y)$	$\#G$
M_{11}	5	4	8	2,000	7,920
M_{12}	11	5	10	26,620	95,040
M_{22}	11	5	5	33,275	443,520
M_{23}	11	5	15	299,475	10,200,960
M_{24}	11	10	20	532,400	244,823,040
J_1	7	6	6	12,348	175,560
J_2	7	6	24	197,568	604,800
J_3	17	8	8	314,432	50,232,960
J_4	23	22	264	847,991,232	$> 8{\cdot}6 \cdot 10^{19}$
$0{\cdot}1$	23	11	66	52,999,452	$> 4,1 \cdot 10^{18}$
$0{\cdot}2$	23	11	11	1,472,207	$> 4{\cdot}2 \cdot 10^{13}$
$0{\cdot}3$	23	11	22	5,888,828	495,766,656,000
Hs	7	6	36	444,528	44,352,000
Mc	11	5	750	748,687,500	898,128,000
Sz	11	5	40	26,240,827,392	448,345,497,600
He	17	8	16	1,257,728	4,030,387,200
Ly	37	18	18	16,411,572	$> 5{\cdot}1 \cdot 10^{16}$
Ru	29	14	28	19,120,976	145,926,144,000
$O'N$	11	10	20	532,400	460,815,505,920
Fi_{22}	13	6	155520	$< 5{\cdot}4 \cdot 10^{13}$	$> 6{\cdot}4 \cdot 10^{13}$
Fi_{23}	17	16	32	5,030,912	$> 4{\cdot}0 \cdot 10^{18}$
Fi_{24}	17	16	32	5,030,912	$> 1{\cdot}2 \cdot 10^{24}$
F_5	19	9	27	5,000,211	$> 2{\cdot}7 \cdot 10^{14}$
F_3	19	18	72	35,557,056	$> 9 \cdot 10^{16}$
F_2	31	15	3600	386,091,360,000	$> 4{\cdot}1 \cdot 10^{33}$
F_1	41	40	400	11,027,360,000	$> 8 \cdot 10^{53}$

elements of the order equal to that of y (denoted as m), the corresponding value of the function $f(x, y)$ which is the upper bound for the number of elements in $E(x, y)$ (cf. Lemma 5.1) and, finally, the order of G.

7. Conjecture (FSQ) and the additive properties of normal subgroups in D^*

Thinking about the possibilities of generalizing the given proof of Theorem 2.6 to somewhat more general cases, one cannot help noticing that the only spot in the argument where D itself was used, is Proposition 3.4 which establishes property (C) for D^* if D is of degree 3. However, in further argument this fact is employed in a much weaker form, viz. as property (C) for the finite quotients D^*/M which probably can be proved in a more general set-up. (Just to compare: the centralizer of $x \in D$ is basically the maximal subfield $K(x)$, while the centralizer of the coset xN in D^*/N contains the image of $K(x)$ as well as the image of the following set: $N + xN$ which may be quite big.) A suggestive surprisingly general result in this direction was proved in [3], [18]: if D is an infinite division ring then for any subgroup $N \subset D^*$ of finite index we have $N - N = D$ (so-called property $N - N$). Obviously, the quotient $F = D^*/N$ cannot be a simple group if for some $x \in D^* \backslash N$ we have $N + xN = D$ since then the coset xN belongs to the center of F. Unfortunately, this property $(N + xN)$ which is apparently very much like $(N - N)$, does not always hold true, even for p-adic division algebras. So, the real issue now is to find (say, using technique of [3], [18]) some additive property of normal subgroups $N \subset D^*$ of finite index that would enable us to establish property (C) or its modification in general case. At the present stage it is difficult to predict the form of such a property, so we limit ourself to stating the following rather straightforward consequence of the above.

PROPOSITION 7.1

Let D be a division ring, $N \subset D^*$ be a normal subgroup of finite index such that for any $x, y \in D^*$ we have

$$D^* = (N + yN + yxN)(N + xN)(N + yN + yxN). \tag{7.1}$$

Then the quotient $F = D^*/N$ cannot be a simple group.

Proof. As we mentioned above, the image of $N + xN$ in F is contained in the centralizer $C(\bar{x}) = C_F(\bar{x})$ of $\bar{x} = xN$ (indeed, if $g = n_1 + xn_2$ then $gn_1^{-1} = 1 + x(n_2 n_1^{-1})$ commutes with $x(n_2 n_1^{-1})$ in D^*, and therefore, $\bar{g} = \overline{gn_1}^{-1}$ commutes with $\bar{x} = \bar{x}(\bar{n}_2 \bar{n}_1^{-1})$ in F). Repeating this argument, we obtain that the image of $N + yN + yxN$ in F is contained in $C_F(\bar{x}, \bar{y})$. In other words (7.1) for any $x, y \in D^*$ implies property (C) for F. But then, according to Theorem 3.5, F cannot be a simple group. ■

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A theorem of the Wiener–Tauberian type for $L^1(H^n)$

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Abstract. The Heisenberg motion group $HM(n)$, which is a semi-direct product of the Heisenberg group H^n and the unitary group $U(n)$, acts on H^n in a natural way. Here we prove a Wiener–Tauberian theorem for $L^1(H^n)$ with this $HM(n)$-action on H^n i.e. we give conditions on the "group theoretic" Fourier transform of a function f in $L^1(H^n)$ in order that the linear span of $\{{}^g f: g \in HM(n)\}$ is dense in $L^1(H^n)$, where ${}^g f(z,t) = f(g\cdot(z,t))$, for $g \in HM(n)$, $(z,t) \in H^n$.

Keywords. Heisenberg group; Gelfand pairs; class-1 representations; elementary spherical functions.

1. Introduction

The celebrated Wiener–Tauberian theorem (see theorem 9·4 in [13]) asserts that for $f \in L^1(\mathbb{R}^n)$, the closed subspace generated by the translates of f is all of $L^1(\mathbb{R}^n)$ if and only if $\hat{f}$, the Euclidean Fourier transform of f, is nowhere vanishing. However there is also a larger group, i.e., the group of rigid motions $M(n)$ acting on $\mathbb{R}^n$. Therefore we can ask for conditions on $\hat{f}$ under which the rigid motion translates of f, i.e., ${}^\sigma f$, $\sigma \in M(n)$, (where ${}^\sigma f(y) = f(\sigma \cdot y)$), generate a dense subspace of $L^1(\mathbb{R}^n)$. Using a slightly stronger version of Wiener's theorem it can be shown easily that $\overline{\operatorname{Span}\{{}^\sigma f:\ \sigma \in M(n)\}} = L^1(\mathbb{R}^n)$ if and only if $\hat{f}(0) \neq 0$ and $\hat{f}$ is not identically zero on each C_α, $\alpha > 0$, where $C_\alpha = \{v \in \mathbb{R}^n: \|v\| = \alpha\}$ (see for example [6], [12]).

In this paper we are interested in the corresponding question when we replace $\mathbb{R}^n$ by the Heisenberg group H^n and $M(n)$ by the "Heisenberg motion group" $HM(n)$ (see § 3). For $f \in L^1(H^n)$, we have the notion of the group-theoretic Fourier transform (§ 4). So we would like to get conditions on the group-theoretic Fourier transform of f which guarantee that $\overline{\operatorname{Span}\{{}^g f: g \in HM(n)\}} = L^1(H^n)$.

In order to answer this question, we make crucial use of a theorem of Hulanicki–Ricci [10] about the ideals in the commutative Banach algebra of "radial" L^1-functions on H^n.

Finally we should mention that the analogue of Wiener's theorem for the two sided action of H^n on itself has been known for sometime – see for example [11] and [16].

This paper is organized as follows: In § 2 we collect the relevant facts about H^n, its representation theory, special Hermite functions and twisted convolution. In § 3, we discuss the class-1 representations of the Gelfand pair $(HM(n), U(n))$, the corresponding elementary spherical functions and the connection between these representations and the representations of H^n. In § 4 we state the Wiener–Tauberian theorem of Hulanicki and Ricci precisely and prove the analogue of the Wiener–Tauberian theorem in our set up.

2. H^n, its representations, special Hermite functions and twisted convolution

Let $H^n = \mathbb{C}^n \times \mathbb{R}$ denote the n-dimensional Heisenberg group endowed with the group law

$$(z,t)(w,s) = (z + w, t + s + \tfrac{1}{2}\mathrm{Im}\, z.\bar{w}).$$

Here $z.\bar{w} = \Sigma_{j=1}^{n} z_j.\bar{w}_j$, for $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n)$.

For each $\lambda \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$, we have an irreducible unitary representation π_λ of H^n realized on $L^2(\mathbb{R}^n)$, the action being

$$\pi_\lambda(z,t)\phi(\xi) = \mathrm{e}^{i\lambda t}\mathrm{e}^{i\lambda((x.y/2) + x,\xi)}\phi(\xi + y),$$

for $z = x + iy$, $\phi \in L^2(\mathbb{R}^n)$, $\xi \in \mathbb{R}^n$. Up to unitary equivalence these π_λ give all the infinite dimensional irreducible unitary representations of H^n (see [7]). We also have another family of one-dimensional irreducible unitary representations χ_w, $w \in \mathbb{C}^n$, given by

$$\chi_w(z,t) = \mathrm{e}^{i\mathrm{Re}\, w.\bar{z}}, \quad (z,t) \in H^n.$$

The representations π_λ for $\lambda \in \mathbb{R}^*$ together with χ_w for $w \in \mathbb{C}^n$ exhaust all the irreducible, pairwise inequivalent, unitary representations of H^n.

Throughout this paper we follow the convention that $\mathbb{N}$, the set of natural numbers, also includes zero. Let us take the orthonormal basis $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ of $L^2(\mathbb{R}^n)$ consisting of normalized Hermite functions. These Hermite functions can be given explicitly as follows: $\Phi_\alpha(x) = \Pi_{j=1}^{n} h_{\alpha_j}(x_j)$, for $x = (x_1, \ldots, x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $h_k(y) = (2^k k! \sqrt{\pi})^{-1/2}(-1)^k \mathrm{d}^k/\mathrm{d}y^k(\mathrm{e}^{-y^2})\mathrm{e}^{y^2/2}$, $y \in \mathbb{R}$, $k = 0, 1, 2, \ldots$. Moreover Φ_α is an eigenfunction for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ with the eigenvalue $(2|\alpha| + n)$. Here $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The special Hermite functions $\Phi_{\alpha\beta}$ are defined as follows:

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-n/2}\langle \pi(z)\Phi_\alpha, \Phi_\beta \rangle_{L^2(\mathbb{R}^n)},$$

where $\pi(z) = \pi_1(z, 0)$. The system $\{\Phi_{\alpha\beta}\}_{\alpha,\beta}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$. The functions $\Phi_{\alpha\beta}$ can be expressed in terms of Laguerre polynomials (see [15]) from which it follows that $\Phi_{\mu\mu}(z_1, \ldots, z_n) = \Phi_{\mu\mu}(|z_1|, \ldots, |z_n|)$. Hence each $\Phi_{\mu\mu}$ is real-valued. Further let

$$\phi_k(z) = L_k^{n-1}(\tfrac{1}{2}|z|^2)\mathrm{e}^{-1/4|z|^2}$$

denote the kth Laguerre function, where L_k^{n-1} denotes the kth Laguerre polynomial of type $n - 1$. Then

$$\phi_k(z) = (2\pi)^{n/2} \sum_{|\alpha| = k} \Phi_{\alpha\alpha}(z).$$

If $F_1, F_2 \in L^1(\mathbb{C}^n)$ and $\lambda \in \mathbb{R}^*$, we define $F_1 *_\lambda F_2$, the λ- twisted convolution of F_1 and F_2 by

$$F_1 *_\lambda F_2(z) = \int_{\mathbb{C}^n} F_1(z - w)F_2(w)\mathrm{e}^{i(\lambda/2)\mathrm{Im}\, z\cdot\bar{w}}\mathrm{d}w.$$

Then it can be seen that

$$\Phi_{\alpha\beta}^\lambda *_\lambda \Phi_{\mu\nu}^\lambda = (2\pi)^{n/2}\delta_{\beta\mu}\Phi_{\alpha\nu}^\lambda,$$

where $\Phi_{\alpha\beta}^\lambda(z) = |\lambda|^{n/2}\Phi_{\alpha\beta}(|\lambda|^{1/2}z)$.

A reference for results quoted for Hermite functions, special Hermite functions, twisted convolutions etc is [15].

3. The Gelfand pair ($HM(n)$, $U(n)$)

The compact group $U(n)$, of $n \rtimes n$ unitary matrices with entires in $\mathbb{C}$, acts on H^n via the automorphism

$$\sigma(z,t)=(\sigma z,t), \quad \sigma\in U(n), \quad (z,t)\in H^n.$$

Therefore we can form the Heisenberg motion group $HM(n)=H^n \rtimes U(n)$, as a semi-direct product of H^n and $U(n)$. The group law in $HM(n)$ is given by

$$(\sigma,z,t)(\tau,w,s)=(\sigma\tau,\sigma w+z,s+t+\tfrac{1}{2}\mathrm{Im}\,\sigma w\cdot\bar{z})$$

for (σ,z,t), $(\tau,w,s)\in HM(n)$. The group $HM(n)$ acts on H^n in the following way

$$(\sigma,z,t)(w,s)=(\sigma w+z,s+t+\tfrac{1}{2}\mathrm{Im}\,\sigma w\cdot\bar{z}).$$

The group $U(n)$ is a maximal compact subgroup of $HM(n)$.

Henceforth we also write G for $HM(n)$ and K for $U(n)$.

Let $L^1(H^n)^\#$ be the closed subalgebra of K-invariant functions in $L^1(H^n)$. As shown in [1], $L^1(H^n)^\#$ is a *commutative* Banach $*$-algebra with respect to the usual convolution on H^n. (K-invariant functions on H^n are sometimes referred to as "radial functions".) Note that functions on H^n can be identified with functions on G that are right K-invariant. Thus $L^1(H^n)^\#$ can be identified with $L^1(K\backslash G/K)$, the subalgebra of $L^1(G)$ consisting of all K-bi-invariant functions on G. Further for f, $g\in L^1(H^n)^\#=L^1(K\backslash G/K)$, $f*g=f*_G g$ where $*$, $*_G$ denote the convolutions in H^n and G respectively. Hence $L^1(K\backslash G/K)$ is also a commutative Banach $*$-algebra and therefore (G,K) is a Gelfand pair. (See [9] for details about Gelfand pairs in general and [2], [3] and [4] for the Gelfand pairs associated with the Heisenberg group in particular.)

Let N be any locally compact topological group and K_0 be a compact subgroup of N. Let $\pi\colon N\mapsto\mathscr{U}(\mathscr{H})$ be an irreducible unitary representation of N on a Hilbert space $\mathscr{H}$. We say that π is a class-1 representation for the pair (N,K_0) if the restriction of π to K_0 contains the trivial representation of K_0, i.e., the space $H_0=\{v\in\mathscr{H}:\pi(k)v=v,\ \forall k\in K_0\}\neq(0)$.

In case (N,K_0) is a Gelfand pair, i.e., if the algebra $\{f\in L^1(N):f(k_1xk_2)=f(x),\ k_1,k_2\in K_0,\ x\in N\}$, is commutative with respect to usual convolution on N, it is known that, for π, $\mathscr{H}$, H_0 as above, $\dim H_0=1$. The function $x\mapsto\langle\pi(x)v_0,v_0\rangle$, $x\in N$ where $v_0\in H_0$ is such that $\|v_0\|=1$ is called the elementary spherical function corresponding to π. For more details on Gelfand pairs, elementary spherical functions etc. see [8], [9].

A family $\{\rho_{\lambda,k}\}_{\lambda\in\mathbb{R}^*,k\in\mathbb{N}}$ of class-1 representations for the pair (G,K) (see [14]) is defined as follows:

For $\lambda\in\mathbb{R}^*$ and $k\in\mathbb{N}$, define

$$\tilde{H}_{\lambda,k}=\left\{f:H^n\to\mathbb{C}\text{ smooth: }\mathscr{L}f=|\lambda|(2k+n)f,\ Tf=i\lambda f,\ \int_{\mathbb{C}^n}|f(z,0)|^2\,dz<\infty\right\},$$

where $\mathscr{L}$ is the Heisenberg sublaplacian and $T=\partial/\partial t$ (see [14]). An inner product $(.,.)$

on $\tilde{H}_{\lambda,k}$ is given as follows

$$(f,g)=(2\pi)^{-n}|\lambda|^n\int_{\mathbb{C}^n} f(z,0)\overline{g(z,0)}\,\mathrm{d}z.$$

Let $H_{\lambda,k}$ be the completion of $\tilde{H}_{\lambda,k}$ with respect to $(.,.)$. Let $\Phi_\alpha^\lambda(x)=|\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x)$, $x\in\mathbb{R}^n$. Then the functions $E_{\alpha\beta}^\lambda(z,t)=\langle\pi_\lambda(z,t)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\rangle$, $\alpha,\beta\in\mathbb{N}^n$, with $|\beta|=k$, form an orthonormal basis for $H_{\lambda,k}$. Define

$$\rho_{\lambda,k}(\sigma,z,t)f(w,s)=f((\sigma,z,t)^{-1}(w,s)),$$

for $(\sigma,z,t)\in G$, $f\in H_{\lambda,k}$, $(w,s)\in H^n$. Then $\rho_{\lambda,k}$ is a unitary representation of G. The following can be essentially found in [14]:

Theorem 3.1. *The representation $\rho_{\lambda,k}$ defined above is an irreducible unitary class-1 representation of G. The corresponding bounded elementary spherical function $e_{\lambda,k}$ is given as*

$$e_{\lambda,k}(\sigma,z,t)=\frac{k!(n-1)!}{(k+n-1)!}\mathrm{e}^{-i\lambda t}\phi_k(|\lambda|^{1/2}z),$$

$(\sigma,z,t)\in G$. The restriction of $\rho_{\lambda,k}$ to H^n breaks up as the sum of $(k+n-1)!/k!(n-1)!$ irreducible representations, each of which is equivalent to the representation π_λ of H^n. Moreover, for $\lambda,\lambda_1\in\mathbb{R}^*$, $k,k_1\in\mathbb{N}$, $\rho_{\lambda,k}$ is equivalent to ρ_{λ_1,k_1} if and only if $\lambda=\lambda_1$, $k=k_1$.

The irreducibility and pairwise inequivalence of $\rho_{\lambda,k}$'s are proved in [14]. Also the fact that the restriction of $\rho_{\lambda,k}$ to H^n breaks up as the sum of $(k+n-1)!/k!(n-1)!$ irreducible representations, each of which is equivalent to the representation π_λ of H^n has been observed in [14]. To see that $\rho_{\lambda,k}$ is class-1 for each $\lambda\in\mathbb{R}^*$, $k\in\mathbb{N}$, note that the function

$$\begin{aligned}
E_k^\lambda(z,t)&=N_k^{-1/2}\sum_{|\beta|=k}E_{\beta\beta}^\lambda(z,t),\quad\text{where } N_k=\frac{(k+n-1)!}{k!(n-1)!}\\
&=N_k^{-1/2}\mathrm{e}^{i\lambda t}(2\pi)^{n/2}\sum_{\beta=k}\Phi_{\beta\beta}(|\lambda|^{1/2}z)\\
&=N_k^{-1/2}\mathrm{e}^{i\lambda t}\phi_k(|\lambda|^{1/2}z)\\
&=N_k^{-1/2}\mathrm{e}^{i\lambda t}L_k^{n-1}(\tfrac{1}{2}|\lambda||z|^2)\mathrm{e}^{-1/4|\lambda||z|^2}
\end{aligned}$$

(using results quoted in §2) is the essentially unique K-fixed vector in $H_{\lambda,k}$. The corresponding elementary spherical function $e_{\lambda,k}$ is therefore given by

$$\begin{aligned}
\mathrm{e}_{\lambda,k}(\sigma,z,t)&=\langle\rho_{\lambda,k}(\sigma,z,t)E_k^\lambda,E_k^\lambda\rangle_{H_{\lambda,k}}\\
&=\langle\rho_{\lambda,k}(e,z,t)E_k^\lambda,E_k^\lambda\rangle_{H_{\lambda,k}},
\end{aligned}$$

where e is the identity element in $U(n)$. Hence the above expression becomes

$$\begin{aligned}
&(2\pi)^{-n}|\lambda|^n\int_{\mathbb{C}^n}E_k^\lambda((e,z,t)^{-1}(w,0))\overline{E_k^\lambda(w,o)}\,\mathrm{d}w\\
&\quad=(2\pi)^{-n}|\lambda|^n\int_{\mathbb{C}^n}E_k^\lambda((w,0)(z,t)^{-1})\overline{E_k^\lambda(w,o)}\,\mathrm{d}w
\end{aligned}$$

$$= (2\pi)^{-n}|\lambda|^n N_k^{-1} \int_{\mathbb{C}^n} \sum_{|\alpha|=k} \langle \pi_\lambda((w,o)(z,t)^{-1})\Phi_\alpha^\lambda, \Phi_\alpha^\lambda \rangle$$

$$\times \sum_{|\beta|=k} \overline{\langle \pi_\lambda(w,0)\Phi_\beta^\lambda, \Phi_\beta^\lambda \rangle} \mathrm{d}w.$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} \int_{\mathbb{C}^n} \mathrm{e}^{i(\lambda/2)\mathrm{Im} z.\bar{w}} \Phi_{\alpha\alpha}^\lambda(w-z)\overline{\Phi_{\beta\beta}^\lambda(w)}\mathrm{d}w.$$

Since $\Phi_{\mu\mu}^\lambda(z) = \Phi_{\mu\mu}^\lambda(-z)$ and $\Phi_{\mu\mu}^\lambda$ is real-valued, the above expression is equal to

$$N_k^{-1} \mathrm{e}^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} \int_{\mathbb{C}^n} \mathrm{e}^{i(\lambda/2)\mathrm{Im} z.\bar{w}} \Phi_{\alpha\alpha}^\lambda(z-w)\Phi_{\beta\beta}^\lambda(w)\mathrm{d}w.$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} \Phi_{\alpha\alpha}^\lambda *_\lambda \Phi_{\beta\beta}^\lambda(z)$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} (2\pi)^{n/2} \delta_{\alpha\beta} \Phi_{\alpha\beta}^\lambda(z)$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} (2\pi)^{n/2} \sum_{|\alpha|=k} \Phi_{\alpha\alpha}^\lambda(z)$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} \phi_k(|\lambda|^{1/2} z).$$

Since $e_{\lambda,k}(\sigma,z,t)$ is independent of the choice of σ, we also write $e_{\lambda,k}(z,t)$ for $e_{\lambda,k}(\sigma,z,t)$ for any $\sigma \in U(n)$.

We now describe the other set of class-1 representations of (G,K). Consider the one dimensional representation $\chi_w(z,t) = \mathrm{e}^{i\mathrm{Re} w.\bar{z}}$, $w \in \mathbb{C}^n \setminus \{0\}$, $(z,t) \in H^n$, of H^n. Let $K_0 = \{k \in K : k.w = w\}$. Then K_0 is a closed subgroup of K. Let ρ_w be the induced representation obtained by inducing $\chi_w \otimes 1$ from $H^n \rtimes K_0$ to $H^n \rtimes K = G$. Here 1 denotes the trivial representation of K_0. The representation space of ρ_w is described as follows: Let

$$\tilde{H}_w = \{f : G \to \mathbb{C} \text{ continuous}: f(g_0 g) = (\chi_w \otimes 1)(g_0) f(g),\ g_0 \in H^n \rtimes K_0,\ g \in G\}.$$

Therefore for $f \in \tilde{H}_w$, $(\sigma,z,t) \in G$,

$$\begin{aligned} f(\sigma,z,t) &= f((e,z,t)(\sigma,0,0)) \\ &= (\chi_w \otimes 1)(e,z,t) f(\sigma,0,0) \\ &= \mathrm{e}^{i\mathrm{Re} w.\bar{z}} f(\sigma,0,0), \end{aligned}$$

and hence f can be viewed as a function on K. Let H_w be the completion of $\tilde{H}_w$ with respect to the inner product

$$(f,g) = \int_K f(k)\overline{g(k)}\mathrm{d}k,\ f,g \in \tilde{H}_w.$$

The induced representation ρ_w is given by

$$\rho_w(\sigma,z,t) f(\tau,w,s) = f((\tau,w,s)(\sigma,z,t)), \quad f \in H_w, (\sigma,z,t), (\tau,w,s) \in G.$$

Then ρ_w is an irreducible unitary representation of G, with $f_0(\sigma,z,t) = \Omega_{2n-1}^{-1/2} \mathrm{e}^{i\mathrm{Re} w.\bar{z}}$ as the essentially unique K-fixed vector. Here Ω_{2n-1} is the total surface measure of the unit

sphere in $\mathbb{R}^{2n}$ (see [2] for details). The corresponding elementary spherical functions η_τ can be computed to be the following

$$\eta_\tau(\sigma, z, t) = \frac{2^{n-1}(n-1)!\,J_{n-1}(\tau|z|)}{(\tau|z|)^{n-1}},$$

where $\tau = |w| > 0$ and J_{n-1} is the Bessel function of order $n-1$. Also ρ_w is equivalent to $\rho_{w'}$ if and only if $|w| = |w'|$. For $w = 0$, χ_0, the trivial representation of G is clearly a class-1 representation with the elementary spherical function $\eta_0 \equiv 1$ on G.

We also write $\eta_\tau(z,t)$ for $\eta_\tau(\sigma,z,t)$ for any $\sigma \in U(n)$. Since we know that $e_{\lambda,k}$ with $\lambda \in \mathbb{R}^*$, $k \in \mathbb{N}$ and η_τ with $\tau \geqslant 0$ are all the bounded elementary spherical functions for the pair (G, K), (see for example [1]), the above discussion completes the description of these in terms of class-1 representations of (G, K). The connection between representations and elementary spherical functions for Gelfand pairs associated with solvable Lie groups has been studied in detail in [2].

4. Wiener–Tauberian theorem for $L^1(H^n)$ with the $HM(n)$ action

We first state the Wiener–Tauberian theorem for $L^1(H^n)^\#$ due to Hulanicki and Ricci [10].

Theorem 4.1. *(Hulanicki and Ricci) Let J be a closed ideal in $L^1(H^n)^\#$ and suppose that*

(1) *For any $\lambda \in \mathbb{R}^*$, $k \in \mathbb{N}$, there exists some $f \in J$ such that*

$$\int f(z,t)\, e_{\lambda,k}(z,t)\,\mathrm{d}z\,\mathrm{d}t \neq 0.$$

(2) *For any $\tau \geqslant 0$, there exists some $f \in J$ such that*

$$\int f(z,t)\,\eta_\tau(z,t)\,\mathrm{d}z\,\mathrm{d}t \neq 0.$$

Then $J = L^1(H^n)^\#$.

To state the analogue of the Wiener–Tauberian theorem for the action of G on H^n, we set up some notation.

Let $\hat{H}^n$ denote the equivalence classes of irreducible unitary representations of H^n. For $h \in L^1(H^n)$, we define the "group theoretic" Fourier transform on $\hat{H}^n$ as follows: Let π be in $\hat{H}^n$ with $\mathscr{H}_\pi$ as the corresponding representation space. Then $\pi(h)$ is the bounded operator defined by

$$\pi(h) = \int_{H^n} h((z,t)^{-1})\,\pi(z,t)\,\mathrm{d}z\,\mathrm{d}t,$$

where the integral is to be interpreted suitably. The assignment $\pi \mapsto \pi(h)$, defined on $\hat{H}^n$ is known as the "group theoretic" Fourier transform of h. Thus for each $\lambda \in \mathbb{R}^*$, $\pi_\lambda(h)$ acts on the Hilbert space $L^2(\mathbb{R}^n)$ and for each $w \in \mathbb{C}^n$, $\chi_w(h)$ (is a scalar and) acts on the 1-dimensional space $\mathbb{C}$.

For each $\lambda \in \mathbb{R}^*$ and $k \in N$, let $P_{\lambda,k}$ be the projection on the kth eigenspace $M_{\lambda,k} = \operatorname{Span}\{\Phi_\alpha^\lambda : |\alpha| = k\}$ of the scaled Hermite operator $H_\lambda = -\Delta + |\lambda|^2|x|^2$ on $\mathbb{R}^n$. Recall $\Phi_\alpha^\lambda(x) = |\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x)$, $x \in \mathbb{R}^n$. We remark that if we take the Fock space model for describing the infinite dimensional representations of H^n, then the λ-dilated Hermite function Φ_α^λ corresponds to a nonzero multiple of the polynomial $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$. Hence

the subspace $M_{\lambda,k}$ of $L^2(\mathbb{R}^n)$ can be identified with the space of homogeneous polynomials of degree k in n-variables $z_1,\ldots,z_n$. The natural action, $u\cdot p(z)=p(u^{-1}\cdot z)$, $u\in U(n)$, of $U(n)$ on this space is irreducible. Thus $L^2(\mathbb{R}^n)=\oplus M_{\lambda,k}$ can be thought of as the decomposition of the representation space of π_λ into irreducible subspaces for the K-action, after $L^2(\mathbb{R}^n)$ is identified with the Fock space model.

For a function h on H^n, define ${}^g h(z,t)=h(g\cdot(z,t))$, for $g\in G$, $(z,t)\in H^n$. We are now in a position to give conditions under which $\overline{\mathrm{Span}\{{}^g f: g\in G\}}=L^1(H^n)$, for a given function $f\in L^1(H^n)$. These conditions are purely in terms of the group theoretic Fourier transform of f.

Theorem 4.2. *Let $f\in L^1(H^n)$ be such that*

(1) $\pi_\lambda(f)\,P_{\lambda,k}\neq 0$ *for each* $\lambda\in\mathbb{R}^*$ *and* $k\in\mathbb{N}$.
(2) *For each* $r>0$, *there exists* $w\in\mathbb{C}^n$ *with* $|w|=r$ *such that* $\chi_w(f)\neq 0$.
(3) $1(f)\neq 0$, *where* 1 *is the trivial representation of* H^n.

Then $\overline{\mathrm{Span}\{{}^g f:\ g\in G\}}=L^1(H^n)$.

Remark. If we define $f_0(z)=\int f(z,t)\mathrm{d}t$, $z\in\mathbb{C}^n$ then the condition (2) above can be rewritten as follows: For each $r>0$, $\hat{f}_0$, the Euclidean Fourier transform of f_0, does not vanish identically on S_r, the sphere of radius r in $\mathbb{R}^{2n}$. Also condition (3) is equivalent to $\hat{f}_0(0)\neq 0$.

Proof. By our remark in §3, the given function f on H^n can be thought of as a right K-invariant function on G via $f(\sigma,z,t)=f(z,t)$, $(\sigma,z,t)=G$. Define $f^*(\sigma,z,t)=\bar{f}(\sigma,z,t)^{-1}=f(-\sigma^{-1}z,t),(\sigma,z,t)\in G$. Then f^* is a left K-invariant function on G. Hence $f^* *_G f$ is a K-bi-invariant function on G. Equivalently, it can be viewed as a K-invariant function on H^n.

We claim that the closed ideal generated by $f^* *_G f$ in $L^1(H^n)^\#$ is the full algebra $L^1(H^n)^\#$. Note that once we establish the claim, the theorem follows from the observations

(a) $h*(f^* *_G f)\in\overline{\mathrm{Span}\{{}^g f:\ g\in G\}}$, for $h\in L^1(H^n)^\#$ and hence $L^1(H^n)^\#=L^1(K\backslash G/K)$ $\subseteq\overline{\mathrm{Span}\{{}^g f:\ g\in G\}}\subseteq L^1(G/K)=L^1(H^n)$.
(b) The smallest closed subspace of $L^1(G/K)$ containing $L^1(K\backslash G/K)$ and invariant under the (left) G-action, is the full space $L^1(G/K)$.

To prove the claim consider

$$
\begin{aligned}
&\int(f^* *_G f)(z,t)e_{\lambda,k}(z,t)\mathrm{d}z\mathrm{d}t\\
&=\int(f^* *_G f)(z,t)\langle\rho_{\lambda,k}(e,z,t)E_k^\lambda,E_k^\lambda\rangle\mathrm{d}z\mathrm{d}t\\
&=\int(f^* *_G f)(z,t)\mathrm{e}^{-i\lambda t}N_k^{-1}\phi_k(|\lambda|^{1/2}z)\mathrm{d}z\mathrm{d}t\\
&=\int(f^* *_G f)(z,t)\mathrm{e}^{-i\lambda t}N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\langle\pi_\lambda(z)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\rangle\mathrm{d}z\mathrm{d}t\\
&=N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\langle\pi_\lambda(f^* *_G f)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\rangle.
\end{aligned}
$$

Again an easy computation shows that

$$\begin{aligned}\pi_\lambda(f^* *_G f) &= \int (f^* *_G f)(z,t)^{-1}\pi_\lambda(z,t)\,dz\,dt \\ &= \int_{H^n}\int_{U(n)} (f * f^*)(-z,-t)\pi_\lambda(u.z,t)\,du\,dz\,dt \\ &= \int_{U(n)} \pi_{\lambda,u}(f * f^*)\,du,\end{aligned}$$

where $\pi_{\lambda,u}(z,t) = \pi_\lambda(u.z,t)$, $(z,t)\in H^n$, $u\in U(n)$. Therefore,

$$\begin{aligned}&\int (f^* *_G f)(z,t)e_{\lambda,k}(z,t)\,dz\,dt \\ &= N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\int_{U(n)} \langle \pi_{\lambda,u}(f * f^*)\Phi_\alpha^\lambda, \Phi_\alpha^\lambda\rangle\,du \\ &= N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\int_{U(n)} \langle \pi_{\lambda,u}(f^*)\pi_{\lambda,u}(f)\Phi_\alpha^\lambda, \Phi_\alpha^\lambda\rangle\,du \\ &= N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\int_{U(n)} \|\pi_{\lambda,u}(f)\Phi_\alpha^\lambda\|^2\,du.\end{aligned}$$

Hence $\int(f^* *_G f)(z,t)e_{\lambda,k}(z,t)\,dz\,dt = 0 \Leftrightarrow \|\pi_{\lambda,u}(f)\Phi_\alpha^\lambda\| = 0$, for all α such that $|\alpha| = k$ and a.e. $u\in U(n)$. Now as the irreducible representation $\pi_{\lambda,u}$ has the same central character as π_λ, by the Stone-von Neumann theorem (see [7]), $\pi_{\lambda,u}$ is equivalent to π_λ. Also for each $u\in U(n)$, $\lambda\in\mathbb{R}^*$ we can choose an intertwining unitary operator $m_\lambda(u)\colon L^2(\mathbb{R}^n)\mapsto L^2(\mathbb{R}^n)$ such that $m_\lambda(u)\pi_\lambda(z,t) = \pi_{\lambda,u}(z,t)m_\lambda(u)$, for all $(z,t)\in H^n$ and $u\mapsto m_\lambda(u)\colon U(n)\mapsto\mathcal{U}(L^2(\mathbb{R}^n))$ is a continuous projective representation of $U(n)$. Therefore, the condition $\int(f^* *_G f)(z,t)e_{\lambda,k}(z,t)\,dz\,dt = 0$ is equivalent to $m_\lambda(u)\pi_\lambda(f)m_\lambda(u)^{-1}\Phi_\lambda^\alpha = 0$, for all $u\in U(n)$, α such that $|\alpha| = k$. As for each λ and k, $m_\lambda(u)$ sends $M_{\lambda,k}$ onto $M_{\lambda,k}$, the above is equivalent to $\pi_\lambda(f)P_{\lambda,k} = 0$. The condition (1) in the hypothesis implies that this is not the case. Hence $\int(f^* *_G f)(z,t)e_{\lambda,k}(z,t)\,dz\,dt \neq 0$, for any $\lambda\in\mathbb{R}^*$, $k\in\mathbb{N}$. Also for $\tau > 0$, we have

$$\int (f^* *_G f)(z,t)\eta_\tau(z,t)\,dz\,dt = \int_{H^n}\int_{U(n)} (f^* *_G f)(z,t)e^{i\mathrm{Re}((u.w).\bar{z})}\,du\,dz\,dt,$$

where $w\in\mathbb{C}^n$ is such that $|w| = \tau$. Again using the fact that $(f^* *_G f)(z,t) = \int_{U(n)}(f * f^*)(u.z,t)\,du$, we have $\int(f^* *_G f)(z,t)\eta_\tau(z,t)\,dz\,dt = \text{const.}\int_{U(n)}|\chi_{u.w}(f)|^2\,du$. Therefore,

$$\int (f^* *_G f)(z,t)\eta_\tau(z,t)\,dz\,dt = 0$$

if and only if $\chi_{u.w}(f) = 0$, for all $u\in U(n)$. This in turn is equivalent to $\hat{f}_0 \equiv 0$ on the sphere S_τ of radius τ in $\mathbb{C}^n$. But the condition (2) in the hypothesis implies that this is not the case. Hence $\int(f^* *_G f)(z,t)\eta_\tau(z,t)\,dz\,dt \neq 0$, for $\tau > 0$. For $\tau = 0$, $\int(f^* *_G f)(z,t)\,dz\,dt = |\int f(z,t)\,dz\,dt|^2 \neq 0$. Hence the Wiener–Tauberian theorem holds for the closed ideal $\overline{\{h*(f^* *_G f)\colon h\in L^1(H^n)^\#\}}$ generated by $f^* *_G f$ in $L^1(H^n)^\#$, i.e., $\overline{\{h*(f^* *_G f)\colon h\in L^1(H^n)^\#\}} = L^1(H^n)^\#$.

Remark. The conditions in Theorem 4.2 are also necessary for $f \in L^1(H^n)$ to have the property that $\overline{\mathrm{Span}\{{}^g f\colon g \in G\}} = L^1(H^n)$. This can be seen, for example, by taking the function $h(z,t) = e^{-|z|^2} e^{-t^2}$, $(z,t) \in H^n$, which is in $L^1(H^n)$ but fails to be in $\overline{\mathrm{Span}\{{}^g f : g \in G\}}$ if any of the condition (1), (2) or (3) is violated.

5. Concluding remarks

One can also consider analogues of Wiener's theorem for other Gelfand pairs. The result referred to in the introduction is actually about the Gelfand pair $(M(n), SO(n))$. In another direction, one can consider the pair (G, K) where G is a connected semisimple Lie group of the noncompact type with finite centre and K is a maximal compact subgroup of G. There is a whole body of literature devoted to this set up. For some very recent results, see for example [5].

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Multiplicity formulas for finite dimensional and generalized principal series representations

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Abstract. The article presents two results. (1) Let $\mathfrak{a}$ be a reductive Lie algebra over $\mathbb{C}$ and let $\mathfrak{b}$ be a reductive subalgebra of $\mathfrak{a}$. The first result gives the formula for multiplicity with which a finite dimensional irreducible representation of $\mathfrak{b}$ appears in a given finite dimensional irreducible representation of $\mathfrak{a}$ in a general situation. This generalizes a known theorem due to Kostant in a special case. (2) Let G be a connected real semisimple Lie group and K a maximal compact subgroup of G. The second result is a formula for multiplicity with which an irreducible representation of K occurs in a generalized representation of G arising not necessarily from fundamental Cartan subgroup of G. This generalizes a result due to Enright and Wallach in a fundamental case.

Keywords. Reductive Lie algebra and subgroup; maximal compact group; weight; partition function; multiplicity; generalized principal series.

1. Introduction and notation

This article contains two results. They are as follows:

Suppose $\mathfrak{a}$ is a reductive Lie algebra over $\mathbb{C}$ and $\mathfrak{b}$ is a reductive subalgebra of $\mathfrak{a}$. Suppose $\mathfrak{s}$ and $\mathfrak{t}$ are Cartan subalgebras of $\mathfrak{a}$ and $\mathfrak{b}$ respectively such that $\mathfrak{t} \subset \mathfrak{s}$. Our first result (Theorem 2.3) is a formula for multiplicity with which a finite dimensional irreducible $\mathfrak{b}$-module occurs in a given finite dimensional irreducible $\mathfrak{a}$-module. This result generalizes the corresponding result of Kostant [12] proved under the assumption that $\mathfrak{t}$ has an element which is regular for the root system of the pair $(\mathfrak{a}, \mathfrak{s})$. Some easy but interesting deductions of the above result are given in Corollary 2.4.

The second result (Theorem 3.14) is a nontrivial application of the first one and is as follows:

Suppose G is a connected real semi-simple Lie group with finite centre, K a maximal compact subgroup of G, Q a suitably chosen cuspidal parabolic subgroup of G, $Q = MAN$ its Langlands decomposition, Q^0 (respectively M^0) the connected component of identity in Q (respectively M), σ a discrete series or limit of discrete series representation of M^0 and ν is in $\mathfrak{a}^*$ (the dual of the Lie algebra $\mathfrak{a}$ of A). In theorem 3.14 we give a Blattner type formula for multiplicity with which a (finite dimensional) irreducible representation of K appears in the representation $U(Q^0, \sigma, \nu) = \mathrm{ind}_{Q^0}^G (\sigma \otimes \exp \nu \otimes 1)$ of G induced from the representation $\sigma \otimes \exp \nu \otimes 1$ of Q^0. The representation $U(Q^0, \sigma, \nu)$. (more generally $U(Q, \sigma, \nu) = \mathrm{ind}_Q^G(\sigma \otimes \exp \nu \otimes 1)$ with σ in the limit of discrete series of M) have been studied in the work of Langlands, Knapp and others (see [11] and references therein). These representations, in the fundamental case, are studied in the work of Enright–Varadrajan and Wallach [3–7] using algebraic techniques only. These algebraic methods not only answer the ineducibility questions but also yield the knowledge of composition factors for most of σ and ν (in the

fundamental case). It is therefore desirable to extend the E–V–W-theory to nonfundamental cases too. The second result of this paper (Theorem 3.14) below is just the first step in that direction. It gives a formula for K-multiplicity for $U(Q^0, \sigma, \nu)$ in the general case. It generalizes the corresponding result of Enright and Wallach proved in fundamental case. The arguments here are combinational in nature but much more work is required in the general case than the fundamental case.

Next step, as in E–V–W-theory should be an explicit algebraic construction of Lie algebra modules infinitesimally equivalent to $U(Q^0, \sigma, \nu)$. In this regard we have constructed certain Lie algebra modules which enjoy the same K-multiplicity formulas as that for $U(Q^0, \sigma, \nu)$ given by Theorem 3.14; identical K-multiplicities along with the fact that certain K-multiplicities are precisely one may lead to their equivalence with $U(Q^0, \sigma, \nu)$. These results will be discussed in another paper. Full development of this approach, may not only establish equivalence of analytic and algebraic methods but may produce more information and theorem 3.14 is a basic important step for it.

[I add that the referee has pointed out that certain results, (viz 6.3.12 and 6.5.9) in Vogan's book: Representations of reductive groups, Birkhauser, on standard modules, would yield the result 3.14, provided one makes a transition from cohomological induction to classical parabolic induction. Such a transition he has pointed out, would require comparison of distribution characters, is not immediate and is not done in Vogan's book.

The standard modules are constructed by cohomological induction from principal series and proof of Blattner type K-multiplicity formulas (6.3.12 and 6.5.9) in Vogan's book on well understood ideas in homological algebras. Nevertheless the proof of 6.3.12, as Vogan remarks, is a little complicated and indirect. On the other hand, the g.p.s modules are constructed from usual parabolic induction from discrete series and proof of Blattner type formula 3.14 given here relies only on combinatorial arguments about elements in the root system, and it is very direct. Vogan comments in his book on p. 334 that there is no analogue of 6.2.14 (Frobenius reciprocity) which plays a crucial role in Kazhdan–Lusztig's conjecture and is at the heart of all the results on structure of $\tau^* V$, in Enright's construction; perhaps 3.14 would be taken as Frobenius reciprocity in that direction.

Our main aim is to see if we can push Enrights machinary further from fundamental to non-fundamental case. In this respect, the construction we are testing also has K-multiplicities given by the same formula 3.14. In other words, the formulation of 3.14 is also important. It differs from that of 6.5.9 in Vogan's book. The construction of modules, we are interested, is based upon different circle of ideas than those for standard modules and hence in spite of the referee's feeling that the task of finding algebraic construction is finished by Zuckerman's cohomological construction, we have a feeling that there is still some scope for more work in that direction, it may help in simplifying the theory and may sharpen the results.]

We now describe the general notation we will use throughout the paper.

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real and complex numbers. For a real vector space V_0 we drop the suffix to denote its complexification. For a vector space V over a field F, we write V^* for its dual.

For any Lie algebra $\mathfrak{m}$ over $\mathbb{C}$ let $U(\mathfrak{m})$ denote the universal enveloping algebra of $\mathfrak{m}$. If $\mathfrak{m}$ is a reductive Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{m}$, let $\Delta(\mathfrak{m}, \mathfrak{h})$ denote the set of roots of the pair $(\mathfrak{m}, \mathfrak{h})$. For each α in $\Delta(\mathfrak{m}, \mathfrak{h})$ we have the one

dimensional root space $\mathfrak{m}_\alpha$ and the root space decomposition

$$\mathfrak{m} = \mathfrak{h} + \sum_{\alpha\in\Delta(m,\mathfrak{h})} \mathfrak{m}_\alpha.$$

For α in $\Delta(\mathfrak{m},\mathfrak{h})$, let $\overline{H_\alpha}$ be the unique element in $[\mathfrak{m}_\alpha, \mathfrak{m}_{-\alpha}]$ such that $\alpha(\overline{H_\alpha}) = 2$. If root vectors X_α in $\mathfrak{m}_\alpha$ and $X_{-\alpha}$ in $\mathfrak{m}_{-\alpha}$ are such that $[X_\alpha, X_{-\alpha}] = \overline{H_\alpha}$ then $\{X_\alpha, X_{-\alpha}, \overline{H_\alpha}\}$ is called standard triple associated with α. λ in $\mathfrak{h}^*$ is called $\Lambda(\mathfrak{m},\mathfrak{h})$ integral if $\lambda(\overline{H_\alpha})\in\mathbb{Z}$ for every α in $\Lambda(\mathfrak{m},\mathfrak{h})$. Suppose P is a positive system for $\Lambda(\mathfrak{m},\mathfrak{h})$ then $\lambda\in\mathfrak{h}^*$ is called P-dominant if $\lambda(\overline{H_\alpha})$ is a nonnegative real for every α in P. For any λ in $\mathfrak{h}^*$ we denote by $V_{\mathfrak{m},P,\lambda}$ the Verma module of P-highest weight λ. If λ in $\mathfrak{h}^*$ is a P-dominant integral we denote by $F^\lambda_{\mathfrak{m},P}$ the finite dimensional irreducible $\mathfrak{m}$-module of P-highest weight λ. We denote by $W(\mathfrak{m},\mathfrak{h})$ the Weyl group of the pair $(\mathfrak{m},\mathfrak{h})$. For α in $\Delta(\mathfrak{m},\mathfrak{h})$ we denote by σ_α the corresponding reflection in $W(\mathfrak{m},\mathfrak{h})$. For σ in $W(\mathfrak{m},\mathfrak{h})$, $\varepsilon(\sigma)$ will denote the determinant of the linear transformation $\sigma:\mathfrak{h}\to\mathfrak{h}$. If $\mathfrak{t}$ is a subspace of $\mathfrak{h}$ and if λ is in $\mathfrak{h}^*$ we denote by $\lambda\,|\mathfrak{t}$ the restriction of λ to $\mathfrak{t}$.

Finally suppose V is a vector space, and $E = \{\alpha_1,\dots,\alpha_m\}$ is a sequence of elements of V^*. Assume there is some X in V such that $\alpha_i(X) > 0$ for $1 \leqslant i \leqslant m$. Then for any λ in V^*, $\mathbb{P}_E(\lambda)$ is defined to be the number of m-tuples $(n_1,\dots,n_m)$ of non-negative integers such that $\lambda = \Sigma_{1\leqslant i\leqslant m} n_i\alpha_i$. We also make the convention that $\mathbb{P}_\phi(\lambda) = 0$ if λ in V^* is non-zero and $\mathbb{P}_\phi(\lambda) = 1$ if $\lambda = 0$.

2. Multiplicity formula for finite dimensional representations

Suppose $\mathfrak{a}$ is a reductive Lie algebra over $\mathbb{C}$, $\mathfrak{b}$ is a reductive subalgebra of $\mathfrak{a}$, $\mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{a}$ and $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{b}$ such that $\mathfrak{t}\subset\mathfrak{s}$.

In this secton, we construct a formula for multiplicity with which a given finite dimensional irreducible $\mathfrak{b}$-module occurs in a given finite dimensional irreducible $\mathfrak{a}$ module. Our formula is a generalization of Kostant's formula (cf [12]) proved under the assumption that $\mathfrak{t}$ has a point at which no root in $\Delta(\mathfrak{a},\mathfrak{s})$ vanishes.

We fix some notation. Set

$$\Delta_\mathfrak{a} = \Delta(\mathfrak{a},\mathfrak{s}),\ \Delta_\mathfrak{b} = \Delta(\mathfrak{b},\mathfrak{t}),\ W_\mathfrak{a} = W(\mathfrak{a},\mathfrak{s}),\ W_\mathfrak{b} = W(\mathfrak{b},t)$$

$$\Delta^0_\mathfrak{a} = \{\alpha\in\Delta_\mathfrak{a} \,|\, \alpha|_\mathfrak{t} \equiv 0\}, \quad \Delta^*_\mathfrak{a} = \Delta\backslash\Delta^0_\mathfrak{a}.$$

Let $\mathfrak{c} = \mathfrak{s} + \Sigma_{\alpha\subset\Delta^0_\mathfrak{a}}\mathfrak{a}_\alpha$. Then $\mathfrak{c}$ is a reductive subalgebra of $\mathfrak{a}$, $\mathfrak{s}$ is its Cartan subalgebra and $\Delta^0_\mathfrak{a} = \Delta(\mathfrak{c},\mathfrak{s})$ which we also denote by $\Delta_\mathfrak{c}$.

DEFINITION 2.1

Let $P_\mathfrak{a}$ be a positive system for $\Delta_\mathfrak{a}$ and let $P_\mathfrak{b}$ be a positive system for $\Delta_\mathfrak{b}$. We say $P_\mathfrak{a}$ is compatible with $P_\mathfrak{b}$ if $\mathfrak{t}$ contains a point, say H_0, such that, $\alpha(H_0)$ is a real positive number for every α in $P_\mathfrak{b}$ as well for every α in $P_\mathfrak{a}\cap\Delta^*_\mathfrak{a}$.

PROPOSITION 2.2

Let $P_\mathfrak{b}$ be a positive system for $\Delta_\mathfrak{b}$. Then $\Delta_\mathfrak{a}$ has a positive system which is compatible with $P_\mathfrak{b}$.

Proof. First, let $\langle .,. \rangle$ be a nondegenerate form on $\mathfrak{a} \times \mathfrak{a}$ whose restriction to $[\mathfrak{a}, \mathfrak{a}] \times [\mathfrak{a}, \mathfrak{a}]$ is the killing form of $[\mathfrak{a}, \mathfrak{a}]$ and let $\mathfrak{u}$ be the ortho-complement of $\mathfrak{t}$ in $\mathfrak{s}$.

Select H_1 in $\mathfrak{t}$ such that $\alpha(H_1)$ is a positive real number for each α in $P_{\mathfrak{b}}$ and that $\alpha(H_1) \neq 0$ for each α in $\Delta_{\mathfrak{a}}^*$. Select H_2 in $\mathfrak{u}$ such that $\alpha(H_2)$ is a positive real number for each α in $\Delta_{\mathfrak{a}}$ nonvanishing on $\mathfrak{u}$. Let γ be a positive real number such that $\gamma. |\alpha(H_2)| < |\alpha(H_1)|$ for each α in $\Delta_{\mathfrak{a}}^*$. Set $H_0 = \gamma H_2 + H_1$ and let $P_{\mathfrak{a}} = \{\alpha \in \Delta_{\mathfrak{a}} / \alpha(H_0) > 0\}$. It is then easy to check that $P_{\mathfrak{a}}$ is a positive system for $\Delta_{\mathfrak{a}}$ and it is complatible with $P_{\mathfrak{b}}$.

We now fix a positive system $P_{\mathfrak{b}}$ for $\Delta_{\mathfrak{b}}$ and also a $P_{\mathfrak{b}}$-compatible positive system $P_{\mathfrak{a}}$ for $\Delta_{\mathfrak{a}}$. Let $H_0 \in \mathfrak{t}$ be such that $\alpha(H_0)$ is positive real for each α in $P_{\mathfrak{b}}$ and also for each $\alpha \in P_{\mathfrak{a}} \cap \Delta_{\mathfrak{a}}^*$. Let $P_{\mathfrak{c}} = P_{\mathfrak{a}} \cap \Delta_{\mathfrak{a}}^0$. Let $\rho_{\mathfrak{a}}, \rho_{\mathfrak{b}}$ and $\rho_{\mathfrak{c}}$ denote half the sum of roots in $P_{\mathfrak{a}}, P_{\mathfrak{b}}$ and $P_{\mathfrak{c}}$ respectively. If w is a $\mathfrak{t}$-weight of the cannonical $\mathfrak{b}$-module $\mathfrak{a}/\mathfrak{b}$, call w positive if $w(H_0) > 0$ and let T denote the family of positive weights of the $\mathfrak{b}$-module $\mathfrak{a}/\mathfrak{b}$ counted according to the multiplicity. For each λ is $\mathfrak{s}^*$ write $\bar{\lambda}$ for its restriction to $\mathfrak{t}$. In the above setting we now prove the main result of this section.

Theorem 2.3. *Let μ in $\mathfrak{s}^*$ be $P_{\mathfrak{a}}$-dominant integral and let w in $\mathfrak{t}^*$ be $P_{\mathfrak{b}}$-dominant integral. Let $\mathfrak{m}(\mu, w) = \dim \mathrm{Hom}_{\mathfrak{b}}(F^w_{\mathfrak{b},P_{\mathfrak{b}}}, F^\mu_{\mathfrak{a},P_{\mathfrak{b}}})$. Then*

(a) $m(\mu, w) = \sum\limits_{\lambda \in \mathfrak{s}^*} \sum\limits_{\sigma \in W_{\mathfrak{a}}} \varepsilon(\sigma) \mathbb{P}_T(\overline{\sigma(\mu + \rho_{\mathfrak{a}})} - (w + \overline{\rho_{\mathfrak{a}}})) \mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu + \rho_{\mathfrak{a}}) + \lambda)$.

(b) *If $\tau \in W_{\mathfrak{b}}$ then*

$$\begin{aligned} &m(\mu, w)\tau(\mathfrak{b}) \\ &= \Sigma_{\lambda \in \mathfrak{s}^*} \Sigma_{\sigma \in W_{\mathfrak{a}}} \varepsilon(\sigma) \mathbb{P}_T(\overline{\sigma(\mu + \rho_{\mathfrak{a}}))} - \tau(w + \rho_{\mathfrak{b}}) \\ &\quad - \rho_{\mathfrak{b}} + \overline{\rho_{\mathfrak{a}}}) \, \mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu + \rho_{\mathfrak{a}}) + \lambda). \end{aligned}$$

(c) *If η in $\mathfrak{t}^*$ is singular with respect to $\Delta_{\mathfrak{b}}$ then*

$$0 = \sum_{\lambda \in \mathfrak{s}^*} \sum_{\sigma \in W_{\mathfrak{a}}} \varepsilon(\sigma) \mathbb{P}_T(\overline{\sigma(\mu + \rho_{\mathfrak{a}}))} - (\eta - \rho_{\mathfrak{b}} + \overline{\rho_{\mathfrak{a}}}). \mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu + \rho_{\mathfrak{a}}) + \lambda).$$

Remark. The above formula take simple forms when $\Delta_{\mathfrak{a}}^0 = \emptyset$ i.e. when no root in $\Delta_{\mathfrak{a}}$ vanish on $\mathfrak{t}$ e.g. in this case (a) becomes

(a)′ $$m(\mu, w) = \Sigma_{\sigma \in W_{\mathfrak{a}}} \varepsilon(\sigma) \mathbb{P}_T(\overline{\sigma(\mu + \rho_{\mathfrak{a}})} - (w + \overline{\rho_{\mathfrak{a}}})).$$

This is clear by virtue of our definition of $P_\emptyset$.

Proof of Theorem 2.3. For a vector space $\mathfrak{p} (= \mathfrak{s}$ or $\mathfrak{t})$, and any λ in $\mathfrak{p}^*$, let e^λ denote the unique function on $\mathfrak{p}^*$ which takes the value 1 at λ and zero everywhere else. For any $\mathbb{Z}$-valued function f on $\mathfrak{p}^*$ write $f = \Sigma_{\lambda \in \mathfrak{p}^*} f(\lambda) e^\lambda$. For any function f on $\mathfrak{p}^*$, let support of f stand for the set of those $\lambda \in \mathfrak{p}^*$ such that $f(\lambda) \neq 0$.

Let $F_{\mathfrak{s}}$ denote the family of all $\mathbb{Z}$-valued functions on $\mathfrak{s}^*$ whose support is contained in a finite union of sets of the type

$$\left\{ \lambda - \sum_{\alpha \in P_{\mathfrak{a}}} \mathfrak{m}_\alpha \alpha / m_\alpha \in \mathbb{Z}^+ \right\},$$

where λ is any element of $\mathfrak{s}^*$. Similarly, let $F_{\mathfrak{t}}$ denote the family of all $\mathbb{Z}$ valued functions on $\mathfrak{t}^*$ whose support is contained in a finite union of sets of the type $\{\theta - \Sigma_{\alpha \in P_{\mathfrak{a}}} m_\alpha \bar{\alpha} / m_\alpha \subset \mathbb{Z}^+\}$.

Define addition in $F_{\mathfrak{p}}$ (for $\mathfrak{p} = \mathfrak{s}$ or $\mathfrak{t}$) pointwise and multiplication by convolution i.e. if f and g are in $F_{\mathfrak{p}}$ then $f+g(\lambda) = f(\lambda)+g(\lambda)$ and $fg(\lambda) = \Sigma_{\eta\in\mathfrak{p}^*} f(\eta)g(\lambda-\eta)$. For f and g in $F_{\mathfrak{p}}$, we write $g = f^{-1}$ or $g = 1/f$ if $gf = e^0$.

For any f in $F_{\mathfrak{s}}$ with finite support, define $\bar{f}$ on $\mathfrak{t}^*$ by $\bar{f}(\theta) = \Sigma_{\lambda\in\mathfrak{s}^*, \bar{\lambda}=\theta} f(\lambda)$. Then (i) $\overline{e^{\lambda}} = e^{\bar{\lambda}}$ and (ii) $\overline{fg} = \bar{f}\bar{g}$ for any f and g in $F_{\mathfrak{s}}$ with finite support. Both (i) and (ii) are straightforward and we check them only at the end of the section.

We now proceed with the Proof. Let $P_{\mathfrak{a}}^* = P_{\mathfrak{a}} \cap \Delta_{\mathfrak{a}}^*$. Then $\alpha \mapsto \bar{\alpha}$ is a bijection from $P_{\mathfrak{a}}^*$ onto $P_{\mathfrak{b}} \cup T$. Hence we can find two disjoint subsets, say $P_{\mathfrak{b}}^*$ and P_T^* of $P_{\mathfrak{a}}$ such that $P_{\mathfrak{a}}^* = P_{\mathfrak{b}}^* \cup P_T^*$ and $\alpha \mapsto \bar{\alpha}$ is a bijection from $P_{\mathfrak{b}}^*$ (respectively P_T^*) onto $P_{\mathfrak{b}}$ (respectively T). (2.3.1)

Now, if V is a vector space, α is in V^* and A is a family of elements in V^* set,

$$d_\alpha = e^{\alpha/2} - e^{-\alpha/2},\ D_A = \pi_{\alpha\in A} d_\alpha$$

and

$$\rho_A = \tfrac{1}{2} \sum_{\alpha\in A} \alpha$$

and write $D_{\mathfrak{a}} = D_{P_{\mathfrak{a}}}, D_{\mathfrak{b}} = D_{P_{\mathfrak{b}}}, D_{\mathfrak{c}} = D_{P_{\mathfrak{c}}}, D_{\mathfrak{a}}^* = D_{P_{\mathfrak{a}}^*}, D_{\mathfrak{b}}^* = D_{P_{\mathfrak{b}}^*}$ and $D_T^* = D_{P_T^*}$. Then, from (2.3.1), (i) and (ii) we get

$$\bar{D}_{\mathfrak{b}}^* = D_{\mathfrak{b}}, \quad \bar{D}_T^* = D_T \tag{2.3.2}$$

and

$$\bar{\rho}_{\mathfrak{a}} = \tfrac{1}{2} \sum_{\alpha\in P_{\mathfrak{a}}} \bar{\alpha} = \tfrac{1}{2} \sum_{\alpha\in P_{\mathfrak{a}}^*} \bar{\alpha} = \rho_{\mathfrak{b}} + \rho_T. \tag{2.3.3}$$

Since $P_{\mathfrak{a}} = P_{\mathfrak{c}} \cup P_{\mathfrak{a}}^*$ and $P_{\mathfrak{a}}^* = P_{\mathfrak{b}}^* \cup P_T^*$ are disjoint unions, we get

$$D_{\mathfrak{a}} = D_{\mathfrak{c}} \cdot D_{\mathfrak{a}}^* \text{ and } D_{\mathfrak{b}}^* \cdot D_T^* = D_{\mathfrak{a}}^*. \tag{2.3.4}$$

Now, let χ_μ (respectively χ_w) denote the character of $F_{\mathfrak{a},P_{\mathfrak{a}}}^{\mu}$ (respectively $F_{\mathfrak{b},P_{\mathfrak{b}}}^{w}$). Then

$$D_{\mathfrak{a}} \cdot \chi_\mu = \sum_{\sigma\in W_{\mathfrak{a}}} \varepsilon(\sigma) e^{\sigma(\mu+\rho_{\mathfrak{a}})}; \quad D_{\mathfrak{a}} = \sum_{\sigma\in W_{\mathfrak{a}}} \varepsilon(\sigma) e^{\sigma\rho_{\mathfrak{a}}}, \tag{2.3.5}$$

$$D_{\mathfrak{b}} \cdot \chi_w = \sum_{\tau\in W_{\mathfrak{b}}} \varepsilon(\tau) e^{\tau(w+\rho_{\mathfrak{b}})}; \quad D_{\mathfrak{b}} = \sum_{\tau\in W_{\mathfrak{b}}} \varepsilon(\tau) e^{\tau\rho_{\mathfrak{b}}}. \tag{2.3.6}$$

(cf. 24.3 in [10]). Note also,

$$D_{\mathfrak{c}}^{-1} = \sum_{\lambda\in\mathfrak{s}^*} \mathbb{P}_{P_{\mathfrak{c}}}(\lambda) e^{-(\lambda+\rho_{\mathfrak{c}})} \quad \text{and} \quad D_T^{-1} = \sum_{w\in\mathfrak{t}^*} \mathbb{P}_T(w) e^{-(w+\rho_T)}. \tag{2.3.7}$$

Combining (2.3.4) through (2.3.7), we get

$$\begin{aligned} D_{\mathfrak{b}}^* \cdot \chi_\mu &= (D_T^* \cdot D_{\mathfrak{c}})^{-1} \cdot D_{\mathfrak{a}} \chi_\mu \\ &= (D_T^*)^{-1} \cdot \sum_{\lambda\in\mathfrak{s}^*} \sum_{\sigma\in W_{\mathfrak{a}}} \varepsilon(\sigma) \mathbb{P}_{P_{\mathfrak{c}}}(\lambda) e^{-(\lambda+\rho_{\mathfrak{c}})+\sigma(\mu+\rho_{\mathfrak{a}}))} \\ &= (D_T^*)^{-1} \cdot \sum_{\eta\in\mathfrak{s}^*} \sum_{\sigma\in W_{\mathfrak{a}}} \varepsilon(\sigma) \mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu+\rho_{\mathfrak{a}}) - (\eta+\rho_{\mathfrak{c}})) \cdot e^{\eta}. \end{aligned} \tag{2.3.8}$$

Now if $\eta\in\mathfrak{s}^*$ and $\sigma\in W_{\mathfrak{a}}$ then

$$\mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu+\rho_{\mathfrak{a}}) - (\eta+\rho_{\mathfrak{c}})) e^{\bar{\eta}} = \mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu+\rho_{\mathfrak{a}}) - (\eta+\rho_{\mathfrak{c}})) e^{\overline{\sigma(\mu+\rho_{\mathfrak{a}})}}. \tag{2.3.9}$$

This is because, if $\mathbb{P}_{P_c}(\sigma(\mu+\rho_a)-(\eta+\rho_c))\neq 0$ then $\bar{\eta}=\sigma\overline{(\mu+\rho_a)}$ as every α in P_c vanishes on t. Using (2.3.2), (2.3.7), (2.3.8) and (2.3.9) we get

$$D_b\overline{\chi_\mu}=\overline{D_b^*\cdot\chi_\mu}=1/D_T^*\sum_{\eta\in s^*}\sum_{\sigma\in W_a}\varepsilon(\sigma)\mathbb{P}_{P_c}(\sigma(\mu+\rho_a)-(\eta+\rho_c)e^{\overline{\sigma(\mu+\rho_a)}}$$

$$=\sum_{W\in t^*}\sum_{\eta\in s^*}\sum_{\sigma\in W_a}\varepsilon(\sigma)\mathbb{P}_{P_c}(\sigma(\mu+\rho_a)-(\eta+\rho_c))\mathbb{P}_T(w)e^{(-(w+\rho_T)+\overline{\sigma(\mu+\rho_a)})}$$

$$=\sum_{\theta\in t^*}\sum_{\eta\in s^*}\sum_{\sigma\in W_a}\varepsilon(\sigma)\mathbb{P}_{P_c}(\sigma(\mu+\rho_a)-(\eta+\rho_c))\mathbb{P}_T(\overline{\sigma(\mu+\rho_a)}-(\theta+\rho_T))e^{\theta}. \tag{2.3.10}$$

On the other hand, it is easy to see that

$$\bar{\chi}_\mu=\sum_{w\in L}\mathfrak{m}(\mu,w)\chi_w, \tag{2.3.11}$$

where L is the set of P_b^*-dominant integral elements of t^*. Using (2.3.6) and (2.3.11) we get

$$D_b\cdot\overline{\chi_\mu}=\sum_{m\in L}\mathfrak{m}(\mu,w)D_b\cdot\chi_w=\sum_{w\in L}\sum_{\tau\in W_b}\mathfrak{m}(\mu,w)\varepsilon(\tau)e^{(w+\rho_b)}. \tag{2.3.12}$$

Now let $\tau\in W_b$ and compare the co-effecient of $e^{\tau(w+\rho_b)}$ in (2.3.10) and (2.3.12) and use (2.3.3) to get part (b) of the theorem. Part (a) follows from (b) by taking $\tau=1$. If $\eta\in t^*$ is singular with respect to Δ_b then the co-efficient of e^{η} in (2.3.12) is zero. Hence the co-efficient of e^{η} in (2.3.10) must be zero. That proves (c) and completes the proof.

Theorem 2.3 generalizes both Lemma 6.3 in [12] which assumes $t=s$ and theorem 3.4.1 in [1] which assumes existence of Δ_a-regular element in t.

Using the multiplicity formula we will now derive some easy consequences.

Continue with the notation of the last theorem and let u be the ortho complement of t in s with respect to a non-degenerate form on a which is an extension of killing form on $[\mathfrak{a},\mathfrak{a}]$ and set $\mathfrak{d}=\mathfrak{u}+\Sigma_{\alpha\in\Delta_a^0}\mathfrak{a}_\alpha$. Then $\mathfrak{d}$ is a reductive Lie algebra, u is its Cartan subalgebra. Set $\Delta_\mathfrak{d}=\Delta(\mathfrak{d},\mathfrak{u})$ and $P_\mathfrak{d}=\{\alpha|u/\alpha\in P\cap\Delta_a^0\}$. The $P_\mathfrak{d}$ is a positive system for $\Delta_\mathfrak{d}$.

COROLLARY 2.4

(In the above notation). Let μ in $\mathfrak{s}^*$ be P_a-dominant integral. Let v be a P_a-highest weight vector of weight μ in F^{μ}_{a,P_a} and let $W=\cup(\mathfrak{d})v$. Suppose $\bar{\mu}$ is P_b-dominate integral then

(a) *$m(\mu,\bar{\mu})=$ degree of $F^{\bar{\mu}}_{\mathfrak{d},P_\mathfrak{d}}$ where $\bar{\mu}=\mu|\mathfrak{d}$.*
(b) *W is precisely the set of all P_b-highest vectors of weight $\bar{\mu}$ in F^{μ}_{a,P_a}.*
(c) *The* b-module *$U(\mathfrak{b})W$ is precisely the isotypical submodule of F^{μ}_{a,P_a} of type $F^{\bar{\mu}}_{\mathfrak{b},P_\mathfrak{b}}$ (i.e. it is the sum of all* b*-submodules of F^{μ}_{a,P_a} which are equivalent to $F^{\bar{\mu}}_{\mathfrak{b},P_\mathfrak{b}}$).*
(d) *The $\mathfrak{d}$-module W is isomorphic with $F^{\bar{\mu}}_{\mathfrak{d},P_\mathfrak{d}}$.*

Proof. By part (a) of Theorem 2.3 we get

$$m(\mu,\bar{\mu})=\sum_{\lambda\in s^*}\sum_{\sigma\in W_a}\varepsilon(\sigma)\mathbb{P}_T(\overline{\sigma(\mu+\rho_a)}-\overline{(\mu+\rho_a)})\mathbb{P}_{P_c}(\sigma(\mu+\rho_a)+\lambda). \tag{2.4.1}$$

Let $\sigma\in W_{\mathfrak{a}}$. If $\sigma\in W_{\mathfrak{c}}$ then $(\sigma(\mu+\rho_{\mathfrak{a}})-(\mu+\rho_{\mathfrak{a}}))$ being sum of elements in $-P_{\mathfrak{c}}$ vanishes on $\mathfrak{t}$. Hence $\overline{(\sigma(\mu+\rho_{\mathfrak{a}}))-(\mu+\rho_{\mathfrak{a}}))}=0$. Hence

$$\mathbb{P}_T\overline{(\sigma(\mu+\rho_{\mathfrak{a}}))}-\overline{(\mu+\rho_{\mathfrak{a}})}=1$$

in this case. On the other hand, if $\sigma\notin W_{\mathfrak{c}}$ then $\sigma(\mu+\rho_{\mathfrak{a}})-(\mu+\rho_{\mathfrak{a}})$ is a sum of roots in $-P_{\mathfrak{a}}$, atleast one of which is in $-(P_{\mathfrak{a}}\backslash P_{\mathfrak{c}})$ and hence it takes negative value at H_0 and therefore $\mathbb{P}_T\overline{(\sigma(\mu+\rho_{\mathfrak{a}}))}-\overline{(\mu+\rho_{\mathfrak{a}})}=0$ in this case. It therefore follows from (2.4.1) that

$$m(\mu,\bar{\mu})=\sum_{\lambda\in\mathfrak{s}^*}\sum_{\sigma\in W_{\mathfrak{c}}}\varepsilon(\sigma)\mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu+\rho_{\mathfrak{a}})+\lambda). \tag{2.4.2}$$

For $\lambda\in\mathfrak{s}^*$ and $\sigma\in W_{\mathfrak{c}}$, set, $\bar{\lambda}=\lambda|_{\mathfrak{t}}$, $\tilde{\lambda}=\lambda|_{\mathfrak{u}}$, $\bar{\sigma}=\sigma|_{\mathfrak{t}}$ and $\tilde{\sigma}=\sigma|_{\mathfrak{u}}$. Since every root in $P_{\mathfrak{c}}$ vanishes on $\mathfrak{t}$ we get

(1) $\bar{\sigma}=$ identity for each $\sigma\in W_{\mathfrak{c}}$.
(2) $\sigma\mapsto\tilde{\sigma}$ is a bijection from $W_{\mathfrak{c}}$ onto $W_{\mathfrak{b}}$.
(3) $\overline{\sigma\lambda}=\bar{\lambda}$ and $\widetilde{\sigma\lambda}=\tilde{\sigma}(\tilde{\lambda})\,\forall\,\lambda\in\mathfrak{s}^*$ and $\sigma\in W_{\mathfrak{c}}$.
(4) If $\bar{\lambda}=0$ then

$$\mathbb{P}_{P_{\mathfrak{c}}}(\lambda)=\mathbb{P}_{P_{\mathfrak{b}}}(\tilde{\lambda}).$$

Let $\Gamma=\{\lambda\in\mathfrak{s}^*/\bar{\lambda}=\overline{(\mu-\rho_{\mathfrak{a}})}\}$.

(5) If $\lambda\in\Gamma$ and $\sigma\in W_{\mathfrak{c}}$ then $\overline{\sigma(\mu+\rho_{\mathfrak{a}})+\lambda)}=\overline{\mu+\rho_{\mathfrak{a}}}+\bar{\lambda}=0$ hence $\mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu+\rho_{\mathfrak{a}})+\lambda)$ $=\mathbb{P}_{P_{\mathfrak{b}}}(\tilde{\sigma}(\tilde{\mu}+\tilde{\rho}_{\mathfrak{a}})+\tilde{\lambda})$ by (3) and (4).
(6) If $\lambda\in\mathfrak{s}^*\backslash\Gamma$ and $\sigma\in W_{\mathfrak{c}}$ then $\overline{\sigma(\mu+\rho_{\mathfrak{a}})+\lambda)}\neq 0$ and hence $\mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu+\rho_{\mathfrak{a}})+\lambda)=0$.
(7) $\lambda\to\tilde{\lambda}$ is a bijection from Γ onto $\mathfrak{u}^*$. Now

$$\begin{aligned}
m(\mu,\bar{\mu})&=\sum_{\lambda\in\Gamma}\sum_{\sigma\in W_{\mathfrak{c}}}\varepsilon(\sigma)\mathbb{P}_{P_{\mathfrak{c}}}(\sigma(\mu+\rho_{\mathfrak{a}})+\lambda)\ \text{(by (2.4.2) and (6))}\\
&=\sum_{\lambda\in\Gamma}\sum_{\sigma\in W_{\mathfrak{c}}}\varepsilon(\sigma)\mathbb{P}_{P_{\mathfrak{b}}}(\tilde{\sigma}(\tilde{\mu}+\tilde{\rho}_{\mathfrak{a}})+\tilde{\lambda})\ \text{(by (5))}\\
&=\sum_{\eta\in u^*}\sum_{\sigma\in W_{\mathfrak{c}}}\varepsilon(\sigma)\mathbb{P}_{P_{\mathfrak{b}}}(\sigma(\tilde{\mu}+\tilde{\rho}_{\mathfrak{a}})+\eta)\ \text{(by (7))}.
\end{aligned} \tag{2.4.3}$$

Now for $\eta\in u^*$ by Kostant's multiplicity formula (Corollary 7.5.10 of [2]).

$$\sum_{\sigma\in W_{\mathfrak{c}}}\varepsilon(\sigma)\mathbb{P}_{P_{\mathfrak{b}}}(\sigma(\tilde{\mu}+\rho_{\mathfrak{b}})+\eta)$$

is the dimension of the $-(\eta+\rho_{\mathfrak{b}})$ weight space of the $\mathfrak{b}$-module $F^{\tilde{\mu}}_{\mathfrak{b},P_{\mathfrak{b}}}$ and since $\tilde{\rho}_{\mathfrak{a}}=\rho_{\mathfrak{b}}$, it follows from (2.4.3) that $m(\mu,\bar{\mu})=$ degree of $F^{\tilde{\mu}}_{\mathfrak{b},P_{\mathfrak{b}}}$. This proves (a).
Since $\mathfrak{b}=u+\Sigma_{\alpha\in\Delta^{\circ}_{\mathfrak{a}}}a_\alpha$ PBW theorem shows that $U(\delta)$ has a basis, say B, such that (i) every element in B is a $\mathfrak{s}$-weight vector whose weight is zero on $\mathfrak{t}$.

Let $D\in B$ and let $\lambda\in\mathfrak{s}^*$ be its weight. Then $\bar{\lambda}=0$. Let $\alpha\in P_{\mathfrak{b}}$. Let $\beta\in P_{\mathfrak{a}}$ be such that $\bar{\beta}=\alpha$. Now $\lambda+\beta(H_0)=\beta(H_0)>0$. Hence $\lambda+\beta$ is not a sum of elements in $-P_{\mathfrak{a}}$. Therefore $\mu+\lambda+\beta$ can not be a weight of $F^{\mu}_{\mathfrak{a},P_{\mathfrak{a}}}$. Hence $X_\beta\cdot D\cdot v=0$. Therefore, (ii) $\mathfrak{b}_\alpha\cdot D\cdot v=0$.

(i) and (ii) now prove that (iii) each element of $W = U(d)v$ is a $P_{\mathfrak{b}}$ highest weight vector of t-weight $\bar{\mu}$. From (iii) and (a) we get (iv) $\dim W \leqslant \dim \mathrm{Hom}_{\mathfrak{b}} (F^{\bar{\mu}}_{\mathfrak{b},P_{\mathfrak{b}}}, F^{\mu}_{\mathfrak{a},P_{\mathfrak{b}}}) =$ degree $F^{\bar{\mu}}_{\mathfrak{b},P_{\mathfrak{b}}}$.

But W contains a copy of $F^{\bar{\mu}}_{\mathfrak{b},P_{\mathfrak{b}}}$. Hence (iv) shows that $W = F^{\bar{\mu}}_{\mathfrak{b},P_{\mathfrak{b}}}$ which proves (d) and also

$$\dim W = \dim \mathrm{Hom}_{\mathfrak{b}}(F^{\bar{\mu}}_{\mathfrak{b},P_{\mathfrak{b}}}, F^{\mu}_{\mathfrak{a},P_{\mathfrak{b}}})$$

which in conjuction with (iii) establishes (b) and (c) follows from (b) trivially. This completes the proof of the Corollary.

It remains to check (i) and (ii) which appeared in the proof of Theorem 2.3. For this let $\theta = \mathfrak{s}^*$ and let f and g in $F_{\mathfrak{s}}$ have finite support. Then

$$\begin{aligned}
\bar{f}\cdot\bar{g}(\theta) &= \sum_{\eta\in t^*} \bar{f}(\eta)\bar{g}(\theta-\eta)\\
&= \sum_{\eta\in t^*}\ \sum_{\lambda\in\mathfrak{s}^*,\bar{\lambda}=\eta}\ \sum_{\omega\in\mathfrak{s}^*,\bar{\omega}=\theta-\bar{\lambda}} f(\lambda)g(\omega)\\
&= \sum_{\eta\in t^*}\ \sum_{\lambda\in\bar{\mathfrak{s}}^*,\bar{\lambda}=\eta}\ \sum_{\delta\in\bar{\mathfrak{s}}^*,\bar{\delta}=\theta} f(\lambda)g(\delta-\lambda)\\
&= \sum_{\lambda\in\mathfrak{s}^*}\ \sum_{\delta\in\mathfrak{s}^*,\bar{\delta}=\theta} f(\lambda)g(\delta-\lambda)\\
&= \sum_{\delta\in\mathfrak{s}^*,\bar{\delta}=\theta}\ \sum_{\lambda\in\mathfrak{s}^*} f(\lambda)g(\delta-\lambda)\\
&= \sum_{\delta\in\mathfrak{s}^*,\bar{\delta}=\theta} f\cdot g(\delta)\\
&= \overline{f\cdot g}(\theta).
\end{aligned}$$

This proves (ii).

Now let $\lambda\in\mathfrak{s}^*$. Then

$$\overline{e^{\lambda}}(\theta) = \sum_{\delta\in\mathfrak{s}^*,\bar{\delta}=\theta} e^{\lambda}(\delta).$$

It follows that, $\overline{e^{\lambda}}(\theta) = 1$ if $\bar{\lambda} = 0$ and is equal to 0 otherwise. This proves $\overline{e^{\lambda}} = e^{\bar{\lambda}}$.

3. *K*-multiplicities for generalized principal series representations

Let G be a connected real semi-simple Lie group with finite centre, K a maximal compact subgroup of G, Q a suitably chosen cuspidal parabolic subgroup of G, $Q = MAN$ its Langland's decomposition and Q^0 (respectively M^0) the connected component of identity of Q (respectively M). In this section, using theorem 2.3, we built up a Blattner type formula for multiplicity with which a finite dimensional irreducible representation of K appears in the induced representation $\mathrm{ind}_{Q^0}^{G}$ $(\sigma\otimes\exp\nu\otimes 1)$ where σ is a discrete or limit of discrete series representation of M^0 and $\nu\in a^*$;. $\mathfrak{a}_0$-the Lie algebra of A. Our main results are Theorem 3.13 and 3.14.

Let g_0 (respectively k_0) be the Lie algebra of G (repectively K), $g_0 = k_0 \oplus \mathfrak{p}_0$ the Cartan decomposition of g_0, θ the corresponding Cartan involution. We fix b_0,

a Cartan sub-algebra of k_0 and set $h_0 =$ the centralizer of b_0 in g_0. Then h_0 is a fundamental Cartan subalgebra of g_0. Set, once for all, $\Delta = \Delta(g, h)$. Let $\mathfrak{u} = k_0 \oplus \sqrt{-1}\mathfrak{p}_0$. Then u is a compact real form of g. Let σ and τ denote the conjugations of g with respect to g_0 and $\mathfrak{u}$ respectively. Then $\tau\sigma = \sigma\tau$ and $\tau\sigma | g_0 = \theta$. We denote the automorphism $\tau\sigma$ of $\mathfrak{g}$ also by θ. For each α in Δ, we choose a standard triple $\{X_\alpha, X_{-\alpha}, \bar{H}_\alpha\}$ such that $\tau(X_\alpha) = -X_{-\alpha}$ (This is possible; cf. Lemma 1.32 in [14]) and set $T_\alpha = X_\alpha - X_{-\alpha}$ and $Z_\alpha = X_\alpha + X_{-\alpha}$. Then $\tau(T_\alpha) = T_\alpha$ and $\tau(Z_\alpha) = -Z_\alpha$.

A root $\alpha \in \Delta$ is imaginary or complex according as $\alpha | \mathfrak{h} \cap \mathfrak{p}$ is zero or nonzero. If $\alpha \in \Delta$ is imaginary then either $\mathfrak{g}_\alpha \subset k$ in which case α is said to be compact or $\mathfrak{g}_\alpha \subset \mathfrak{p}$ in which case α is said to be noncompact. If α and β are distinct roots in Δ, we say α and β are orthogonal if $\alpha(\bar{H}_\beta) = 0$; we say α and β are strongly orthogonal (SO) if $\alpha \pm \beta$ are not roots; we say α and β are weakly orthogonal (WO) if they are orthogonal but not strongly orthogonal. Strongly orthogonal roots are necessarily orthogonal. If α is in Δ and S is a family of roots in Δ, we say α is orthogonal (respectively SO) to S if α is orthogonal (respectively SO) to each member of S; we say α is weakly orthogonal to S if it is orthogonal but not strongly orthogonal to S. If S is a family of roots in Δ we say S is a strongly orthogonal family if either S has atmost one element or any two distinct members of S are strongly orthogonal. For each noncompact root α in Δ we define an automorphism C_α of $\mathfrak{g}$ by $C_\alpha = \exp(ad(-\pi T_\alpha/4))$ – and call it Cayley transform of α.

We fix, once and for all S – a strongly orthogonal family (possibly empty) of noncompact roots in Δ and define an automorphism C_S of $\mathfrak{g}$ by $C_S =$ Identity if S is empty and $C_S = \Pi_{\alpha \in S} C_\alpha$ otherwise and call it Cayley transform of S.

If S is empty, set $\mathfrak{t}_o = \mathfrak{b}_o$ and $\mathfrak{a}_o = \mathfrak{h}_o \cap \mathfrak{p}_o$; if S is a nonempty set

$$\mathfrak{t}_o = \{H \in \mathfrak{b}_0 / \beta(H) = o \, \forall \beta \in S\}$$

and

$$\mathfrak{a}_0 = \mathfrak{h}_o \cap \mathfrak{p}_o + \sum_{\beta \in S} RZ_\beta.$$

Set $\mathfrak{j}_o = \mathfrak{t}_o \oplus \mathfrak{a}_o$ in either case.

Now if β is in S then $C_\beta(\bar{H}_\beta) = Z_\beta \in \mathfrak{p}_o$ and $C_\alpha(\overline{H_\beta}) = \bar{H}_\beta$ for every $\alpha \neq \beta$ in S. Hence it follows that $C_S(\overline{H_\beta}) = Z_\beta$ for all β in S. Also $C_S | t_o =$ identity. Hence $C_S(\mathfrak{h}) = \mathfrak{j}$ and therefore $\mathfrak{j}_o$ is a Cartan subalgebra of $\mathfrak{g}_0$ which we call as Cartan subalgebra built by C_S.

Let $\mathfrak{l}_o$ be the centralizer of $\mathfrak{a}_o$ in $\mathfrak{g}_o$. Then the restriction of the Cartan killing form of $\mathfrak{g}_o$ to $\mathfrak{l}_o \times \mathfrak{l}_o$ is nondegenerate. Hence if $\mathfrak{m}_o$ is the ortho complement of $\mathfrak{a}_o$ in $\mathfrak{l}_o$ with respect to this form then $\mathfrak{l}_0 = \mathfrak{m}_o \oplus \mathfrak{a}_0$. Both $\mathfrak{l}_o$ and $\mathfrak{m}_o$ are reductive subalgebras of $\mathfrak{g}_o$ and respectively $\mathfrak{j}_o$ and $\mathfrak{t}_o$ are their Cartan subalgebras.

Let Σ denote the set of the roots of the pair $(\mathfrak{g}, \mathfrak{a})$. For λ in Σ let $\mathfrak{g}_o^\lambda$ denote the root space corresponding to λ. Select and fix H^* in $\mathfrak{a}_o$ such that $\lambda(H^*)$ is nonzero for every λ in Σ. Let $\Sigma^+ = \{\lambda \in \Sigma / \lambda(H^*) > 0\}$ and set $\mathfrak{n}_o = \Sigma_{\lambda \in \Sigma^+} \mathfrak{g}_o^\lambda$ and $\mathfrak{q}_o = \mathfrak{l}_o \oplus \mathfrak{n}_o$. Then $\mathfrak{q}_o$ is a (cuspidal) parabolic subalgebra of $\mathfrak{g}_o$ and $\mathfrak{q}_o = \mathfrak{m}_o \oplus \mathfrak{a}_o \oplus \mathfrak{n}_o$ is its Langlands decomposition. We call $\mathfrak{q}_o$ as a parabolic subalgebra built by C_S.

It is clear that $\Delta(\mathfrak{a}, \mathfrak{j}) = \{\alpha \circ C_S^{-1} / \alpha \in \Delta\}$. Since $\Delta(\mathfrak{l}, \mathfrak{j}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{j}) / \alpha | a \equiv 0\} = \{\alpha \circ C_S^{-1} / \alpha \in \Delta, \alpha$ imaginary and orthogonal to $S\}$. We identify $\Delta(\mathfrak{l}, \mathfrak{j})$ with $\{\alpha \in \Delta / \alpha$ imaginary and orthogonal to $S\}$. Since, $\alpha | a \equiv 0$ for each α in $\Delta(\mathfrak{l}, \mathfrak{j})$ and $\{\alpha | \mathfrak{t} / \alpha \in \Delta(\mathfrak{l}, \mathfrak{j})\} = \Delta(\mathfrak{m}, \mathfrak{t})$ we identify $\Delta(\mathfrak{m}, \mathfrak{t})$ with $\Delta(\mathfrak{l}, \mathfrak{j})$.

If α in Δ is imaginary and orthogonal to S then $C_S(\mathfrak{g}_\alpha) = \mathfrak{m}_\alpha$ and either $\mathfrak{m}_\alpha \subset \mathfrak{k}$ in which case we say α in $\mathfrak{m}$-compact or $\mathfrak{m}_\alpha \subset \mathfrak{p}$ in which case we say α is $\mathfrak{m}$-noncompact.

The following results are known (cf. ch. 2 of [13]) and also easy to check by direct computation.

PROPOSITION 3.1

Let $\alpha \in \Delta$ be orthogonal to S. Then

(1) *α is strongly orthogonal to all except possibly one member of S.*
(2) *If α is strongly orthogonal to S then $C_S(X_\alpha) = X_\alpha$.*
(3) *If α is strongly orthogonal to all but one member say β of S then $C_S(X_\alpha) = cX_{\alpha-\beta} + dX_{\alpha+\beta}$ for some nonzero numbers c and d.*
(4) *If α is imaginary and strongly orthogonal to S then α is $\mathfrak{m}$-compact if and only if α is compact. If α is imaginary and weakly orthogonal to S then α is $\mathfrak{m}$-compact if and only if it is noncompact.*

Set $\Delta_{\mathfrak{k}} = \Delta(\mathfrak{k}, \mathfrak{b})$, $\Delta_{\mathfrak{m}} = \Delta(\mathfrak{m}, \mathfrak{t})$, $\Delta_{\mathfrak{m},\mathfrak{k}} = \Delta(\mathfrak{m} \cap \mathfrak{k}, t)$ and let W, $W_{\mathfrak{k}}$ $W_{\mathfrak{m}}$, $W_{\mathfrak{m},\mathfrak{k}}$ denote the Weyl groups of the root systems $\Delta, \Delta_{\mathfrak{k}}$, $\Delta_{\mathfrak{m}}$, $\Delta_{\mathfrak{m},\mathfrak{k}}$ respectively. Fix, once for all, $P_{\mathfrak{m}}$ a positive system for $\Delta_{\mathfrak{m}}$ and set $P_{\mathfrak{m},\mathfrak{k}} = P_{\mathfrak{m}} \cap \Delta_{\mathfrak{m},\mathfrak{k}}$. Then $_{\mathfrak{m},\mathfrak{k}}$ is a positive system for $\Delta_{\mathfrak{m},\mathfrak{k}}$.

Now refer to [5] and [6]. Using the terminology and results there in, we easily see that $(\mathfrak{m}, \mathfrak{m} \cap \mathfrak{k})$ is a regular pair and $P_{\mathfrak{m}}$ is an admissible positive system. Further for every $\Delta_{\mathfrak{m},\mathfrak{k}}$ integral element λ in $\mathfrak{t}^*$ the Verma-module $V_{\mathfrak{m},-P_{\mathfrak{m}},\lambda}$ admits a lattice of $\mathfrak{m}$-modules $A_s (s \in W_{\mathfrak{m},\mathfrak{k}})$ above it and if we set $W_{-P_{\mathfrak{m}},\lambda} = A_l / \Sigma_{s \neq l} A_s$ then $W_{-P,\lambda}$ is an admissible $(\mathfrak{m}, \mathfrak{m} \cap \mathfrak{k})$-module. Call $W_{-P_{\mathfrak{m}}}, \lambda$ the fundamental series module of the pair $(\mathfrak{m}, \mathfrak{m} \cap \mathfrak{k})$ parametrized by $-P_{\mathfrak{m}}$ and λ.

For $\Delta_{\mathfrak{m},\mathfrak{k}}$-integral λ in $\mathfrak{t}^*$ and ν in $\mathfrak{a}^*$, consider $W_{-P_{\mathfrak{m}},\lambda}$ as a $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ module by allowing H in $\mathfrak{a}$ to act by $\nu(H)$ and X in $\mathfrak{n}$ to act by zero. The $\mathfrak{q}$-module thus obtained is denoted by $W_{-P_{\mathfrak{m}},\lambda} \otimes \nu \otimes 0$.

If W is any $\mathfrak{q}$-module, we induce it in the following way to get a $\mathfrak{g}$-module. Let $E \mapsto E^+$ be the unique anti automorphism of $U(\mathfrak{q})$ such that $X^+ = X$ for all X in $\mathfrak{q}$ and let $\mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), W)$ denote the space of all linear functions f: $U(\mathfrak{g}) \mapsto W$ such that $f(DE) = E^+ f(D)$ for all $D \in U(\mathfrak{g})$ and E in $U(\mathfrak{q})$. This space is a $\mathfrak{g}$-module under the action $Xf(Y) = f(X^+ Y) (f \in \mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), W), X$ in $\mathfrak{g}$. The set of all $\mathfrak{k}$-finite vectors in $\mathrm{Hom}_{U(q)}(U(\mathfrak{g}), W)$ is a $\mathfrak{g}$-module which we denote by $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(W)$ and call as $\mathfrak{g}$-module induced by the $\mathfrak{q}$-module W.

For λ and ν as above, call the module $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}(W_{P_{\mathfrak{m}},\lambda} \otimes \nu \otimes 0)$ as generalized principal series (gps)-module parametrized by $P_{\mathfrak{m}}, \lambda$ and ν.

Our aim is to construct a $\mathfrak{k}$-multiplicity formula for these gps modules. We need some, preparation. Set

$$\Delta^0 = \{\alpha \in \Delta / \alpha | \mathfrak{t} \equiv 0\}, \Delta^* = \Delta \backslash \Delta^0, \Delta_{\mathfrak{k}}^0 = \{\alpha \in \Delta_{\mathfrak{k}} / \alpha | \mathfrak{t} \equiv 0\}$$

and $\Delta_{\mathfrak{k}}^* = \Delta \backslash \Delta_{\mathfrak{k}}^0$. Call the strongly orthogonal family S simple if $\Delta^0 = S \cup -S$. We note that S is simple if it has atmost one element.

PROPOSITION 3.2

Following conditions are equivalent.

(i) $\Delta_{\mathfrak{k}}^0 = \phi$
(ii) *S is simple*
(iii) *The only real roots of $\Delta(\mathfrak{g}, \mathfrak{j})$ are $\alpha \circ C_S^{-1}$ with $\pm \alpha \in S$.*

Proof. First note if $\alpha\in\Delta^0$. Then $\alpha=\Sigma_{\beta\subset S}d_\beta\beta$ for some scalars d_β and further $\alpha\notin\pm S$ if and only if atleast two co-efficients $d'_\beta s$ are nonzero and in this case atleast one d_β will be non integral.

Now if $\Delta^0=S\cup -S$ then $\Delta^0_K=\Delta^0\cap\Delta_K=(S\cup -S)\cap\Delta_K=\phi$ as every β in $\pm S$ is noncompact. Thus (ii)$\Rightarrow$(i). Conversely, if $\Delta^0\neq S\cup -S$, let $\alpha\in\Delta^0\backslash(S\cup -S)$. Write $\alpha=\Sigma_{\beta\in S}d_\beta\cdot\beta$. Then atleast two of $d_\beta s$ are nonzero. Let $d_{\beta_1}\neq 0\neq d_{\beta_2}$ for some β_1,β_2 in S. Now $\alpha(H_\beta)=2d_\beta\forall\beta$ in S. Therefore if $d_{\beta_1}>0$ (respectively $d_{\beta_1}<0$) then $\alpha-\beta_1$ (respectively $\alpha+\beta_1$) is a root in Δ^0 and it is compact if α is noncompact. Thus either $\alpha\in\Delta^0_\mathfrak{k}$ or $\alpha\notin\Delta^0$ in which case $\alpha-\beta_1$ or $\alpha+\beta_1$ (according to whether d_β is +ve or −ve) is in $\Delta^0_\mathfrak{k}$. Hence $\Delta^0_\mathfrak{k}\neq\phi$. Thus (i)$\Rightarrow$(ii). Hence (i)$\leftrightarrow$(ii). We note that $\Delta(\mathfrak{g},\mathfrak{j})=\{C_S(\alpha)=\alpha\circ C_S^{-1}/\alpha\in\Delta\}$ and $C_S\alpha|_t=\alpha|_t$ for all $\alpha\in\Delta$. Now the root $C_S(\alpha)$ is real $\leftrightarrow C_S(\alpha)|\mathfrak{t}\equiv 0\Leftrightarrow\alpha|\mathfrak{t}=0$ i.e. iff $\alpha\in\Delta^0$ from this the equivalence of (ii) and (iii) follows immediately. This completes the proof.

DEFINITION 3.3

(1) A positive system P for Δ is said to be S-admissible if $S\subset P$ and there exists H_0 in $\sqrt{-1}\mathfrak{t}_o$ such that $\alpha(H_0)>0$ for each α in P unless $\alpha|_t=0$.
(2) Let $P_\mathfrak{m}$ be a positive system for $\Delta_\mathfrak{m}$ and P be a positive system for Δ. We say P is $P_\mathfrak{m}$-compatible if $P_\mathfrak{m}\subset P$, i.e. if $P_\mathfrak{m}=\{\alpha|_t\alpha\in P/\alpha$ imaginary and orthogonal to $S\}$.

PROPOSITION 3.4([1])

Let $P_\mathfrak{m}$ be a positive system for $\Delta_\mathfrak{m}$. Then there exists a positive system P for Δ which is S-admissible and $P_\mathfrak{m}$-compatible.

Proof. Set

$$\mathfrak{p}=\{H\in\sqrt{-1}\mathfrak{t}_o/\alpha(H_o)>0 \text{ for each } \alpha \text{ in } P_\mathfrak{m}\}$$

and

$$\mathfrak{m}=\{H\in\sqrt{-1}\mathfrak{t}_o/\alpha(H)\neq 0 \text{ for each } \alpha \text{ in } \Delta^*\}.$$

Then $\mathfrak{p}$ is open and $\mathfrak{m}$ is dense in $\sqrt{-1}\mathfrak{t}_o$ Let Δ^0_+ be any positive system for Δ^0 such that $S\subset\Delta^0_+$. (This is possible as $\overline{H_\beta}(\beta\in S)$ is a basis for $\mathfrak{u}$. Any lexicographic ordering on Δ^0 will give such a positive system.) Let H_1 in $\mathfrak{u}$ be such that $\alpha(H_1)>0$ for each α in Δ^o_+. Select any H_2 in $\mathfrak{b}\cap\mathfrak{m}$. Let r be a positive real number such that $r.|\alpha(H_2)|>\alpha(H_1)$ for each α in Δ^*. Let $H_o=rH_2$ and $H=H_1+H_o$. Set

$$P=\{\alpha\in\Delta/\alpha(H)>0\}.$$

Then it is easily checked that P is an S-admissible, $P_\mathfrak{m}$-compatible positive system for Δ. This completes the proof. Following proposition characterizes simple system S.

PROPOSITION 3.5

Let P be a S-admissible positive system for Δ. Then every β in S is P-simple if and only if S is simple.

Proof. Let $H_o\in\sqrt{-1}\mathfrak{t}_o$ be such that $\alpha(H_o)>0$ for all α in $P\cap\Delta^*$. Now suppose S is simple then $\alpha(H_o)>0$ for all α in $P\backslash S$ and $\alpha(H_o)=0$ for all α in S. Hence if β is in S then

β can neither be a sum of roots in $P\backslash S$ nor a sum of a root in P and a root in S nor a sum of roots in S. Hence every β in S is P-simple. Conversely suppose every β in P is P-simple. Let $\alpha\in\Delta$ be such that $\bar{\alpha}=0$. Then there exists scalars d_β such that $\alpha=\Sigma_{\beta\in S}d_\beta\cdot\beta$. Assume without loss of generality that α is in P, then all d_β are non-negative. Suppose $\alpha\notin S$. Then atleast two $\dot{d}_\beta s$ say d_β and d_r are nonintegral. Now $\overline{\alpha(H_\beta)}=2\cdot d_\beta$. Hence $2d_\beta=1$ or 3. If $2d_\beta=1$, then $\alpha-\beta$ is a root and it is in P as β is P-simple. But

$$\alpha-\beta=\sum_{\delta\in S\backslash\{\beta\}} d_\delta\cdot\delta-\tfrac{1}{2}\beta.$$

This contradicts that elements in S are simple as the co-efficient of r in the expression for $\alpha-\beta$ is positive whereas co-efficient for β is negative. Hence $2d_\beta\neq 1$. If $2d_\beta=3$ then $\alpha-2\beta$ is a root in P and

$$\alpha-2\beta=\sum_{\delta\in S\backslash\{\beta\}} d_\delta\cdot\delta-\tfrac{1}{2}\beta.$$

This again gives a contradiction. This forces α to be an element of S. Hence $\Delta^*=S\cup -S$ i.e. S is simple. This completes the proof.

We fix, once for all, an S-admissible, $P_{\mathfrak{m}}$-compatible positive system P for Δ and set

$$P_{\mathfrak{k}}=\{\alpha\in\Delta_{\mathfrak{k}}|\alpha=\beta|\mathfrak{b}\text{ for some }\beta\in P\},$$

$$P_{\mathfrak{m},\mathfrak{k}}=\{\alpha\in P_{\mathfrak{m}}|\alpha\ \mathfrak{m}\text{-compact}\},$$

$$P_{\mathfrak{m},\mathfrak{n}}=P_{\mathfrak{m}}\backslash P_{\mathfrak{m},\mathfrak{k}}=\{\alpha P_{\mathfrak{m}}|\alpha\ \mathfrak{m}\text{-noncompact}\}$$

Let ρ_c (respectively $\rho_{\mathfrak{m},c}$) equal half the sum of roots in $P_{\mathfrak{k}}$ (respectively $P_{\mathfrak{m},\mathfrak{k}}$).

Following lemma is basic to the proof of multiplicity formula for gps modules. Its proof is similar to that of lemma 4.1 of [7] and hence we omit it.

Lemma 3.6. *Let W be any $\mathfrak{q}$-module. Let $\mu\in\mathfrak{b}^*$ be $P_{\mathfrak{k}}$-dominant integral. Then*

$$\dim \mathrm{Hom}_{\mathfrak{k}}(F^{\mu}_{\mathfrak{k},P_{\mathfrak{k}}},\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}(W))=$$
$$\Sigma_w \dim \mathrm{Hom}_{\mathfrak{m}\cap\mathfrak{k}}(F^{w}_{\mathfrak{m}\cap\mathfrak{k},P_{\mathfrak{m},\mathfrak{k}}},F^{\mu}_{\mathfrak{k},P_{\mathfrak{k}}})\dim \mathrm{Hom}_{\mathfrak{m}\cap\mathfrak{k}}(F^{w}_{\mathfrak{m}\cap\mathfrak{k},P_{\mathfrak{m},\mathfrak{k}}}W), \tag{3.6.1}$$

where summation is over all $P_{\mathfrak{m},\mathfrak{k}}$-dominant integral elements w in $\mathfrak{t}^*$.

Next taking $W=W_{-\bar{P}_{\mathfrak{m}},\lambda}$ in lemma 3.6, we will determine the two factors which appear in summation in (3.6.1) separately.

PROPOSITION 3.7

Let $\lambda\in\mathfrak{t}^*$ be $\Lambda_{\mathfrak{m},\mathfrak{k}}$ integral. Assume $(\lambda-\rho_{\mathfrak{m},c})\overline{(H_\alpha)}$ is not positive integer for any $\alpha\in -P_{\mathfrak{m},\mathfrak{k}}$. Let $w\in\mathfrak{t}^*$ be $P_{\mathfrak{m},\mathfrak{k}}$-dominant integral. Then

$$\dim \mathrm{Hom}_{\mathfrak{m}\cap\mathfrak{k}}(F^{w}_{\mathfrak{m}\cap\mathfrak{k},P_{\mathfrak{m},\mathfrak{k}}},W_{-P_{\mathfrak{m},\lambda}})$$
$$=\Sigma_{\sigma\in W_{\mathfrak{m},\mathfrak{k}}}\varepsilon(\sigma)\mathbb{P}_{P_{\mathfrak{m},\mathfrak{n}}}(\sigma(w+\rho_{\mathfrak{m},c})-(\lambda-\rho_{\mathfrak{m},c})).$$

Proof. This follows from Theorem 5.1(b) of [7] which is a restatement of Theorem 4.9 of [6] and a stronger version of Theorem 14.5 of [5].

Next we will first determine certain weights of the cannonical $\mathfrak{m}\cap\mathfrak{k}$-module $\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k}$ and then using Theorem 2.3 we will compute

$$\dim \mathrm{Hom}_{\mathfrak{m}\cap\mathfrak{t}}(F^{w}_{\mathfrak{m}\cap\mathfrak{t},P_{\mathfrak{m},\mathfrak{t}}}, F^{\mu}_{\mathfrak{t},P_{\mathfrak{t}}}).$$

We need some notation. Set

$$P^{o} = \{\alpha\in P/\alpha|\mathfrak{t}\equiv 0\}$$

and

$$P^{*} = P\backslash P^{o}.$$

For λ in b^* or h^* we will write $\bar{\lambda}$ for $\lambda|_{\mathfrak{t}}$. Fix H_o in $\sqrt{-1}\,\mathfrak{t}_o$ such that $\alpha(H_o)>0$ for all α in P^*.

Call a $\mathfrak{t}$-weight w of the $\mathfrak{m}\cap\mathfrak{k}$-module $\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k}$ positive if $w(H_o)>0$.

From now on, let T denote the family of positive weights of the $\mathfrak{m}\cap\mathfrak{k}$- module $\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k}$ each counted with its multiplicity.

We will determine T. We need some more terminology. Suppose α and β are two roots in Δ. Say α is related to β if either α and β are strongly orthogonal or $\alpha(\overline{H_\beta})=-2$. Otherwise say α is unrelated to β.

PROPOSITION 3.8

Let $X\mapsto X^1$ denote the canonical projection of $\mathfrak{k}$ onto $\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k}$. For each α in $\Delta(\mathfrak{g},\mathfrak{h})$ set $E_\alpha = X_\alpha$ if α imaginary and $E_\alpha = X_\alpha + \theta X_\alpha$ if α complex. If $\mathfrak{h}\cap\mathfrak{p}\neq(0)$, fix a point H_* in $\mathfrak{h}\cap\mathfrak{p}$ such that $\alpha(H_*)\neq 0$ for every complex $\alpha\in\Delta$. If $\mathfrak{h}\cap\mathfrak{p}=(0)$ take $H_*=0$. Set $P^{\#}=\{\alpha\in P^*/\alpha$ compact and unrelated to some member of $S\}\cup\{\alpha\in P^*/\alpha$ complex and $\alpha(H_*)>0\}$. Let

$$B=\{E^1_\alpha/\alpha\in P^{\#}\},$$

$T=$ family of positive $\mathfrak{t}$-weights of the $\mathfrak{m}\cap\mathfrak{k}$ module $\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k}$ counted with multiplicity and $(\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k})^+=$ the subspace of $\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k}$ spanned by positive weight vectors.

Then B is a basis for $(\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k})^+$ and hence T is precisely the family of $\alpha|_t=\bar{\alpha}(\alpha\in P^{\#})$.

Proof. First we will prove that B is linearly independent. Suppose not. Then there exist a nonempty subset C of $P^{\#}$ and nonzero scalars $a_\alpha(\alpha$ in $C)$ such that

$$\sum_{\alpha\in C} a_\alpha E^1_\alpha = 0 \text{ i.e. } \left[\sum_{\alpha\in\beta} a_\alpha E_\alpha, \mathfrak{a}\right] = \{o\}. \tag{3.8.1}$$

Let D be the subset of C consisting of all imaginary roots in C.

Suppose $D\neq\varnothing$. Then (3.8.1) shows that if β is in S then

$$\sum_{\alpha\in D} a_\alpha[X_\alpha, X_\beta+X_{-\beta}] + \sum_{\alpha\in C\backslash D} a_\alpha[X_\alpha+\theta X_\alpha, X_\beta+X_{-\beta}]=0. \tag{3.8.2}$$

Linear independence of $X'_\alpha s$ now show that

$$\sum_{\alpha\in D} a_\alpha[X_\alpha, X_\beta+X_{-\beta}]=0 \text{ for every } \beta\in S. \tag{3.8.3}$$

Using (3.8.3) we will deduce several properties of D. Suppose β is in S and α in D. Now if $\alpha+\beta$ is a root then (3.8.3) shows that $\alpha+\beta=r-\beta$ for some r in D. In particular $\alpha+2\beta=r$ is a root in D. Thus we have,

(pi) If $\alpha \in D, \beta \in S$ and if $\alpha + \beta$ is a root then $\alpha + 2\beta$ is a root in D. Similar consideration shows that

(pii) If $\alpha \in D, \beta \in S$ and if $\alpha - \beta$ is a root then $\alpha - 2\beta$ is a root in D. Combining (pi) and (pii) and using the fact that β string through α can not have more than four elements we easily deduce that

(piii) If $\alpha \in D$, $\beta \in S$ then either $\alpha + \beta$ is not a root or $\alpha - \beta$ is not a root.

Now suppose $\alpha \in D$, $\beta \in S$ and $\alpha + 2\beta$ is a root then first of all $\alpha + 2\beta$ is in D. Now if $\alpha + 3\beta$ is a root then by (pi) $\alpha + 4\beta$ is also a root which can not happen and hence $\alpha + 3\beta$ is not a root. Therefore using (pi) and (piii) we deduce that

(p1) If $\alpha \in D$, $\beta \in S$ and $\alpha + \beta$ is a root then $\alpha + 2\beta$ is a root in D, further β string through α is $\{\alpha, \alpha + \beta, \alpha + 2\beta\}$ and therefore $\alpha(\overline{H_\beta}) = -2$. Similarly we also deduce that

(p2) If $\alpha \in D$, $\beta \in S$ and $\alpha - \beta$ is a root then $\alpha - 2\beta$ is a root in D, further β string through α is $\{\alpha, \alpha - \beta, \alpha - 2\beta\}$ and therefore $\alpha(\overline{H_\beta}) = +2$.

Now let $\alpha \in D$ and set

$$S_\alpha = \{\beta \in S / \alpha \text{ unrelated to } \beta\}.$$

First note that α is an imaginary root and belongs to $P^{\#}$ hence $S_\alpha \neq \phi$.

Let $\beta \in S_\alpha$ then $\alpha(\overline{H_\beta}) \neq -2$ and also α and β are not strongly orthogonal hence at least one of $\alpha + \beta$ and $\alpha - \beta$ is a root. Using (p1) and (p2) we see that

(p3) If $\alpha \in D, \beta \in S_\alpha$ then $\alpha - 2\beta \in D$ and β strong through α is $\{\alpha - 2\beta, \alpha - \beta, \alpha\}$ and hence $\alpha(\overline{H_\beta}) = 2$.

Now fix $\alpha \in D$ and let $\beta_1, \beta_2, \ldots, \beta_r$ be an enumeration of roots in S_α. For each $i (1 \leqslant i \leqslant r)$ set

$$\alpha_i = \alpha - \sum_{1 \leqslant j \leqslant i} 2\beta_j.$$

We claim α_i is a root in $D (1 \leqslant i \leqslant r)$.

Clearly by (p3) α_1 is a root in D. Assume α_i is in D for some i, $1 < i < r$. It is easy to see that $\alpha_i(\bar{H}_{\beta_{i+1}}) = \alpha(\bar{H}_{\beta_{i+1}}) = 2$ (by (p3)). In particular α_i is unrelated to β_{i+1}. Hence by (p3), $\alpha_i - 2\beta_{i+1}$ is a root in D i.e. α_{i+1} is a root in D. Thus induction proves that the claim is true. Hence we get α_r in D. Now take any β in S_α. Then by (p3), $\alpha_r(\bar{H}_\beta) = 2$. On the other hand for any i, $1 \leqslant i \leqslant r$ we have

$$\alpha_i(\bar{H}_{\beta_i}) = \left(\alpha - 2 \sum_{1 \leqslant J \leqslant r} B_j\right)(\bar{H}_{\beta_i}) = \alpha(\bar{H}_{\beta_i}) - 2\beta_i(\bar{H}_{\beta_i}) = -2$$

by (p3). Hence $\beta \neq \beta_i$ for any i, $1 \leqslant i \leqslant r$. Thus $\beta \notin S_\alpha$. Now $\alpha = \alpha_r + 2\Sigma_{1 \leqslant i \leqslant r} \beta_i$. Further β and β_i for each i are strongly orthogonal (being distinct members of S). Hence $\alpha(\overline{H_\beta}) = \alpha_r(\overline{H_\beta}) = 2$. Hence $\beta \in S_\alpha$. This contradiction proves that D must be empty. Now if rank $\mathfrak{g} =$ rank $\mathfrak{k}$ then there are no complex roots. Hence $C = D$. Hence $C = \phi$. This contradicts $C \neq \phi$. Hence in this case we conclude that B is linearly independent. If rank $\mathfrak{g} \neq$ rank $\mathfrak{k}$ then using (3.8.1), the fact that $D = \phi$ and H_* is in $\mathfrak{h} \cap \mathfrak{p}$ we get $\Sigma_{\alpha \in C} a_\alpha [X_\alpha + \theta X_\alpha, H_*] = 0$ i.e.

$$\sum_{\alpha \in C} a_\alpha \alpha(H_*)(X_\alpha - \theta X_\alpha) = 0. \tag{3.8.4}$$

Now since $D = \phi$, every α in C is a complex root in $P^\#$ and hence $\alpha(H_*) > 0$ for all α in C. Also $X'_\alpha s$ are linearly independent. Hence from (3.8.4) we conclude that $C = \phi$. This contradiction proves that B is linearly independent.

We next want to prove that B spans $(\mathfrak{k}/\mathfrak{m} \cap \mathfrak{k})^+$. Set

$$P^*_{cc,+} = \{\alpha \in P^* | \alpha \text{ compact or complex}, \alpha(H_*) \geqslant 0\}.$$

Note $P^\# \subset P^*_{cc,+}$. Since $\Delta_\mathfrak{k} = \Delta_\mathfrak{k}^0 \cup \Delta_\mathfrak{k}^*$, one checks easily that

$$\mathfrak{k} = \mathfrak{b} + \sum_{\alpha \in \Delta_\mathfrak{k}^0} \mathfrak{k}_\alpha + \sum_{\alpha \in \pm P^*_{cc,+}} \mathbb{C}E_\alpha. \tag{3.8.5}$$

It follows immediately that

$$(\mathfrak{k}/\mathfrak{m} \cap \mathfrak{k})^+ = \sum_{\alpha \in P^*_{cc,+}} \mathbb{C}E^1_\alpha. \tag{3.8.6}$$

Let $\alpha \in P^*_{cc,+}$. If $\alpha \in P^\#$ then $E^1_\alpha \in B$. If $\alpha \notin P^\#$ then α is compact, imaginary and related to every member of S. We have two cases:

Case (i): α strongly orthogonal to S then $E_\alpha = X_\alpha = C_s(X_\alpha) \in \mathfrak{m} \cap \mathfrak{k}$. Hence $E^1_\alpha = 0$.

Case (ii): α not strongly orthogonal to S. Let $\{\beta_1, \ldots, \beta_r, \ldots, \beta_m\}$ be an enumeration of elements in S such that $\alpha(\overline{H_{\beta_i}}) = -2, 1 \leqslant i \leqslant r$ and α is strongly orthogonal to β_i if $r < i \leqslant m$. Set

$$\gamma = \alpha + \sum_{1 \leqslant i \leqslant r} \beta_i.$$

Then one checks easily that γ is an imaginary root. It is weakly orthogonal to $\beta_i (1 \leqslant i \leqslant r)$ and strongly orthogonal to $\beta_i (r < i < m)$. By Proposition 3.1(a) we get $r = 1$. Set $\beta = \beta_1$ then $\gamma = \alpha + \beta$. Using Proposition 3.1 we get

$$C_S(X_\gamma) = C_\beta(X_\gamma) = cX_{\gamma-\beta} + dX_{\gamma+\beta} \text{ for some nonzero constants } c \text{ and } d. \tag{3.8.7}$$

Now γ is noncompact. Hence it follows from Proposition 3.1(d) that $C_S(X_\gamma)$ is in $\mathfrak{m} \cap \mathfrak{k}$. Thus $C_S(X_\gamma)^1 = 0$. Therefore (3.8.7) gives

$$X^1_\alpha = X^1_{\gamma-\beta} = c^{-1} dX^1_{\gamma+\beta} = c^{-1} dX^1_{\alpha+2\beta}. \tag{3.8.8}$$

Now $\alpha + 2\beta$ is strongly orthogonal to $\{\beta_2, \ldots, \beta_m\} = S \backslash \{\beta\}$ and $(\alpha + 2\beta)(\overline{H_\beta}) = 2$. Now $\alpha + 2\beta$ is compact, $\alpha + 2\beta$ is unrelated to β (as $(\alpha + 2\beta)(\overline{H_\beta}) = 2$) and $\alpha + 2\beta \in P^*$ as $(\alpha \in P^*)$. Hence $\alpha + 2\beta \in P^\#$ and $X^1_{\alpha+2\beta} \in B$. Hence from (3.8.8) we get $X^1_\alpha \in B$ i.e. $E^1_\alpha \in B$. Thus in any case $E^1_\alpha \in B$ for every α in $P^*_{cc,|}$. Hence from (3.8.6) we deduce that subspace spanned by B is precisely $(\mathfrak{k}/\mathfrak{m} \cap \mathfrak{k})^+$. Hence B is a basis for $(\mathfrak{k}/\mathfrak{m} \cap \mathfrak{k})^+$. The second assertion in the statement in the proposition is now obvious. This completes the proof.

The following Lemma is an important step in the proof of the $\mathfrak{k}$-multiplicity formula for $\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{q}}(W_{-\Gamma_m,\lambda} \otimes \nu \otimes 0)$.

Lemma 3.9. *Set*

$$P_{\mathrm{m},n} = \{\alpha \in P_m \mid \alpha \text{ m noncompact}\}$$

$$P^*_{\mathrm{nc},+} = \{\alpha \in P^* \mid \text{either } \alpha \text{ noncompact or complex and } \alpha(H_*) > 0\}$$

and let $P^{\#}$ *be as in Proposition* 5.10. *That is*

$$P^{\#} = \{\alpha \subset P^* \mid \text{either } \alpha \text{ complex and } \alpha(H_*) > 0, \text{ or}$$
$$\alpha \text{ compact and unrelated to some member of } S\}.$$

Identify $P_{\mathrm{m},n}$ *with*

$$\left\{\alpha \in P \,\middle|\, \begin{array}{l}\text{either } \alpha \text{ compact and weakly orthogonal to } S \text{ or } \alpha \text{ non-}\\ \text{compact and strongly orthogonal to } S\end{array}\right\}.$$

Then there exists a bijection $f: P_{\mathrm{m},n} \cup P^{\#} \to P^*_{\mathrm{nc},+}$ *such that* $\overline{f(\alpha)} = \bar{\alpha}$ *for every* $\alpha \in P_{\mathrm{m},n} \cup P^{\#}$.

Proof. Let $\{\beta_1, \ldots, \beta_m\}$ be an enumeration of elements of S. For $\alpha \in \Delta$ and any i, $(1 \leqslant i \leqslant m)$ set $m(\alpha, i) = \overline{\alpha(H_{\beta_i})}$ and define $n(\alpha, i)$ in different cases as follows:

(a) $n(\alpha, i) = m(\alpha, i)$ if $\alpha = \pm \beta_i$.
(b) $n(\alpha, i) = 0$ if α and β_i are distinct and strongly orthogonal.
(c)

$$n(\alpha, i) = \begin{cases} m(\alpha, i) & \text{if } m(\alpha, i) = \pm 3 \text{ or } \pm 1 \\ 0 & \text{if } m(\alpha, i) = -2 \text{ but } \alpha \neq -\beta_i \\ 1 & \text{if } m(\alpha, i) = 2 \text{ but } \alpha \neq \beta_i \\ -1 & \text{if } \alpha \text{ and } \beta_i \text{ are weakly orthogonal.} \end{cases}$$

In table 1, we give the values of $n(\alpha, i)$ against $m(\alpha, i)$ in the case when $\pm\alpha$ and β_i are distinct and not strongly orthogonal.

Define a map $s: \Delta \to \Delta$ by

$$s(\alpha) = \alpha - \sum_{1 \leqslant i \leqslant m} n(\alpha, i)\beta_i. \tag{3.9.1}$$

(Root string consideration shows that $s(\alpha)$ defined by (3.9.1) is a root.) For $\alpha \in \Delta$ and $1 \leqslant i \leqslant m$ the following properties of $s(\alpha)$ can be easily verified.

(P1) $s(\alpha)$ is complex if and only if α is complex.
(P2) $\overline{s(\alpha)} = \bar{\alpha}$.
(P3) $s(\alpha) = -\alpha$ if $\alpha \in S$; $s(\alpha) \notin \pm S$ if and only if $\alpha \notin \pm S$.
(P4) $s(\alpha)$ is SO to β_i if and only if α is SO to β_i and in this case $m(\alpha, i) = n(\alpha, i) = m(s(\alpha), i) = n(s(\alpha), i)$.
(P5) $s(\alpha) \in \Delta^o$ if and only if $\alpha \in \Delta^o$.

Table 1.

$m(\alpha, i)$	-3	-2	-1	0	1	2	3

(P6) If $\alpha, \beta \in \Delta$ are different from $\pm\beta_i$, none of α, β strongly orthogonal to β_i and $m(\beta, i) = m(\alpha, i) - 2n(\alpha, i)$. Then $n(\beta, i) = -n(\alpha, i)$. (This can be easily checked from table 1).

(P7) $m(s(\alpha), i) = m(\alpha, i) - 2n(\alpha, i)$

(P8) $n(s(\alpha, i) = -n(\alpha, i)$. This follows from (P3), (P4), (P6) and (P7).

(P9) $s(s(\alpha)) = \alpha$. This follows from (P8).

(P10) $m(s(\alpha), i) = -2$ if and only if either $\alpha = \beta_i$ or $m(\alpha, i) = -2$. This follows from (P3), (P7) and table 1.

(P11) If $\alpha \neq \beta_i$ then α is unrelated to β_i if and only if $s(\alpha)$ is unrelated to β_i. This follows from (P6) and (P10).

(P12) If $\alpha \in P_{m,n}$ ($P_{m,n}$ considered as a subsystem of P) then $s(\alpha) \in P^*_{nc,+}$. (If α is not SO to S then by Proposition 3.10 there exists a unique $\beta = \beta_i$ such that α and β_i are weakly orthogonal. In this case $s(\alpha) = \alpha + \beta_i$ which is noncompact.)

(P13) $n(\alpha, i)$ is odd if and only if α is unrelated to β_i. This can be observed from the table 1.

(P14) Set

$$\gamma_i = s(\alpha) + n(\alpha, i)\beta_i = \alpha - \sum_{1 \leqslant j \neq i \leqslant n} n(\alpha, j)\beta_j.$$

Then (i) $m(\gamma_i, j) = m(s(\alpha), j)$ for $j \neq i$, (ii) $m(\gamma_i, i) = m(\alpha, i)$ hence $n(\gamma_i, i) = n(\alpha, i)$, (iii) γ_i is unrelated to β_j if and only if α is unrelated to $\beta_j (1 \leqslant j \leqslant m)$, (iv) $s(\gamma_i) = \alpha - n(\alpha, i)\beta_i$. (i) and (ii) are proved by direct computation and (iii) follows from (i), (ii) and (P10). As for (iv) note if $j \neq i$ then (i) implies $n(\gamma_i, j) = n(s(\alpha), j)$ and hence by (P8) we get $n(\gamma_i, j) = -n(\alpha, j)$ and (ii) implies $n(\gamma_i, i) = n(\alpha, i)$. Now $s(\gamma_i) = \gamma_i - \Sigma_{1 \leqslant j \leqslant n} n(\gamma_i, j)\beta_j$ substitution now proves (iv).

(P15) Suppose α is such that for every $i (l \leqslant i \leqslant m)$ either α is strongly orthogonal to β_i or $m(\alpha, i)$ is odd or $\alpha \neq \pm\beta_i$ then $n(\alpha, i) = m(\alpha, i)$ for each i and hence in this case $s(\alpha) = \sigma_{\beta_1} \cdot \sigma_{\beta_2} \cdot \cdots \sigma_{\beta_n} \cdot (\alpha)$, σ_{β_i} denoting the reflection corresponding to β_i. We now define a map $f \colon P_{m_n} \cup P^{\#} \to P^*_{nc+}$ as follows:

(d1) For $\alpha \in P_{m,n}$ and α strongly orthogonal to S let $f(\alpha) = s(\alpha) (= \alpha)$.

(d2) For $\alpha \in P_{m,n}$ and α weakly orthogonal to S let β be the unique element of S which is not strongly orthogonal to α and let $f(\alpha) = s(\alpha) - 2\beta (= \alpha - \beta)$.

(d3) For $\alpha \in P^{\#}$ and α complex let $f(\alpha) = s(\alpha)$.

(d4) For $\alpha \in P^{\#}$, α compact let i be the least integer such that α is unrelated to β_i and set

$$\begin{aligned} f(\alpha) &= s(\alpha) \text{ if } s(\alpha) \text{ is noncompact} \\ &= s(\alpha) + n(\alpha, i)\beta_i \text{ if } s(\alpha) \text{ is compact.} \end{aligned}$$

One checks easily that in all the cases $f(\alpha)$ defined above is an element of $P^*_{nc,+}$. Let $\alpha \in P_{m,n} \cup P^{\#}$. Note the following properties of $f(\alpha)$.

(Pi) $\overline{f(\alpha)} = \bar{\alpha}$.

(Pii) $f(\alpha)$ is complex if and only if α is complex.

(Piii) $f(\alpha)$ is strongly orthogonal to S if and only if α is and in this case $\alpha \in P_{m,n}$.

(Piv) If $\alpha \in P_{m,n}$ then $f(\alpha)$ is related to every $\beta \in S$.

(Pv) If $\alpha \in P^{\#}$ then $f(\alpha)$ is related to β in S if and only if α is related to β. In particular $f(\alpha)$ is unrelated to some β in S and if i is the smallest integer such that α is unrelated to β_i then i is the least such that $f(\alpha)$ is unrelated to β_i

(Pvi) $f(\alpha)$ is unrelated to atleast one β in S if and only if $\alpha \in P^{\#}$.

(Pi), (Pii), (Piii), (Piv) are obvious. (Pv) follows from (P11) and (P14). (Pvi) follows from (Piv) and (Pv). We now prove that the map f defined above is a bijection. First we prove the surjectivity of f.

Let $\alpha \in P^*_{nc,+}$.

Case I. If α is complex then $\alpha \in P^\#$ and $f(\alpha) = \alpha$.

Case II. If α is noncompact we have several subcases:

(1) α strongly orthogonal to S then $\alpha \in P_{m,n}$ and $f(\alpha) = \alpha$.
(2) α not strongly orthogonal to S but related to every member of S.

Assume w.l.g that $m(\alpha, \beta_i) = -2$ for $1 \leqslant i \leqslant l$ and α SO to β_i for $l < i \leqslant m$. Let $\gamma = \alpha + \Sigma_{1 \leqslant i \leqslant l} \beta_i$. Then γ is weakly orthogonal to $\beta_i (1 \leqslant i \leqslant l)$ and strongly orthogonal to $\beta_i (l < i \leqslant m)$ by Proposition 3.1, we get $l = 1$. Set $\beta = \beta_1$.

Thus in this case there exists a unique β in S such that $\alpha(\bar{H}_\beta) = -2$ and then $\gamma = \alpha + \beta$ is compact. Hence $\gamma \in P_{m,n}$. Now by definition (d2) $f(\gamma) = \gamma - \beta = \alpha$.

Case III. α not strongly orthogonal to S and also unrelated to some member of S.

In this case let i be the least such that α is unrelated to β_i. Then by (P11) it follows that i is the least such that $s(\alpha)$ is unrelated to β_i. We have two subcases of this case:

(i) $s(\alpha)$ *compact*. In this case, $s(\alpha) \in P^\#$ and $s(s(\alpha)) = \alpha$ which is noncompact hence by definition (d4) $f(s(\alpha)) = \alpha$.
(ii) $s(\alpha)$ *noncompact*. In this case let $\gamma_i = s(\alpha) + n(\alpha, i)\beta_i$. By (P13) $n(\alpha, i)$ is odd (Hence γ_i is compact) and by (P14) γ_i is unrelated to β_i hence $\gamma_i \in P^\#$ by (P14 (iii)). i is the least such that γ_i is unrelated to β_i. Hence by definition (d4), $f(\gamma_i) = s(\gamma_i) + n(\gamma_i, i)\beta_i$. Now by (P14) (iv) $s(\gamma_i) = \alpha - n(\alpha, i)\beta_i$ and by (P14) (ii), $n(\gamma, i) = n(\alpha, i)$. It follows that $f(\gamma_i) = \alpha$.

Thus we have proved that for $\alpha \in P^*_{nc,+}$, there exist γ in $P_{m,n} \cup P^\#$ such that $f(\gamma) = \alpha$. This proves the surjectivity of f.

We now prove the injectivity of f.

Suppose $\alpha, \gamma \in P_{m,n} \cup P^\#$ are such that $f(\alpha) = f(\gamma)$. To prove that $\alpha = \gamma$.

Case I. $\alpha \in P_{m,n}$. In this case $f(\alpha)$ is related to every β in S (see Piv). Thus $f(\gamma)$ is related to every β in S. Hence $\gamma \in P_{m,n}$ (by Pvi). We now have the following subcases.

Case I(i). *α strongly orthogonal to S*. In this case $(f(\gamma) =) f(\alpha) = \alpha$. Thus $f(\gamma)$ is strongly orthogonal to S. Hence by (Piii) we conclude that γ is strongly orthogonal to S and that $f(\gamma) = \gamma$. Thus $\gamma = f(\gamma) = f(\alpha) = \alpha$.

Case I(ii). *α weakly orthogonal to S*. In this case there exists a unique β in S such that α is strongly orthogonal to all members of $S \backslash \{\beta\}$, α is weakly orthogonal to β. In this case $f(\alpha) = \alpha - \beta$ (by d_2).

Now $\gamma \in P_{m,n}$ and γ can not be strongly orthogonal to S (otherwise interchanging roles of γ and α in case I(i) and we would get $\alpha = \gamma$ which would contradict α is a weakly orthogonal to S) hence γ is also weakly orthogonal to S. Let δ be the unique element of S which is not strongly orthogonal to γ. Then $f(\gamma) = \gamma - \delta$ (by d2). Thus $\alpha - \beta = \gamma - \delta$. If

β were not same as δ then $\gamma - \delta$ would be strongly orthogonal to β(as γ is so) i.e. $\alpha - \beta$ would be strongly orthogonal to β which is not correct hence $\beta = \delta$. Therefore $\alpha = \gamma$.

Case II. $\alpha \in P^{\#}$. Then $\gamma \in P^{\#}$ (otherwise by case I, α will be in $P_{m,n}$ and we would get $\gamma = \alpha$). We consider all the possible cases.

Case II(i). *$s(\alpha)$ noncompact, $s(\gamma)$ noncompact.* In this case $f(\alpha) = s(\alpha)$ and $f(\gamma)$ (by d3). Hence we get $s(\alpha) = s(\gamma)$. Therefore by P9 we get $\alpha = \gamma$.

Case II(ii). *$s(\alpha)$ compact.* In this case, let i be the least such that α is unrelated to β_i. Then by (d4) $f(\alpha) = s(\alpha) + n(\alpha, i)\beta_i$. Therefore by (P14(iv)) we get

$$s(f(\alpha)) = \alpha - n(\alpha, i)\beta_i. \tag{$*$}$$

Now let if possible $s(\gamma)$ be noncompact. Then by definition (d3), $f(\gamma) = s(\gamma)$. Therefore by (P9) we get $\gamma = s(f(\gamma))$. Now using $f(\gamma) = f(\alpha)$ along with ($*$) we get $\gamma = \alpha - n(\alpha, i)\beta_i$. Now by (P13) $n(\alpha, i)$ is odd hence $\alpha - n(\alpha, i)\beta_i$ is noncompact (α being compact, β_i being noncompact). Thus γ is noncompact and this contradicts $\gamma \in P^{\#}$. Therefore $s(\gamma)$ must be compact. Now let j be the least such that γ is unrelated to β_j. Then by (d4) we have $f(\gamma) = s(\gamma) + n(\gamma, j)\beta_j$. Therefore

$$s(f(\gamma)) = \gamma - n(\gamma, j)\beta_j \text{ (by P14 (iv)).} \tag{$**$}$$

Now ($*$) and ($**$) alongwith $f(\gamma) = f(\alpha)$ gives

$$\gamma - n(\gamma, j)\beta_j = \alpha - n(\alpha, i)\beta_i. \tag{$***$}$$

Now by (Piv), i is the least such that $f(\alpha)$ is unrelated with β_i and j is the least such that $f(\gamma)$ is unrelated with β_i. Therefore $f(\gamma) = f(\alpha)$ gives $i = j$. Now by P14(ii), $n(f(\alpha), i) = n(\alpha, i)$ and $n(f(\gamma), j) = n(\gamma, j)$. But $f(\alpha) = f(\gamma)$ and $i = j$. Therefore we get $n(\alpha, i) = n(\gamma, j)$. This, in conjuction with ($***$) now gives $\gamma = \alpha$.

This proves that $\gamma = \alpha$ in every case and therefore it follows that f is injective completing the proof of the Lemma.

DEFINITION 3.10

(a) For $w \neq 0$ in t^* let $\mathfrak{p}[[w]]$ be defined by

$$\mathfrak{p}[[w]] = \{X \in \mathfrak{p} / [H, X] = w(H)X \,\forall\, H \in t\}.$$

(b) Let P_n^S denote the family of those nonzero w in t^* with the properties that (i) $\mathfrak{p}[[w]] \neq \{0\}$ and (ii) $w = \bar{\alpha}$ for some α in each P, w being counted in P_n^S $\dim \mathfrak{p}[[w]]$ number of times.

Remark 3.11. P_n^ϕ is usually written as P_n. It then follows that P_n^S is precisely the family of nonzero $\bar{\alpha} = \alpha|_t$ with $\alpha \in P_n$.

PROPOSITION 3.12

$\alpha \mapsto \bar{\alpha}$ is a bijection from $P^*_{\mathrm{nc},+}$ onto the family P_n^S.

Proof. Let $w \in t^*$. Then

$$\mathfrak{p} = h \cap \mathfrak{p} + \sum_{\alpha \text{ non-compact}} \mathbb{C}X_\alpha + \sum_{\alpha \text{ complex} \alpha(H_*) > 0} \mathbb{C}(X_\alpha + \theta X_\alpha).$$

It follows that $w \in P_n^S$ if and only if $w = \bar{\alpha}$ for some $\alpha \in P$ and then

$$\mathfrak{p}[[w]] = \sum_{\alpha \in P, \alpha \text{ non-compact}, \alpha = w} \mathbb{C}X_\alpha + \sum_{\alpha \in P, \alpha \text{ complex } \bar{\alpha} = w, \alpha(H_*) > 0} \mathbb{C}(X_\alpha + \theta X_\alpha)$$

$$= \sum_{\alpha \in P^*_{nc} \text{ and } \bar{\alpha} = w} E_\alpha.$$

From this the proposition follows immediately.

We now collect various assumptions and notation and prove the main result.

Let $\mathfrak{g}_0$ be a real semi-simple Lie algebra, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ its Cartan decomposition, θ corresponding to Cartan involution, $\mathfrak{b}_0$, the Cartan subalgebra of $\mathfrak{k}_0, \mathfrak{h}_0$ the centralizer of $\mathfrak{b}_0$ in $\mathfrak{g}_0$, S the strongly orthogonal family of noncompact roots in $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, C_S Cayley transform of $S, j_0 = \mathfrak{t}_0 \oplus a_0$, a θ invariant Cartan subalgebra of $\mathfrak{g}_0$ built by C_S so that $\mathfrak{t}_0 = \{H \in \mathfrak{b}_0/\beta(H) = o \,\forall \beta \text{ in } S\}$, $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ some parabolic subalgebra of $\mathfrak{g}_0$ with $\mathfrak{a}_0$ as its split part. Let P_m be a positive for $\Delta(\mathfrak{m}, \mathfrak{t})$. Let P be an S-admissible and $P_{\mathfrak{m}}$-compatible positive system for Δ set. $P_{\mathfrak{t}} = \{\alpha \in \Delta(\mathfrak{k}, \mathfrak{b})/\alpha = \beta | b \text{ for some } \beta \in P\}$, $P_{\mathfrak{m},\mathfrak{t}} = \{\alpha \in P_m/\alpha - \mathfrak{m} \text{ compact}\}$, $P_c = \{\alpha \in P_k | \alpha|_{\mathfrak{t}} \equiv 0\}$. $\mathfrak{u} =$ ortho-complement of $\mathfrak{t}$ in $\mathfrak{s}$ with respect to killing form of $\mathfrak{k}, \mathfrak{d} = \mathfrak{u} + \Sigma_{\alpha \in \pm P_c} \mathfrak{k}_\alpha$, $P_{\mathfrak{d}} = \{\alpha | \mathfrak{u}/\alpha \in P_{\mathfrak{t}}; \alpha|_{\mathfrak{t}} \equiv 0\}$. Let $\rho, \rho_{\mathfrak{m},c}, \rho_c, \rho_{\mathfrak{d}}$ denote half the sum of roots in $P, P_{\mathfrak{m},\mathfrak{t}}, P_{\mathfrak{t}}$ and $P_{\mathfrak{d}}$ respectively. Let P_n^S be as in Definition 3.10. For η in $\mathfrak{b}^*$, let $\bar{\eta} = \eta|_{\mathfrak{t}}$ and $\tilde{\eta} = \eta|_{\mathfrak{u}}$. $\Delta_{\mathfrak{m},\mathfrak{t}} = \Delta(\mathfrak{m} \cap \mathfrak{k}, \mathfrak{t})$ and $W_{\mathfrak{t}} = W(\mathfrak{k}, \mathfrak{b})$. Then we have the following theorem.

Theorem 3.13. *Let λ in $\mathfrak{t}^*$ be $\Lambda_{\mathfrak{m},\mathfrak{t}}$-integral. Suppose $(\lambda - \rho_{\mathfrak{m},c})(\overline{H_\alpha})$ is not a positive integer for all α in $P_{\mathfrak{m},\mathfrak{t}}$. Let ν in $\mathfrak{a}^*$ and μ in $\mathfrak{b}^*$ be $P_{\mathfrak{t}}$-dominant integral. Let $m(\lambda, \nu, \mu)$ denote the multiplicity with which finite dimensional irreducible k module with highest weight μ occurs in* $\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(W_{-P_m,\lambda} \otimes \nu \otimes 0)$. Then

(a)

$$m(\lambda, \nu, \mu) =$$

$$\sum_{w \in \mathfrak{b}^*} \sum_{\sigma \in W_{\mathfrak{t}}} \varepsilon(\sigma) \mathbb{P}_{P_n^S}(\overline{\sigma(\mu + \rho_c)} - (\lambda + \bar{\rho}_c - 2\rho_{m,c})) \mathbb{P}_{P_c}(\sigma(\mu + \rho_c) + w).$$

(a′) *In particular, when S is simple*

$$m(\lambda, \nu, \mu) = \sum_{\sigma \in W_{\mathfrak{t}}} \varepsilon(\sigma) \mathbb{P}_{P_n^S}(\overline{\sigma(\mu + \rho_c)} - \lambda + \bar{\rho}_c - 2\rho_{m,c})).$$

(b) *If μ is such that $\bar{\mu} = \lambda - 2\rho_{\mathfrak{m},n}$ then $m(\lambda, \nu, \mu) =$ degree of $F^{\tilde{\mu}}_{\mathfrak{d},P_{\mathfrak{d}}}$ the finite dimensional irreducible $\mathfrak{d}$-module with $P_{\mathfrak{d}}$-highest weight $\tilde{\mu}$. In particular if S is simple then in this case $m(\lambda, \nu, \mu) = 1$.*

Proof. Let L denote the set of all $P_{\mathfrak{m},\mathfrak{t}}$-dominant integral elements in $\mathfrak{t}^*$. For w in L, set

$$m(\mu, w) = \dim \operatorname{Hom}_{\mathfrak{m} \cap \mathfrak{t}}(F^w_{\mathfrak{m} \cap \mathfrak{k}, P_{\mathfrak{m},\mathfrak{t}}}, F^\mu_{\mathfrak{k}, P_{\mathfrak{t}}})$$

and

$$m[\lambda, \nu, \mu] = \dim \operatorname{Hom}_{\mathfrak{m} \cup \mathfrak{t}}(F^w_{\mathfrak{m} \cap \mathfrak{k}, P_{\mathfrak{m},\mathfrak{t}}}, W_{-P_m,\lambda} \otimes \nu \otimes 0).$$

The Lemma 3.6 and Proposition 3.7 give

$$m(\lambda, v, \mu) = \sum_{w \in L} m(\mu, w) m[\lambda, v, w] .$$

$$= \sum_{w \in L} \sum_{\tau \in W_{\mathfrak{m},t}} \varepsilon(\tau) m(\mu, w) \mathbb{P}_{P_{\mathfrak{m},n}}(\tau(w + \rho_{\mathfrak{m},c}) - (\lambda - \rho_{\mathfrak{m},c})). \tag{3.13.1}$$

Using Theorem 2.3 (b) and (3.13.1) we get

$$m(\lambda, v, \mu) = \sum_{w} \sum_{\tau} \sum_{\gamma} \sum_{\sigma} \varepsilon(\sigma) \mathbb{P}_T(\overline{\sigma(\mu + \rho_c)} - (\overline{\tau(w + \rho_{m,c})} - \rho_{m,c} + \overline{\rho_c})$$

$$\times \mathbb{P}_{P_c}(\sigma(\mu + \rho_c) + \gamma) \mathbb{P}_{P_{\mathfrak{m},n}}(\tau(w + \rho_{\mathfrak{m},c}) - (\lambda - \rho_{\mathfrak{m},c})), \tag{3.13.2}$$

where Σ_w, Σ_τ, Σ_γ and Σ_σ denote summations over w in L, τ in $W_{\mathfrak{m},k}$, γ in b^* and σ in $W_{\mathfrak{k}}$ respectively and T is the set of positive weights of the canonical $\mathfrak{m} \cap \mathfrak{k}$ module $\mathfrak{k} | \mathfrak{m} \cap \mathfrak{k}$. Now $\tau(w + \rho_{\mathfrak{m},c})$ with w in L and τ in $W_{\mathfrak{m},k}$ exhaust all $W_{\mathfrak{m},t}$ regular elements of $\mathfrak{t}^*$. Therefore it follows from Theorem 2.3 (c) and (3.13.2) that

$$m(\lambda, v, \mu) = \sum_{\eta} \sum_{\gamma} \sum_{\sigma} \varepsilon(\sigma) \mathbb{P}_T((\overline{\sigma(\mu + \rho_c)} - \overline{(\eta - \rho_{\mathfrak{m},c})} + \overline{\rho_c})$$

$$\times \mathbb{P}_{P_{\mathfrak{m},n}}(\eta - (\lambda - \rho_{\mathfrak{m},c})) \mathbb{P}_{P_c}(\sigma(\mu + \rho_c) + \gamma)$$

$$= \sum_{\gamma \in b^*} \sum_{\sigma \in W_t} \varepsilon(\sigma) \mathbb{P}_{T \cup P_{\mathfrak{m},n}}(\overline{\sigma(\mu + \rho_c)} - (\lambda + \overline{\rho_c - 2\rho_{\mathfrak{m},c}}))$$

$$\times \mathbb{P}_{P_c}(\sigma(\mu + \rho_c) + \gamma), \tag{3.13.3}$$

where Σ_η, Σ_γ and Σ_σ denote summations over η in t^*, $\gamma \in b^*$ and σ in $W_{\mathfrak{m},t}$. But, $\mathbb{P}_{T \cup P_{\mathfrak{m},n}}(\eta) = \mathbb{P}_{P_n^s}(\eta)$ for every $\eta \in \mathfrak{t}^*$ by Lemma 3.9 and Propositions 3.12. Part (a) now follows from (3.13.3) and part (b) follows immediately from (a) by arguments same as those used in proof of Corollary 2.4 (a).

We now work on group level.

Let Q be the parabolic subgroup of G with $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ as its Lie algebra and let $Q = MAN$ be its Langlands decomposition. Let Q^0 (respectively M^0) be the connected component of 1 in Q (respectively M). Then Q^0 (respectively M^0 is the connected subgroup of G with Lie algebra q_0 (respectively m_0) and $Q^0 = M^0AN$. Let $T = \exp t_0$. Then T is a Cartan subgroup of M^0. Let Λ denote the lattice of the differentials of characters of T. Let λ in t^* be P_m-dominant. Suppose $\lambda \in \Lambda + \rho_{\mathfrak{m}}$ and $\lambda(H_\alpha) \neq 0$ for each $P_{\mathfrak{m}}$-simple root α in $\Delta_{\mathfrak{m},t}$. According to Harish Chandra [8, 9] and Schmiod [13] there exists a unique invariant eigen distribution $\theta_{P_{\mathfrak{m}},\lambda}$ called as 'discrete' or Mock discrete series character of M^0 as described in ch. 15 and 16 of [5].

Then by Theorem 16.2 of [5] there exists a nonzero irreducible unitary representation $(\sigma_{P_{\mathfrak{m}},\lambda}, H_{P_{\mathfrak{m}},\lambda})$ of M^0 which has character $\theta_{P_{\mathfrak{m}},\lambda}$ and is infinitesimally equivalent to $W_{-P_{\mathfrak{m}},\lambda + \rho_{\mathfrak{m}}}$. Let as usual $v \in a^*$. Let $\mathrm{ind}_{Q^0}^G(\sigma_{P_{\mathfrak{m}},\lambda} \otimes \exp v \otimes 1)$ be the representation of G induced by the representation $(\sigma_{P_{\mathfrak{m}},\lambda} \otimes \exp v \otimes 1)$ of $M^0AN = Q^0$ to G in the usual way; G operating by left translate. Set

$$\sigma_{P_{\mathfrak{m}},\lambda,v} = \mathrm{ind}_{Q^0}^G(\sigma_{P_{\mathfrak{m}},\lambda} \otimes \exp v \otimes 1).$$

Call $\sigma_{P_{\mathfrak{m}},\lambda,\nu}$ the generalized principal series representation of G with parameters $P_{\mathfrak{m}}, \lambda, \nu$. Then $\sigma_{P_{\mathfrak{m}},\lambda,\nu}$ is infinitesimally equivalent with $\mathrm{ind}_{q^0}^{g}(W_{-P_{\mathfrak{m}},\lambda} \otimes \nu \otimes 0)$.

Theorem 3.14. *Let λ in $\mathfrak{t}^*$ be $P_{\mathfrak{m}}$ dominant. Suppose $\lambda \in \Lambda + \rho_{\mathfrak{m}}$ and also $\overline{\lambda(H_\alpha)} \neq 0$ for every P-simple α in $P_{\mathfrak{m},\mathfrak{t}}$. Let μ in $\mathfrak{b}^*$ be $P_{\mathfrak{t}}$ dominant integral. Let $M(\lambda,\nu,\mu)$ be the multiplicity with which $F^{\mu}_{\mathfrak{t},P_{\mathfrak{t}}}$ appears in $\sigma_{P_{\mathfrak{m}},\lambda,\nu}$. Then*

(a) $\mathrm{M}(\lambda,\nu,\mu) = \sum\limits_{w\in\mathfrak{b}^*} \sum\limits_{\sigma\in W_{\mathfrak{t}}} \varepsilon(\sigma)\mathbb{P}_{P_n^s}(\overline{\sigma(\mu+\rho_c)} - (\lambda + \overline{\rho - 2\rho_c}))\mathbb{P}_{P_c}(\sigma(\mu+\rho_c)+w)$.

(a′) *In particular, if S is simple then*

$$M(\lambda,\nu,\mu) = \sum_{\sigma\in W_{\mathfrak{t}}} \varepsilon(\sigma)\mathbb{P}_{P_n^s}(\overline{\sigma(\mu+\rho_c)} - (\lambda + \overline{\rho - 2\rho_c})).$$

(b) *If $\bar{\mu} = \lambda + \bar{\rho} - 2\bar{\rho}_c$ then*

$$M(\lambda,\nu,\mu) = \text{degree } F^{\bar{\mu}}_{\mathfrak{b},P_{\mathfrak{b}}}.$$

In particular in this case, if further S is simple then

$$M(\lambda,\nu,\mu) = 1.$$

Proof. We claim

$$\bar{\rho} - 2\bar{\rho}_c = \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m},c}. \tag{$*$}$$

($*$) follows immediately from Proposition 15.2 in [11]. We may also argue as follows: If S is empty, then $\Delta(\mathfrak{m},\mathfrak{t})$ consists of all imaginary roots in Δ. $\Delta(\mathfrak{m}\cap\mathfrak{k},t)$ consists of all compact imaginary roots in Δ and if α in P then $\theta\alpha$ is also in P. From this the ($*$) follows easily in this case.

If S has just one element ($*$) follows from (7.21) in [13] easily. In general case, we can prove ($*$) using (7.21) of [13] and induction on number of elements in S. We omit the easy details. Theorem 3.14 now follows from Theorem 3.13 and the equation ($*$) when we note that $\sigma_{P_{\mathfrak{m}},\lambda,\nu}$ is infinitesimally equivalent with $\mathrm{ind}_q^g(W_{-P_{\mathfrak{m}},\lambda} \otimes \nu \otimes 0)$. This completes the proof.

Remark. Theorem 3.13(a′) and (3.14(a′) have been proved in the fundamental case (i.e. when $S = \phi$) by Enright and Wallach in [7]. These results in the case when S is singleton are proved in [1].

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(a) $\mathrm{M}(\lambda,\nu,\mu)=\sum_{w\in\mathfrak{b}^*}\sum_{\sigma\in W_{\mathfrak{t}}}\varepsilon(\sigma)\mathbb{P}_{P_n^s}(\overline{\sigma(\mu+\rho_c)}-(\lambda+\overline{\rho-2\rho_c}))\mathbb{P}_{P_c}(\sigma(\mu+\rho_c)+w)$.

(a′) *In particular, if S is simple then*

$$M(\lambda,\nu,\mu)=\sum_{\sigma\in W_{\mathfrak{t}}}\varepsilon(\sigma)\mathbb{P}_{P_n^s}(\overline{\sigma(\mu+\rho_c)}-(\lambda+\overline{\rho-2\rho_c})).$$

(b) *If $\bar{\mu}=\lambda+\bar{\rho}-2\bar{\rho}_c$ then*

$$M(\lambda,\nu,\mu)=\text{degree } F^{\bar{\mu}}_{\mathfrak{b},P_{\mathfrak{b}}}.$$

In particular in this case, if further S is simple then

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Generalized parabolic bundles and applications—II

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Abstract. We prove the existence of the moduli space $M(n,d)$ of semistable generalised parabolic bundles (GPBs) of rank n, degree d of certain general type on a smooth curve. We study interesting cases of the moduli spaces $M(n,d)$ and find explicit geometric descriptions for them in low ranks and genera. We define tensor products, symmetric powers etc. and the determinant of a GPB. We also define fixed determinant subvarieties $M_L(n,d)$, L being a GPB of rank 1. We apply these results to study of moduli spaces of torsionfree sheaves on a reduced irreducible curve Y with nodes and ordinary cusps as singularities. We also study relations among these moduli spaces (rank 2) as polarization varies over $[0,1]$.

Keywords. Generalized parabolic bundles; nodal curves; torsionfree sheaves.

1. Introduction

This work is a generalization and continuation of our work in [B3] where (and in [B5]) we introduced the notion of generalized parabolic bundles (GPBs). They are vector bundles with flags (of vector spaces) over effective divisors. In [B3] we studied the special case of flags of length 2 and we consider here flags of sufficiently general type. In § 2 we study the generalities on semistable and stable GPBs and their properties. We prove the existence of the moduli space $M(n,d)$ of semistable GPBs of rank n, degree d of certain general type. In § 3, we study interesting cases of the moduli spaces $M(n,d)$ with flags of length 2. We define 1-stability and 1-semistability, compare these notions with the stability and semistability of GPBs and use them to prove the existence of fine moduli spaces for GPBs. We consider the question of defining tensor products, symmetric powers etc. and the determinant of a GPB. We also define fixed determinant subvarieties $M_L(n,d)$, L being a GPB of rank 1. We describe $M_L(2,d)$ explicitly when X is an elliptic curve. There is interesting geometry associated to this (Remark 4.8). Section 4 is the application of these results to the study of moduli spaces of torsion-free sheaves on a reduced irreducible curve Y with nodes and ordinary cusps as singularities. The case of a single node was considered in [B3]. Let X be the desingularization of Y, $p: X \to Y$ being the natural map. We give a correspondence between GPBs on X and torsionfree sheaves on Y. Unlike the correspondence given by Seshadri (Theorem 17, p. 178 [S]), this correspondence is not bijective, but it preserves rank and degree. Also it maps 1-stable (1-semistable) QPBs to stable (semistable) torsionfree sheaves and vice versa. We also consider relations among various moduli spaces (rank 2) as the weight α varies over $[0,1]$ (see 4.9). These relations are similar to those obtained for stable pairs by Bradlow, Thaddeus, Garcia-Prada and others.

Finally we introduce orthogonal GPBs and study their relation to orthogonal sheaves on Y (5·10, 5·11). We postpone the general case (principal G-bundles) to a future paper. On the one hand GPBs are generalizations of parabolic bundles ([SM], [B2]). On the other hand they generalize the presentation functor giving normalizations of

compactified Jacobians studied in detail by Seshadri, Oda, Kleiman, Altman and others. GPBs are associated to representations of the group $\pi_1(X) * \mathscr{Z} * \cdots * \mathscr{Z}$, where $\pi_1(X)$ is the fundamental group of X and $*$ denotes the free product of groups (Theorems 1, 2 [B4]). GPBs have many applications. They have been used crucially for proving factorization rules of generalized theta functions [RN] and for proving Frobenius splitting for moduli varieties of vector bundles on ordinary curves [MR].

2. Generalized parabolic bundles

2A *Generalities on GPBs*

Let X be an irreducible nonsingular algebraic curve defined over an algebraically closed base field k. Let D be an effective divisor on X. Let E be a vector bundle of rank n and degree d on X.

DEFINITION 2.1

A quasi parabolic structure on E over the divisor D is a flag $\mathscr{F}$ of vector subspaces of $H^0(E \otimes \mathscr{O}_D)$ given by $\mathscr{F}: F_0(E) = H^0(E \otimes \mathscr{O}_D) \supset F_1(E) \supset \cdots \supset F_r(E) = 0$.

DEFINITION 2.2

A quasiparabolic bundle (QPB in short) is a vector bundle E together with quasi-parabolic structures $\mathscr{F}^j$ over finitely many disjoint divisors D_j, $j = 1, \ldots, J$. Let $\underline{\mathscr{F}} = (\mathscr{F}^1, \ldots, \mathscr{F}^J)$. Then a QPB is a pair $(E, \underline{\mathscr{F}})$.

DEFINITION 2.3

An isomorphism of QPBs $(E, \underline{\mathscr{F}})$ and $(E', \underline{\mathscr{F}}')$ is an isomorphism $f: E \to E'$ which maps the flag $\mathscr{F}_j$ to the flag $\mathscr{F}'_j$ for all j.

DEFINITION 2.4

A (generalized) parabolic structure on a vector bundle E over an effective divisor D consists of

(1) a quasiparabolic structure on E over D
(2) real numbers $\alpha_1, \ldots, \alpha_r$ with $0 \leqslant \alpha_1 < \cdots < \alpha_r < 1$ called weights associated to the flag.

DEFINITION 2.5

Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, $m_i = \dim F_{i-1}(E)/F_i(E)$, $i = 1, \ldots, r$. Define $wt_D E = \sum_{i=1}^r m_i \alpha_i$. If we consider parabolic structures over divisors $D_1, \ldots, D_J$, we define $wtE = \sum_J wt_{D_j} E$. Define pardeg $E =$ degree $E + wtE$ and par $\mu(E) =$ pardeg E/rank E. These are called respectively the parabolic degree and the parabolic slope of E.

DEFINITION 2.6

A generalized parabolic bundle (abbreviated as GPB) is a vector bundle E together with parabolic structures over finitely many disjoint divisors. We denote it by a triple $(E, \underline{\mathscr{F}}, \underline{\alpha})$. Here $\underline{\alpha} = (\underline{\alpha}^1, \ldots, \underline{\alpha}^J)$, $\underline{\alpha}^j = (\alpha_1^j, \ldots, \alpha_{rj}^j)$, $j = 1, \ldots, J$.

DEFINITION 2.7

Every subbundle K of E gets a natural structure of a GPB (see 3.2 [B3]). By a subbundle of a GPB, we will mean a subbundle with this induced parabolic structure. A GPB $(E,\mathscr{F},\underline{\alpha})$ is semistable (respectively stable) if for every (respectively proper) subbundle K of E, one has par $\mu(K)\leqslant$ (resp. $<$) par $\mu(E)$.

DEFINITION 2.8

Let $(E,\mathscr{F},\underline{\alpha})$ be a GPB. Let $\alpha^{r_{j+1}}=1$, $\alpha_0^j=\alpha_{r_j}^j-1$. For real number α with $0\leqslant\alpha<1$, define $E_\alpha^j=F_{i-1}^j(E)$ if $\alpha_{i-1}^j<\alpha\leqslant\alpha_i^j$, $i=1,\dots,r_j+1$.

A morphism of GPBs is a homomorphism $f:E\to M$ of underlying vector bundles such that $f(E_\alpha^j)\subseteq M_\alpha^j$ for $0\leqslant\alpha<1$ and $j=1,\dots,J$.

PROPOSITION 2.9

(1) *If $h:I\to K$ is a generic isomorphism of GPBs which is not an isomorphism, then* par $\mu(I)<$ par $\mu(K)$.
(2) *If $f:E_1\to E_2$ is a morphism of semistable GPBs of the same rank and same parabolic degree (the divisors D_j being fixed for all j), then f is of constant rank.*
(3) *If, in addition, one of E_1,E_2 is a stable GPB then f is either zero or an isomorphism.*

Proof. Since (2) and (3) follow from (1) by standard arguments, we only prove (1). For $1\leqslant j\leqslant J$, let N_j denote the kernel of $h|D_j:I|D_j\to K|D_j$. One has $\deg K\geqslant\deg I+\sum_j h^0(N_j)$, because h is a generic isomorphism, with equality if and only if h is an isomorphism away from the union of D_j. Define

$$wtN_j=\sum_{i=0}^{r_j-1}\alpha_i^j(\dim(H^0(N_j)\cap F_i^j(I)/H^0(N_j)\cap F_{i+1}^j(I)).$$

Since $\alpha_i^j<1$ for all i,j one has

$$wtN_j\leqslant\sum_i(\dim H^0(N_j)\cap F_i^j(I)-\dim H^0(N_j)\cap F_{i+1}^j I))=h^0(N_j).$$

Thus $wtN\equiv\sum_j wtN_j\leqslant\sum_j h^0(N_j)$ with equality if and only if all $N_j=0$. Since h is a morphism of GPBs and $h|D_j$ induces an injection $(I|D_j)/N_j\to K|D_j$ it follows that $wtI-wtN\leqslant wtK$. Thus, par $\deg I=\deg I+wtI\leqslant\deg\ I+wtN+wtK\leqslant\deg I+\sum_j h^0(N_j)+wtK\leqslant$ par deg K. The last two inequalities cannot be equalities unless h is an isomorphism. Since I and K have the same rank, (1) follows.

PROPOSITION 2.10

Let $\mathscr{C}$ denote the category of semistable GPBs $(E,\mathscr{F},\underline{\alpha})$ on an irreducible non-singular curve X with parabolic structures over fixed divisors $D_1,\dots,D_J$ on X and with fixed par $\mu=m$. *Then $\mathscr{C}$ is an abelian category whose simple objects are stable GPBs.*

By the Jordan–Holder theorem, for any $(E,\mathscr{F},\underline{\alpha})$ in $\mathscr{C}$, there exists a filtration of $(E,\mathscr{F},\underline{\alpha})$ in $\mathscr{C}$ with successive quotients stable GPBs with par $\mu=m$. The associated graded object for this filtration is unique up to isomorphism. Denote this object by $\mathrm{gr}(E,\mathscr{F},\underline{\alpha})$.

DEFINITION 2.11

We define an equivalence relation in $\mathscr{C}$ by $(E,\underline{\mathscr{F}},\underline{\alpha})$ and $(E',\underline{\mathscr{F}}',\underline{\alpha}')$ are equivalent if $\mathrm{gr}(E,\underline{\mathscr{F}},\underline{\alpha})$ and $\mathrm{gr}(E',\underline{\mathscr{F}}',\underline{\alpha}')$ are isomorphic in $\mathscr{C}$.

2B *Existence and properties of the moduli space*

Our aim is to construct a moduli space for equivalence classes of semistable GPBs of a 'fixed type'.

Theorem 1. *Let X be an irreducible nonsingular projective curve of genus $g(g \geqslant 0)$ defined over an algebraically closed field.*

Let $D_1,\ldots,D_J$ be finitely many (fixed) divisors on X such that their supports are mutually disjoint. Consider the set of semistable GPBs $(E,\underline{\mathscr{F}},\underline{\alpha})$ on X of fixed rank n and fixed degree with $\mathscr{F}^j: F^j_0(E) = H^0(E \otimes \mathscr{O}_{D_j}) \supset \cdots \supset F^j_r(E) = 0$ flags of length r (r independent of j) and weights $\underline{\alpha}^j = (\alpha_1,\ldots,\alpha_r)$ fixed independent of j. For each j, we assume the flag type of $\mathscr{F}^j$ fixed (varying with j). Denote this set modulo the equivalence relation (2.11) by M. Then M has the structure of a normal projective variety of dimension $n^2(g-1)+1+\sum_j \dim G_j$ where G_j is the flag variety of flags of type $\mathscr{F}^j$, $j=1,\ldots,J$. The subset of M corresponding to stable GPBs is a nonsingular open subvariety.

Proof. The construction of the moduli space M is done using geometric invariant theory generalizing [B3]. We only sketch the proof as it is very similar to that in [B3]. We first construct a universal space $\tilde{R}$ for GPBs of the above type with an action of $PGL(N)$ on it. Then we show that there exists a good quotient M of $\tilde{R}$ by $PGL(N)$ in the sense of geometric invariant theory. We denote by S the set of all semistable GPBs $(E,\underline{\mathscr{F}},\underline{\alpha})$ of the above type. Without loss of generality, we may assume that $-n \leqslant \deg E < 0$. Let $C_j = \deg D_j, j = 1,\ldots,J$. For $(E,\underline{\mathscr{F}},\underline{\alpha}) \in S$, let $b = \mathrm{par}\deg(E)$. Since the GPB's $(E,\underline{\mathscr{F}},\underline{\alpha}) \in S$ are semistable, there exists m_0 such that for $m \geqslant m_0$, $h^1(E(m)) = 0$ and the canonical map $H^0(E(m)) \to \oplus_{j=1}^J H^0(E(m) \otimes \mathscr{O}_{D_j})$ is surjective. Given an integer b_0, one can choose $m \gg g$ such that for $F \in S$ (i.e. F such that $(F,\underline{\mathscr{F}},\underline{\alpha}) \in S$) or for $F \subset E, E \in S$ and $\mathrm{par}\deg F > b_0$ one has $H^1(F(m)) = 0$ and the canonical map $H^0(F(m)) \to \oplus_j H^0(F(m) \otimes \mathscr{O}_{D_j})$ is surjective. This can be done by arguments similar to those on p. 226 [SM]. We shall choose b_0 suitably later (depending only on n, g, b and $C_j, j = 1,\ldots,J$). Choose $m \gg g$, $m \geqslant m_0$. Let $n = h^0(E(m))$. Let P be the Hilbert polynomial of $E(m)$ in S. Let $Q = \mathrm{Quot}_X(\mathscr{O}_X^N, P)$ be the Hilbert scheme of coherent sheaves on X which are quotients of $\mathscr{O}_X^N$ and have Hilbert polynomial equal to P. There is a universal sheaf U on $Q \times X$. Let R be the subscheme of Q consisting of points q in Q corresponding to sheaves U_q which are vector bundles generically generated by sections and satisfy $H^0(U_q) \approx \mathscr{O}_X^N$ (by the Riemann Roch theorem, $H^1(U_q) = 0$ for $q \in R$). By our choice of m, R contains the subset of Q determined by $E(m)$ with $(E,\underline{\mathscr{F}},\underline{\alpha}) \in S$. It is well known that R is a nonsingular variety. Let $p_1: R \times X \to R$ be the projection. Define $V_j = (p_1)_*(U|R \times D_j)$. Let $G(V_j)$ be the flag bundle over R of the type determined by the parabolic structure over D_j. Let $G(V)$ be the fibre product of $\{G(V_j)\}$, $j = 1,\ldots,J$ over R. We denote the total space of $G(V)$ by $\tilde{R}$. Obviously $\tilde{R}$ has the local universal property for GPBs. It is the universal space for GPBs which we wanted to construct. Let $\tilde{R}^{ss}$ (respectively $\tilde{R}^s$) denote the subset of $\tilde{R}$ corresponding to semistable (resp. stable) GPBs.

The group $PGL(N)$ acts naturally on $\mathscr{O}_X^N$ and hence on $R, \tilde{R}^s, \tilde{R}^{ss}$. We shall construct a projective variety Y with $PGL(N)$-action such that a good quotient of Y modulo

$PGL(N)$ exists. We shall give an affine injective morphism from $\tilde{R}$ to Y which is $PGL(N)$-equivariant. The existence of a good quotient of $\tilde{R}$ modulo $PGL(N)$ then follows (Proposition 3.12, [N]). Following Gieseker [G2], let us define a 'good pair' (F, φ) to be a flat family $F \to T \times X$ of vector bundles on X such that F_t is generated by its global sections at the generic point of $t \times X$ and $\varphi: \mathcal{O}_X^N \to p_*(F)$ is an isomorphism, p being the projection $T \times X \to T$. Let A denote the Jacobian of X corresponding to line bundles on X of degree equal to deg $E(m)$, $E \in S$. Let M be the Poincaré bundle on $X \times A$ and $g: X \times A \to A$ the projection. Define $Z = \mathbf{P}\,(\mathrm{Hom}(\wedge^n \mathcal{O}_A^N, g_* M)^\vee)$. Given a good pair (F, φ) one defines a morphism $T(F, \varphi): T \to Z$ as follows. Fix $t \in T$. One has a natural map $\psi: \wedge^n H^0(F_t) \to H^0(\wedge^n F_t)$ given by $\psi(s_1 \wedge \cdots \wedge s_n) = s$, $s(x) = s_1(x) \wedge \cdots \wedge s_n(x)$. Also, φ gives a map $\wedge^n \varphi: \wedge^n k^N \to \wedge^n H^0(F_t)$. Define $T(F, \varphi)(t) = \psi \circ \wedge^n \varphi$. It is easy to see that this defines a morphism [G2]. Via the action on $\mathcal{O}_A^N$, $PGL(N)$ acts naturally on Z preserving the fibres over A. Note that the fibre of Z over $L \in A$ is $\mathbf{P}(\mathrm{Hom}(\wedge^n k^N, H^0(X, L))^\vee)$. If we are given a good pair $(\underline{F}, \varphi)$ where $\underline{F} = (F, \underline{\mathscr{F}}, \underline{\alpha})$ is a family of GPBs of fixed type considered in the theorem, parametrized by T, then for every t in T, $F_t | D_j$ has flag

$$\mathscr{F}^j: F_0^j(F_t) = H^0(F_t \otimes \mathcal{O}_{D_j}) \supset F_1^j(F_t) \supset \cdots \supset F_r^j(F_t) = 0.$$

Let $e_j: H^0(F_t) \to H^0(F_t \otimes \mathcal{O}_{D_j})$ be the natural map. Via e_j (by pulling back) the flag $\mathscr{F}^j$ induces a flag on $H^0(F_t)$. Identifying $H^0(F_t)$ with k^N by φ, we get a flag (of fixed type) on k^N:

$$\mathscr{F}^j(k^N): k^N = F_0^j(k^N) \supset F_1^j(k^N) \supset \cdots \supset F_r^j(k^N),$$

with $F_r^j(k^N) = $ kernel of e_j. Let $f_i^j = $ dimension of F_i^j, $i = 0, \ldots, r$, $j = 1, \ldots, J$. Let G_i^j denote the Grassmannian of subspaces of k^N of dimension f_i^j. Let $G = \Pi G_i^j$, $i = 1, \ldots, r-1$, $j = 1, \ldots, J$. Thus $(\underline{F}, \varphi)$ determines a morphism $f: T \to G$. Define a morphism $\tilde{T}(\underline{F}, \varphi): T \to Z \times G$ by $\tilde{T}(\underline{F}, \varphi) = T(F, \varphi) \times f$. Thus we get a morphism $\tilde{T}: \tilde{R} \to Z \times G$. The space Y we wanted to define is the product $Z \times G$ and $\tilde{T}$ is the required $PGL(N)$-equivariant morphism. Let $\delta_0 = b + [n(m + 1 - g - \alpha_r \sum_j C_j)]$, $\delta_i = n(\alpha_{i+1} - \alpha_i)$ for $i = 1, \ldots, r-1$. Let $L_0 = \mathcal{O}_Z(1)$, $L_{ij} = $ generator of Pic G_i^j for all i, j. Let N_1 be an integer such that $N_1 \delta_i$ is an integer for all i. Let $q_Z: Z \times G \to Z$, $q_{ij}: Z \times G \to G_i^j$ be the projections. Define a line bundle L on $Z \times G$ by

$$L = (q_z)^* L^{N_1 \delta_0} \otimes (\otimes_{i=1}^{r-1} \otimes_{j=1}^{J} (q_{ij})^* L_{ij}^{N_1 \delta_i}).$$

On $Y = Z \times G$ we take the linearlization of the $PGL(N)$-action given by L. Let Y^{ss}, (Y^s) denote the set of semistable (stable) points of Y.

PROPOSITION 2.12

(a) $q \in \tilde{R}^{ss} \Rightarrow \tilde{T}(q) \in Y^{ss}$.
(b) $q \in \tilde{R}^{s} \Rightarrow \tilde{T}(q) \in Y^{s}$.
(c) $q \in \tilde{R}$, $\tilde{T}(q) \in Y$, $q \notin \tilde{R}^{ss} \Rightarrow \tilde{T}(q) \notin Y^{ss}$.
(d) $q \in \tilde{R}^{ss} - \tilde{R}^{s} \Rightarrow \tilde{T}(q) \notin Y^{s}$.

Proof. Similar to 3.12 [B3].

PROPOSITION 2.13

The morphism $\tilde{T}$ is a proper injective morphism.

Proof. Similar to 3.13 [B3] or 3 [B1].

Since $\tilde{T}$ is affine and a good quotient of Y^{ss} modulo $PGL(N)$ exists (well known), it follows that a good quotient M of $\tilde{R}^{ss}$ modulo $PGL(N)$ exists (Proposition 3.12 [N]). $\tilde{R}$ is a nonsingular projective variety of dimension $n^2(g-1)+N^2+\sum_j \dim G_j$, G_j being the flag variety of flags of type $\mathscr{F}^j$, $j=1,\dots,J$. Hence M is a normal projective variety of dimension $n^2(g-1)+1+\sum_j \dim G_j$. Also, if $\tilde{R}^{ss}=\tilde{R}^s$ and $\mathrm{Aut}(E,\underline{\mathscr{F}},\underline{\alpha})=k$ (scalars), then M is a geometric quotient and is nonsingular. We end the proof of existence of M with this remark.

PROPOSITION 2.14

Let h denote the canonical morphism from $\tilde{R}^{ss}$ onto the quotient M. Let $\mathscr{E}$ denote the universal family of GPBs on $\tilde{R}^{ss}\times X$. Then for $p,q\in\tilde{R}^{ss}$, one has $h(p)=h(q)$ if and only if $\mathrm{gr}(\mathscr{E}_p)\approx\mathrm{gr}(\mathscr{E}_q)$.

Proof. Similar to 3.15, [B3].

3. Interesting special cases of the moduli spaces of GPBs

3A *α-stability, α-semistability of QPBs*

Notation 3.1. In this section we study QPBs $(E,\underline{\mathscr{F}}),\underline{\mathscr{F}}=(\mathscr{F}^1,\dots,\mathscr{F}^J)$ with $\mathscr{F}^j{:}F_0^j(E)\supset F_1^j(E)\supset 0$. Let $\dim F_1^j(E)=a_j$, $a=\sum_j a_j$, $j=1,\dots,J$. If $J=1$, then we may often denote $(E,\mathscr{F})$ by $(E,F_1(E))$. The results of this section are also needed later for applications.

DEFINITION 3.2

Let α be a real number with $0\leqslant\alpha\leqslant 1$. A QPB$(E,\underline{\mathscr{F}})$ with $\mathscr{F}^j{:}F_0^j(E)\supset F_1^j(E)\supset 0$ is called α-semistable (respectively α-stable) if for any proper subbundle F of E with induced quasiparabolic structure, one has

$$\frac{\deg F+\sum_j\alpha\dim F_1^j(F)}{\operatorname{rank}F}\leqslant(<)\frac{\deg E+\sum_j\alpha\dim F_1^j(E)}{\operatorname{rank}E}.$$

Note that for $0\leqslant\alpha<1$, α-semistability (or α-stability) is the same as semistability (or stability) of the GPB$(E,\underline{\mathscr{F}},\underline{\alpha})$ with $\alpha^j=(0,\alpha)$ for all j.

PROPOSITION 3.3

Let $(E,\underline{\mathscr{F}})$ be a QPB with $\mathscr{F}^j{:}F_0^j(E)\supset F_1^j(E)\supset 0$, $a=\sum_j\dim F_1^j(E)$.

(1) *Suppose that $1-1/[a(n-1)]<\alpha<1$. Then if $(E,\underline{\mathscr{F}})$ is α-semistable, it is also 1-semistable. If $(E,\underline{\mathscr{F}})$ is 1-stable, it is also α-stable.*
(2) *Suppose that rank and degree of E are coprime and a is an integral multiple of rank E. Then $(E,\underline{\mathscr{F}})$ is 1-stable if and only if it is 1-semistable.*
(3) *If the conditions of* (1) *and* (2) *are satisfied then α-stability is equivalent to α-semistability and the moduli space M is nonsingular.*

Proof. (1) Let F be a proper subbundle of E of rank r with induced (quasi)parabolic structure. Define

$$B(F)=n\deg(F)-r\deg(E),\ A(F)=ra-n\textstyle\sum_j\dim F_1^j(F).$$

The condition for α-stability (α-semistability) can then be written as $B(F) < (\leqslant) \alpha A(F)$, for all subbundles F of E. Define $\delta = 1 - \alpha$. If $A(F) \geqslant 0$ then $B(F) \leqslant \alpha A(F) \Rightarrow B(F) \leqslant A(F)$ since $\alpha \leqslant 1$. If $A(F) < 0$, then $[B(F) \leqslant A(F) - \delta A(F)] \Rightarrow B(F) \leqslant A(F)$ if and only if $-\delta A(F) < 1$. Since $\sum \dim F_1^j(F) \leqslant a$, one has $-A(F) \leqslant a(n-r)$. Thus $-A(F) \leqslant a(n-1)$ for any F. Hence if $\delta < 1/a(n-1)$, for $0 < 1 - \delta = \alpha < 1$, α-semistability implies 1-semistability. Suppose now $(E, \underline{\mathscr{F}})$ is 1-stable. For $A(F) \leqslant 0$, if $B(F) < A(F)$, then $B(F) < \alpha A(F)$ as $\alpha > 0$. For $A(F) > 0$, $B(F) < A(F) \Rightarrow B(F) < \alpha A(F) = A(F) - \delta A(F)$ if and only if $\delta A(F) < 1$. Since $\dim F_1^j(F) \geqslant 0$, $A(F) \leqslant ra \leqslant a(n-1)$ for all F. Thus for $1/\delta > a(n-1)$, $0 < \alpha = 1 - \delta < 1$, 1-stability implies α-stability.

(2) Proof as in the vector bundle case.

(3) The first assertion is clear from (1) and (2). The second assertion follows from Theorem 1.

Theorem 2. *Let $M(n,d)$ denote the moduli space of stable GPBs $(E, \underline{\mathscr{F}}, \underline{\alpha})$ of rank n and degree d on a nonsingular curve X of genus g satisfying the following conditions.*

(1) $\mathscr{F} = (\mathscr{F}^1, \ldots, \mathscr{F}^J)$, $\mathscr{F}^j : F_0^j(E) \supset F_1^j(E) \supset 0$, $\dim F_1^j(E) = a_j$ *is a fixed integer depending on j*, $a = \sum_j \dim F_1^j(E)$.

(2) $\underline{\alpha} = (\underline{\alpha}^1, \ldots, \underline{\alpha}^J)$ *with* $\underline{\alpha}^j = (0, \alpha)$ *for all* j, $1 - 1/a(n-1) < \alpha < 1$.

(3) *The rank n and degree d are coprime and $a = \sum_j a_j$ is an integral multiple of n.*

Then $M(n,d)$ is a fine moduli space.

Proof. Similar to 3.16 [B3].

3B *Operations on QPBs*

In this section we assume that $D_j = x_j + z_j$, for all j and the QPBs are of the type defined in 3.1. We denote such a QPB by $(M, F_1^j(M))$.

3.4. *Direct sum and tensor product of two QPBs.* The direct sum of two QPBs $(M, F_1^j(M))$ and $(N, F_1^j(N))$ is the QPB $(M \oplus N, F_1^j(M) \oplus F_1^j(N))$. The tensor product of the two QPBs is the QPB$(M \otimes N, F_1^j(M \otimes N))$ where $F_1^j(M \otimes N)$ is the image of $F_1^j(M) \otimes F_1^j(N)$ under the projection map $(M_{x_j} \oplus M_{z_j}) \otimes (N_{x_j} \oplus N_{z_j}) \to (M \otimes N)_{x_j} \oplus (M \otimes N)_{z_j}$. We remark that if the projections from $F_1^j(M)$ (respectively $F_1^j(N)$) to M_{x_j} and M_{z_j} (resp. N_{x_j} and N_{z_j}) are isomorphisms then a similar statement is true for $F_1^j(M \otimes N)$.

3.5. Let T denote an operation on vector spaces such that for $V_1 \subset V_2$, $T(V_1) \subset T(V_2)$ and for any two vector spaces V_1, V_2 there is a map $\mathrm{pr}: T(V_1 \oplus V_2) \to T(V_1) \oplus T(V_2)$. For example, $T(V) = \mathrm{End}(V), \otimes^m(V), S^m(V), \Lambda^m(V)$ etc. In these examples $T(V_1) \oplus T(V_2)$ is a direct summand of $T(V_1 \oplus V_2)$ (characteristic zero), hence there is a canonical projection map pr. T which induces a corresponding operation on vector bundles which we again denote by T. We want to extend T to QPBs. Note that $\mathrm{pr}\, T(F_1^j(E)) \subset T(E_{x_j}) \oplus T(E_{z_j})$. For a QPB $(E, F_1^j(E)$, define $T(E, F_1^j(E)) = (T(E), \mathrm{pr}(T(F_1^j(E)))$.

3.6. In particular when $T = \Lambda^n$, the top exterior product, then $T(E, F_1^j(E))$ is a rank 1 QBP called the determinant. Let $(E, \underline{\mathscr{F}}, \underline{\alpha})$ be a GPB with $E, \underline{\mathscr{F}}$ as in 3.1. Then we define

the determinant of $(E, \underline{\mathscr{F}}, \underline{\alpha})$, denoted by $\det(E, \underline{\mathscr{F}}, \underline{\alpha})$, to be the rank one GPB (det $E, \underline{\mathscr{F}}_0, \underline{\alpha}$).

PROPOSITION 3.7

Let p_j and q_j be the projections $F_1^j(E) \to E_{x_j}$ and $F_1^j(E) \to E_{z_j}$ respectively. Let M' be the subset of the moduli space $M(n, d)$ corresponding to GPBs satisfying the condition that at least one of p_j or q_j is an isomorphism for every $j = 1, \dots, J$. Then $\det : M' \to M(1, d)$ *defined by $(E, \underline{\mathscr{F}}, \underline{\alpha}) \mapsto \det(E, \underline{\mathscr{F}}, \underline{\alpha})$ is a morphism.*

Proof. Let $(\mathscr{E}, \underline{\mathscr{F}}, \underline{\alpha}) \to T \times X$ be a (flat) family of GPBs. It suffices to show that we can globalize the construction in 3.6 to $(\mathscr{E}, \underline{\mathscr{F}}, \underline{\alpha})$ replacing $(E, \underline{\mathscr{F}}, \underline{\alpha})$. By definition, $\underline{\mathscr{F}}$ gives a rank n subbundle $F_1^j(\mathscr{E})$ of $\mathscr{E}_j = p_{T_*}(\mathscr{E} | T \times D_j)$ for all j. Let $\wedge^n \mathscr{E} = \mathscr{L}$. We have

$$(p_T)_*(\mathscr{L} | T \times D_j) = (p_T)_*(\mathscr{L} | T \times x_j) \oplus (p_T)_*(\mathscr{L} | T \times z_j)$$
$$= \wedge^n(p_{T_*}(\mathscr{E} | T \times x_j)) \oplus \wedge^n(P_{T_*}(\mathscr{E} | T \times z_j)).$$

Let $p \colon \wedge^n \mathscr{E}_j \to \wedge^n(p_{T_*}(\mathscr{E} | T \times x_j)) \oplus \wedge^n(p_{T_*}(\mathscr{E} | T \times z_j))$ be the natural projection. The rank n bundle $F_1^j(\mathscr{E})$ determines a rank 1 subbundle F^j of $\wedge^n \mathscr{E}_j$. Then $p(F^j)$ is a rank 1 subbundle of $(p_T)_*(\mathscr{L} | T \times D_j)$ giving the parabolic structure over D_j on $\mathscr{L}$. Thus we get a family $(\mathscr{L}, \underline{\mathscr{F}}_0, \underline{\alpha})$ of rank one GPBs on $T \times X$.

Notation 3.8. Fix a GPB$\bar{L} = (L, \underline{\mathscr{F}}_0, \underline{\alpha})$ of rank 1 and degree d. Let $M'_{\bar{L}}$ denote the subset of M' consisting of GPBs $(E, \underline{\mathscr{F}}, \underline{\alpha})$ such that $\det(E, \underline{\mathscr{F}}, \underline{\alpha}) = (L, \underline{\mathscr{F}}_0, \underline{\alpha})$. Let $M_{\bar{L}}$ denote the closure of $M'_{\bar{L}}$ in $M(n, d)$. Notice that $M'_{\bar{L}}$ is closed in M'.

3.9. Let V_1, V_2 be two vector spaces of dimension n. Let G be the Grassmannian of n-dimensional subspaces of $V_1 \oplus V_2$. Let $e_1, \dots, e_n$ be a basis of V_1 and let $e_{n+1}, \dots, e_{2n}$ be a basis of V_2. G is embedded in $\mathbf{P}(\wedge^n(V_1 \oplus V_2))$ by the Plucker embedding. Let $\{P_{i_1, \dots, i_n}\}$, $1 \leqslant i_1 < \cdots < i_n \leqslant 2n$ be the Plucker coordinates. Let H be the hyperplane defined by $a.P_{1\dots n} - b.P_{n+1\dots(2n)} = 0$ (a, b both nonzero and fixed). The following Lemma seems to be known.

Lemma 3.10. *$G \cap H$ is nonsingular.*

Remark 3.11. The result of the lemma does not hold if one of a or b is zero as can be seen by taking $n = 2$. In that case $G \cap H$ becomes a cone with base a nonsingular quadric in $\mathbf{P}^3$.

Theorem 3. *Suppose that $\bar{L} = (L, \underline{\mathscr{F}}_0, \underline{\alpha})$ is such that $p_j(F_1^j(L)) \neq 0$, $q_j(F_1^j(L)) \neq 0$ for all j. Then one has the following.*

(1) *$M_{\bar{L}}$ is normal.*
(2) *If $(n, d) = 1$, $\underline{\alpha} = (0, \alpha)$ with $1 - 1/nJ(n-1) < \alpha < 1$, then $M_{\bar{L}}$ is nonsingular.*

Proof. Let (b_j, a_j), $b_j \in L_{x_j}$, $a_j \in L_{z_j}$ be a generator of $F_1^j(L)$, $j = 1, \dots, J$. By our assumption, a_j and b_j are both nonzero for all j. Consider a GPB$(E, \underline{\mathscr{F}}, \underline{\alpha})$ (as in 3.1). Let $V_{1j} = E_{xj}$, $V_{2j} = E_{zj}$, $G_j = \mathrm{Gr}(n, V_{1j} \oplus V_{2j}) \approx G$. We identify $F_1^j(E)$ with the element of

G_j determined by it. Then a semistable $(E, \mathscr{F}, \underline{\alpha})$ corresponds to an element of $M'_{\bar{L}}$ if and only if for all j, $F_1^j(E)$ belongs to the subset of G_j defined by

$$\{a_j P_{1\ldots n} - b_j P_{\{(n+1)\ldots(2n)\}} = 0,\ P_{(n+1)\ldots(2n)} \neq 0,\ P_{1\ldots n} \neq 0\}.$$

The closure of this set is the hyperplane section $G_j \cap H_j$ of G_j; where H_j is defined by $a_j P_{1\ldots n} - b_j P_{(n+1)\ldots(2n)} = 0$. Thus a semistable $\mathrm{GPB}(E, \mathscr{F}, \underline{\alpha})$ corresponds to an element of $M_{\bar{L}}$ if and only if $F_1^j(E) \in G_j \cap H_j$ for all j. Let R, $\tilde{R}$ be the spaces defined in the proof of Theorem 1. $\tilde{R}$ is a bundle over R with fibres $\Pi_j G$, a J-fold product of G. Let R_L denote the subset of R corresponding to vector bundles E with fixed determinant L. R_L is known to be nonsingular. Let $\tilde{R}_{\bar{L}}$ be the fibre bundle over R_L with fibres $\Pi_j G \cap H_j$, which is a subbundle of $\tilde{R}|_{R_L}$. By Lemma 3.10, $\tilde{R}_{\bar{L}}$ is nonsingular. $M_{\bar{L}}$ is the quotient of $\tilde{R}_{\bar{L}}$ by $PGL(N)$ (in the sense of geometric invariant theory), hence $M_{\bar{L}}$ is normal. If the conditions of (2) are satisfied, then by Proposition 3.3 $M_{\bar{L}}$ is a geometric quotient and hence is nonsingular.

3C *Moduli spaces for rank* 2

3.12. Throughout this subsection, we assume that $J = 1, D = x + z, r(E) = 2, 0 < \alpha < 1$. Let (e_1, e_2) and (e_3, e_4) denote bases of E_x and E_z respctively, these will be chosen suitably in different cases. Let G_r denote the Grassmannian of 2-dimensional subspaces of $V = E_x \oplus E_z$, $G_r \subset \mathbf{P}(\wedge^2 V)$. Any element in $\wedge^2 V$ can be written in the form $X_1 e_1 \wedge e_2 + Y_1 e_3 \wedge e_4 + X_2 e_1 \wedge e_4 + Y_2 e_2 \wedge e_3 + X_3 e_3 \wedge e_1 + Y_3 e_2 \wedge e_4$. G_r is defined by $X_1 Y_1 + X_2 Y_2 + X_3 Y_3 = 0$. $F_1(E)$ defines a point in G_r. The subset of G_r corresponding to stable (resp. semistable) QPS $(E, F_1(E))$ will be denoted by G_r^s (resp. G_r^{ss}). Let H denote the hyperplane $hX_1 - Y_1 = 0$, $h \neq 0$.

Lemma 3.13. *Let the assumptions be as above.*

(i) *A QPB$(E, F_1(E))$ of degree* 1 *is α-stable* (= 1-*stable*) *for* $1/2 < \alpha < 1$ *if and only if one of the conditions* (a), (b) *is satisfied,* (a) *E is a stable vector bundle and* $F_1(E) \neq M_x \oplus M_z$ *for any line subbundle M of E of degree zero.* (b) *E has a subbundle* M_1 *of degree* 1 *with* $E/M_1 = M$, $\deg(M_1) = 1$, $\deg(M) = 0$ *and* $F_1(E) \cap ((M_1)_x \oplus (M_1)_z) = 0$, $F_1(E) \neq L_x \oplus L_z$ *for any line subbundle L of E isomorphic to M.*
(ii) *A QPB$(E, F_1(E))$ of degree zero is α-semistable for* $0 < \alpha < 1$ *if and only if E is a semistable vector bundle and for any line subbundle L of E of degree* 0, $F_1(E) \neq L_x \oplus L_z$. *Further, it is α-stable if and only if it satisfies the additional condition* $F_1(E) \cap (L_x \oplus L_z) = 0$ *for L as above.*

Proof. This follows from straightforward computations.

PROPOSITION 3.14

Assumptions as in 3.12. *Assume further that degree* $E = 1, g = 1, 1/2 < \alpha < 1$. (1) *The open subset of $M_{\bar{L}}$ corresponding to QPBs with underlying vector bundle E stable is isomorphic to* $G_r \cap H - X$. (2) *The closed subset of $M_{\bar{L}}$ corresponding to QPBs with E not stable is a fibration over X with fibres isomorphic to* $\mathbf{P}^1$.

Proof. (1) On the elliptic curve X there is a unique vector bundle E of rank 2, degree 1 with a fixed determinant. By Lemma 3.13(i)(a), $(G_r \cap H)^{ss} = G_r \cap H - \mathrm{Pic}\, X$. Since $X \approx \mathrm{Pic}\, X$, the result follows.

(2) By 3.13, $E = M_1 \oplus M$. Fix $M \in \mathrm{Pic}^0 X$, since det E is fixed, this fixes E too. Choose nonzero elements $e_1 \in (M_1)_x, e_2 \in M_x, e_3 \in (M_1)_z, e_4 \in M_z$. Any automorphism of E is of the form $f = \left[\begin{smallmatrix} \lambda & vs \\ 0 & \mu \end{smallmatrix}\right]$ with $\lambda \in k^* = \mathrm{Aut}\, M_1, \mu \in k^* = \mathrm{Aut}\, M, v \in k, s \in \Gamma(M^* \otimes M_1) - \{0\}$. There are two cases depending on zeroes of s.

Case (i). Assume $s(x)$ and $s(y)$ are both nonzero. By suitable choice of the basis elements, one can have $(G_r \cap H)^{ss} = \{Y_3 \neq 0\} - \{X_1 = 0 = X_2 - Y_2\}$ (by 3.13) and $f(e_1) = \lambda e_1, f(e_2) = \mu e_2 + v e_1, f(e_3) = \lambda e_3, f(e_4) = \mu e_4 + v e_3$. Then Aut (E) acts on $\mathbf{P}^5$ by

$$f(X_1, Y_1, X_2, Y_2, X_3, Y_3) = (\lambda\mu X_1, \lambda\mu Y_1, \lambda\mu X_2 + \mu v Y_3, \lambda\mu Y_2 + \mu v Y_3, \lambda^2 X_3 - \lambda v X_2 - \lambda v Y_2 - v^2 Y_3, \mu^2 Y_3).$$

For the normal subgroup G_a defined by $\lambda = \mu = 1$ acting on the cone $C(H) \approx k^5$ (coordinates $(X_1, X_2, Y_2, X_3, Y_3)$) the ring of invariants is generated by $X_1, X_2 - Y_2, Y_3, U = X_2 Y_2 + X_3 Y_3$. The affine cone $C(G_r \cap H)$ is given by $U = -hX_1^2$. Hence the quotient of $(G_r \cap H)^{ss}$ by G_a is $(\mathbf{A}^2 - 0) \subset \mathbf{P}^2$, it is given by the map $(X_1, X_2, Y_2, X_3, Y_3) \mapsto (X_1, X_2 - Y_2, Y_3)$. On this quotient the induced action of $G_m = P(\mathrm{Aut}\, E)/G_a$ is given by multiplication of coordinates by $1, 1, t$ respectively $(t = \mu\lambda^{-1})$. The quotient is $\mathbf{P}^1$, it is given by mapping to $(X_1, X_2 - Y_2)$.

Case (ii). Suppose $s(x) = 0, s(z) \neq 0$. By 3.13, $(G_r \cap H)^{ss} = \{Y_3 \neq 0\} - \{X_1 = 0 = X_2 = X_3\}$. As in case (ii), one has $f(e_1) = \lambda e_1, f(e_2) = \mu e_2, f(e_3) = \lambda e_3, f(e_4) = \mu e_4 + v e_3$. The action of Aut E on $\mathbf{P}^5$ is given by

$$f(X_1, Y_1, X_2, Y_2, X_3, Y_3) = (\lambda\mu X_1, \lambda\mu Y_1, \lambda\mu X_2 \lambda\mu Y_2 + \mu v Y_3, \lambda^2 X_3 - \lambda v X_2, \mu^2 Y_3).$$

The normal subgroup G_a acts on $H \approx k^5$ by

$$v(X_1, X_2, Y_2, X_3, Y_3) = (X_1, X_2, Y_2 + vY_3, X_3 - vX_2, Y_3).$$

The ring of invariants is generated by X_1, X_2, Y_3 and $U = X_2 Y_2 + X_3 Y_3, G_r \cap H$ is defined by $U = -hX_1^2$. The quotient of $(G_r \cap H)^{ss}$ by G_a is $(\mathbf{A}^2 - 0) \subset \mathbf{P}^2$, it is given by projection to (X_1, X_2, Y_3) coordinates, $\mu\lambda^{-1} = t \in G_m = P$ $(\mathrm{Aut}\, E)/G_a$ acts on it by $t(X_1, X_2, Y_3) = (X_1, X_2, tY_3)$. The (required) quotient is $\mathbf{P}^1$ given by projection to coordinates (X_1, X_2).

Remark 3.15. The above calculations indicate that $M_{\bar{L}}$ is obtained by blowing up an elliptic curve (isomorphic to X) in a nonsingular quadric ($G_r \cap H$ above) in $\mathbf{P}^4$.

PROPOSITION 3.16

With the notations of 3.12, *assume that* $g = 1, 0 < \alpha < 1$ *and the determinant of* E *is trivial. Then* $M_{\bar{L}}$ *is a* $\mathbf{P}^2$*-bundle over* $\mathbf{P}^1$.

Proof. Lemma 3.13 (ii) implies that either (a) $E = M \oplus M^{-1}, M \in \mathrm{Pic}^0 X$ or (b) E comes in a nontrivial extension $0 \to M_1 \xrightarrow{g} E \xrightarrow{h} M_2 \to 0, M_1 = M_2 = M \in \mathrm{Pic}^0 X, M^2 = \mathcal{O}$. Up to isomorphism there are four vector bundles of type (b) corresponding to four roots of $\mathcal{O}$. The vector bundles of type (a) are parametrized by $(\mathrm{Pic}^0 X)/(\mathbf{Z}/2) \approx \mathbf{P}^1$.

(a) If $M = M^{-1}$, then $(E, F_1(E))$ is equivalent to a GPB with E of type (b). Therefore, we may assume that $M \neq M^{-1}$. Let e_1, e_2, e_3, e_4 be basis of $M_x, M_x^{-1}, M_z, M_z^{-1}$ respectively. Since any line subbundle of E of degree zero is either M or M^{-1}, by 3.13(ii) the only nonsemistable points in $G_r \cap H$ are $(0, 0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 0, 1)$. Stable points are given by $X_3 Y_3 \neq 0$. $P(\mathrm{Aut}\, E) = P(G_m \times G_m) = G_m . t \in G_m$ acts by $t(X_1, Y_1, X_2, Y_2, X_3, Y_3) = (X_1, Y_1, X_2, Y_2, tX_3, t^{-1} Y_3)$. Projections to (X_1, X_2, Y_2) coordinates show that the quotient of $G_r \cap H$ under G_m is $\mathbf{P}^2$. The semistable but nonstable GPBs correspond to the quadric $hX_1^2 = X_2 Y_2$ in $\mathbf{P}^2$.

(b) Let e_1, e_2, e_3, e_4 be the basis elements of $(M_1)_x, (M_2)_x, (M_1)_z, (M_2)_z$ respectively. Any automorphism of E is of the form $\lambda Id + \mu g \circ h . \mathbf{P}(\mathrm{Aut}\, E) \approx G_a$. Taking $\mu \lambda^{-1} = t \in G_a$, it acts by $te_1 = e_1, te_3 = e_3, te_2 = e_2 + te_1, te_4 = e_4 + te_3$. Hence one has $t(X_1, Y_1, X_2, Y_2, X_3, Y_3) = (X_1, Y_1, Y_2 + tY_3, X_2 + tY_3, X_3 - t(X_2 + Y_2) - t^2 Y_3, Y_3)$. The ring of invariants for G_a-action on k^6 is generated by $X_1, Y_1, U_1 = X_2 - Y_2, Y_3$ and $U_2 = X_2 Y_2 + X_3 Y_3$. Hence the quotient of the cone $C(G_r \cap H)^{ss}$ is the affine quadric $hX_1^2 = U_2$ in $(k^3 - \{0\}) \times k$, where the latter k has U_2 as coordinate, while coordinates in k^3 are X_1, Y_3, U_1. Note that the nonsemistable points for G_a are $\{X_1 = Y_3 = U_1 = 0\}$. The quotient of $C(G_r \cap H)^{ss}$ by scalar multiplication is $\mathbf{P}^2$, given by projection to (X_1, Y_3, U_1) coordinates. The nonstable GPBs correspond to $Y_3 = 0$ in this $\mathbf{P}^2$. It is not difficult to see that in case $E = M \oplus M, M^2 = \mathcal{O}$, there are no stable GPBs and the semistable GPBs give a $\mathbf{P}^1$ (in the moduli space $M_{\bar{L}}$), which is the same as $\{Y_3 = 0\}$ in $\mathbf{P}^2$ above.

These calculations show that there is a surjective map $h: M_{\bar{L}} \to \mathbf{P}^1$ with fibres $\mathbf{P}^2$ and over $\mathbf{P}^1 - \{4 \text{ points}\}$ this fibration is locally trivial. The result now follows from Tsen's theorem (p. 108, case (d), [M].

4. Applications to curves with nodes and ordinary cusps

4A *Preliminaries*

Let Y be an integral projective curve over an algebraically closed field k. Let $\pi: X \to Y$ be the normalization map. Let (A, m) be the local ring at a singular point y of Y. We assume that Y has only nodes and ordinary cusps as singularities.

PROPOSITION 4.1

Let F be a torsionfree A-module of rank n. Then $F \approx rA \oplus (n-r)m$ (rA denotes the direct sum of r copies of A).

Proof. We assume that y is an ordinary cusp, the nodal case being proved in Proposition 2, Part 8 [S]. By Corollary 6.2 [B], every indecomposable torsion-free A-module is isomorphic to an ideal. Any ideal is isomorphic either to A or m [1.4[D]]. The result follows by induction on rank.

PROPOSITION 4.2

Let A be the local ring at a node. Let m_1, m_2 be the two maximum ideals of the semilocal ring $\bar{A}$, $k_i = \bar{A}/m_i, i = 1, 2$; $k_1 = k_2 = k$. Let $p: \bar{A} \to k_1 \oplus k_2$ be the canonical surjection, $q = \oplus_n p, n > 0$. Let V be a subspace of $k_1^n \oplus k_2^n$ of dimension n. Let $p_i: V \to k_i^n$ be the projection, $a_i =$ dimension of the kernel of p_i. Then $F = q^{-1}(V) \approx (n - a_1 - a_2) A \oplus (a_1 + a_2)m$.

Proof. We assert that there is an automorphism of $\bar{A}^n$ such that the induced automorphism h_1 of $(\bar{A}/m)^n$ maps V onto the subspace $V_1 = k_2^{a_1} \oplus D^{n-a_1-a_2} \oplus k_1^{a_2} \subset (k_1 \oplus k_2)^{a_1} \oplus (k_1 \oplus k_2)^{n-a_1-a_2} \oplus (k_1 \oplus k_2)^{a_2}$, D is the diagonal of $k_1 \oplus k_2$. Since $p^{-1}D = A, p^{-1}k_i \approx m$, one has $p^{-1}(V_1) \approx (n - a_1 - a_2)A \oplus (a_1 + a_2)m$, hence the result.

We now prove our assertion. Let K_i, I_i denote respectively the kernel and image of $p_i, i = 1, 2$. Write $V = K_1 \oplus W \oplus K_2$. Let $e_1, \ldots, e_{a_1}$ (resp. $e_{n-a_2+1}, \ldots, e_n$) be a basis of K_1 (resp. K_2). Let $e_{a_1+1}, \ldots, e_{n-a_2}$ be a basis of W. For $e_i \in W$, write $e_i = e_{i,1} + e_{i,2}, e_{i,j}$ being the component in $k_j^n, j = 1, 2$. Complete the basis $e_1, \ldots, e_{a_1}, e_{a_1+1,2}, \ldots, e_{n-a_2,2}$ of I_2 to a basis $\{e_{i,2}\}, i = 1, \ldots, n$ of k_2^n where $e_{i,2} = e_i, i \leqslant a_1$. Similarly choose a basis $\{e_{i,1}\}$ of k_1^n extending a basis of I_1 with $e_{i,1} = e_i, i = n - a_2 + 1, \ldots, n$. Let $b_i = e_{i,1} + e_{i,2}$ for all i. Since q is a surjection there exist $M_{j,i}$ in $\bar{A}$, $i, j = 1, \ldots, n$, such that $q(M_{1,i}, \ldots, M_{n,i}) = b_i$ for all i. The matrix $M = (M_{ji})$ is the matrix of an endomorphism f of $\bar{A}^n$ which induces an endomorphism h_1 of $(\bar{A}/m)^n$. h_1 maps the canonical basis of $(\bar{A}/m)^n$ to the basis $\{b_i\}$. M modulo m is the matrix of the base change. Hence the determinant of M modulo m is a unit and therefore det M is a unit. Thus f is an automorphism, so is h_1. With respect to the basis $\{b_i\}$, $h_1(V_1) = V$.

4.3. We now assume that A is the local ring at an ordinary cusp y. Then $(\bar{A}, m_x)$ is a local ring. One has $A \cap m_x = m = m_x^2, m^{-1} = \bar{A}$. There is a canonical A-splitting of the exact sequence $0 \to m_x/m \to \bar{A}/m \to \bar{A}/m_x \to 0$ as follows. $\bar{A}$ is a k-algebra, we consider $\bar{A}/m_x = k$ embedded in $\bar{A}$. If $f(x)$ denotes the element of k determined by f, then $f - f(x) \in m_x$. This induces a map $s: \bar{A}/m \to m_x/m$. It is A-linear (as $m_x \cap A = m$) but not $\bar{A}$-linear. Using the splitting given by s we write $\bar{A}/m = k_1 \oplus k_2, k_1 = \bar{A}/m_x, k_2 = m_x/m$, $k_1 \approx k_2 \approx k$.

Lemma 4.4. *Let $p: \bar{A} \to k_1 \oplus k_2$ be the canonical map. Let V be a one dimensional subspace of $k_1 \oplus k_2$, $p_i: V \to k_i$ projections, $F = p^{-1}(V)$. Then one has*

(1) *If $V = k_2$ then $F = m_x$.*
(2) *If p_1 is nonzero then $F \approx A$.*

Proof. It is easy to check (1) and that if $V = k_1, F = A$. For (2) it suffices to show that there is a unit $b \in \bar{A}$ such that multiplication by b induces a linear automorphism h of $k_1 \oplus k_2$ with $h(V) = k_1$. Let t be a uniformizing parameter in $\bar{A}$, $m_x = t\bar{A}$, $m = t^2\bar{A}$. For $f \in \bar{A}$, $f = f_0 + f_1 t \mod m$, $f_0, f_1 \in k$ i.e. $p(f) = (f_0, f_1) \in k_1 \oplus k_2$. Then $h(f_0, f_1) = p(bf) = (b_0 f_0, f_0 b_1 + f_1 b_0)$. Choose b with $b_0 = 1, b_1 = -v_1/v_0$ where (v_0, v_1) is a generator of V.

PROPOSITION 4.5

With A as above let V, q, p_i, F be as in 4.2. Let a be the rank of p_1. Then $F \approx aA \oplus (n - a)m$.

Proof. As in Proposition 4.2 we can find $f, h_1, V_1 = h_1(V)$ (use $M \bmod m_x$ is a unit). Thus we may assume that $V = V_1$. Consider the automorphism of $\bar{A}^n$ whose matrix is a diagonal matrix with first a_1 (diagonal) entries 1, next $n - a_1 - a_2$ entries $b \in \bar{A}$ and the last a_2 entries 1. Choose b with $p(b) = (1, -1) \in \bar{A}/m$. It follows from the proof of Lemma 4.4 that the induced automorphism h_2 of $(\bar{A}/m)^n$ is identity on the first a_1 and last a_2 factors and maps each D in the middle $n - a_1 - a_2$ factors onto k_1. Thus $h_2(V_1) = k_2^{a_1} \oplus k_1^{n-a_1} = k_2^{n-a} \oplus k_1^a$, hence $F \approx q^{-1}(V_2) = (n - a)m \oplus aA$.

4B *Relation between torsionfree sheaves and QPBs*

Notation 4.6. Let $y_1,\dots,y_J$ be the singular points of Y. Define divisors $D_j=\pi^{-1}(y_j)$, $D_j=x_j+z_j$ if y_j is a node, $D_j=2x_j$ if y_j is an ordinary cusp. Let Q denote the set of isomorphism classes of QPBs$(E,\mathscr{F})$ of rank n, degree d on X with $\mathscr{F}^j{:}F_0^j(E)\supset F_1^j(E)\supset 0$, dim $F_1^j(E)=n$ for all j. If y_j is a node let p_j,q_j be the projections from $F_1^j(E)$ to E_{x_j},E_{z_j} respectively and a_j,b_j be the dimensions of their kernels. If y_j is a cusp let p_j,a_j be defined as in 4.5. For $\bar r=(r_1,\dots,r_J)$, $0\leqslant r_j\leqslant n$, define $Q_{\bar r}=\{(E,\mathscr{F})|a_j+b_j=n-r_j$ if y_j is a node and $a_j=n-r_j$ if y_j is a cusp$\}$. Let (A_j,m_j) be the local ring at y_j. Let S denote the set of isomorphism classes of torsion-free sheaves of rank n and degree d on Y. Let $S_{\bar r}=\{F\in S|$ stalk $F_{y_j}\approx r_jA_j\oplus(n-r_j)m_j\}$. Then S (resp. Q) is a disjoint union of $S_{\bar r}$ (resp. $Q_{\bar r}$), $0\leqslant r_j\leqslant n$. Let $Q_n=Q_{(n,\dots,n)}$, $S_n=S_{(n,\dots,n)}$. The latter is the set of locally free sheaves in S.

PROPOSITION 4.7

There exists a map $f{:}Q\to S$ with the following properties.

(1) $f(Q_{\bar r})=S_{\bar r}$.
(2) $f|_{Q_n}{:}Q_n\to S_n$ *is a bijection.*
(3) $(E,\mathscr{F})$ is 1-stable (resp. 1-semistable) *if and only if its image F under f is a stable (resp. semistable) torsionfree sheaf.*

Proof. Let $(E,\mathscr{F})\in Q$. By 4.2, 4.3, $\pi_*(E)\otimes k(y_j)=(k_1\oplus k_2)^n=H^0(E|_{D_j})$. Then $f(E,\mathscr{F})=F$ is defined by the exact sequence $0\to F\to\pi_*(E)\to\oplus_j(\pi_*E\otimes k(y_j))/F_1^j\,(E)\to 0$. Since $\chi(\mathcal{O}_Y)=\chi(\mathcal{O}_X)-J$, $\deg\cdot F=\chi(F)-\text{rank }F\cdot\chi(\mathcal{O}_Y)$, $\deg. E=\chi(E)-n\chi\,(\mathcal{O}_X)$, F and E have the same degree.

(1) In view of Propositions 4.2, 4.5 we have only to show that for $F\in Q_{\bar r}$, there is $(E,\mathscr{F})$ mapping to F under f. Let $E_0=\pi^*\,F/\text{torsion}$. E is given by an extension $0\to E_0\to E\to\oplus_jT_j\to 0$, where $T_j=k(x_j)^{a_j}\oplus k(z_j)^{b_j}$, $a_j+b_j=n-r$ if y_j is a node and $T_j=k(x_j)^{n-r_j}$ if y_j is a cusp. The composite of sheaf inclusions $F\to\pi_*(E_0)\to\pi_*(E)$ induces a linear map $F\otimes k(y_j)\to\pi_*E\otimes k(y_j)$. Define $F_1^j(E)$ to be the image of this linear map.
(2) The inverse f^{-1} of $f|_{S_n}$ is defined as follows. For $F\in S_n$, define $E=\pi^*F$, $F_1^j(E)=F\otimes k(y_j)\subset(F\otimes\pi_{\_}\mathcal{O}_X)\otimes k(y_j)=(\pi_*(E)\otimes k(y_j))$. Since the above inclusion is induced by $\mathcal{O}_Y\subset\mathcal{O}_X$ and $\mathcal{O}_Y$ maps onto k_1 (resp. onto each of k_1,k_2) in $k_1\oplus k_2$ if y_j is a cusp (resp. a node), it follows that p_j (resp. each of p_j,q_j) is of maximum rank for all j.
(3) Similar to 4.2 [B3].

Theorem 4. *Let $M=M(n,d)$ denote the moduli space of semistable GPBs of rank n, degree d on X satisfying 4.6 and weights $(0,\alpha)$, $1-1/[nJ(n-1)]<\alpha<1$. Let $U=U(n,d)$ denote the moduli space of semistable torsionfree sheaves of rank n, degree d on Y.*

(1) *The map f (see 4.7) induces a morphism $f{:}M\to U$.*
(2) $f|_{(M_n)^s}{:}(M_n)^s\to(U_n)^s$ *is an isomorphism, where the superscript* s *denotes stable points. In particular f is birational.*
(3) *f is surjective.*
(4) *If $(n,d)=1$, then $M(n,d)$ is a desingularization of $U(n,d)$.*

Proof. Note first that semistable GPBs are also 1-semistable QPBs and hence map to semistable torsionfree sheaves under f. (1) and (2) now follow since the above

constructions globalize easily to families of bundles. (4) follows from Proposition 3.3. (3) We show that the image of f contains the set U^s of stable points in U. If F is a stable torsionfree sheaf on Y sheaf then by 4.7 there exists a 1-stable QPB and hence a stable GPB mapping to F. Since Y has only planar singularities, U is irreducible [R] and U^s is an open dense subset. Since f is proper if follows that f is a surjection.

Theorem 5. *Assume that Y has only J nodes as singularities. Let L be a fixed line bundle on Y. Let U_n^L denote the closed subset of U_n corresponding to vector bundles with fixed determinant L. Let U_L be the closure of U_n^L in U. Let $\bar{L} = f_1^{-1}(L)$, f_1 being the map f in* case $n = 1$. *Let $M_{\bar{L}}$ be as in Theorem 3. Then f induces a birational surjective morphism $M_{\bar{L}} \to U_L$. If $(n, d) = 1$, then M_L is a desingularization of U_L.*

Proof. First note that $\det \circ f = f \circ \det$ where the latter f is the map f in case $n = 1$. Hence $f(M_{\underline{L}}') = U_n^L$, the subset corresponding to vector bundles with fixed determinant L. Since f is proper it follows that $f(M_{\underline{L}}) = U_L$. The rest of the assertions follow from Theorems 3 and 4.

Remark 4.8. *Relation with singular intersection of quadrics.* Assume that $g = 1$, $J = 1$, Y is a hyperelliptic curve with Weierstrass points w_0, w_1, w_2, w_3, w_4; w_0 being the unique node of Y. The desingularization X is an elliptic curve. Let L be a line bundle on Y of degree 5. The linear system $|L|$ gives an embedding of Y in $\mathbf{P}^3$. The linear system $|\pi^* L|$ gives an embedding of X in $\mathbf{P}^4$. The inclusion $H^0(L) \to H^0(\pi_* \pi^* L) = H^0(\pi^* L)$ induces a projection from $\mathbf{P}^4$ to $\mathbf{P}^3$ mapping X onto Y (isomorphically outside w_0). There exists a Cartier divisor W_0 of degree two supported at w_0. Let $W = W_0 + \sum_{i>0} w_i$. On $\mathbf{P}^5 = \mathbf{P}((L \otimes \mathcal{O}_W)^*)$ there is a singular pencil of quadrics of the form $Q_1 = X_1 X_2 + X_3^2 + X_4^2 + X_5^2 + X_6^2$, $Q_2 = X_1^2 + 2a_1 X_1 X_2 + a_3 X_3^2 + a_4 X_4^2 + a_5 X_5^2 + a_6 X_6^2$, a_i being distinct scalars [B6]. $Q = Q_1 \cap Q_2$ is a 3-fold with a unique singular point q. Q is a normalization of $U_L(2, 1)$ and is bijective with it (Main theorem, [NE]). The blow up of Q at q is isomorphic to the blow up of a nonsingular quadric Q_0 in $\mathbf{P}^4$ along X. Q_0 is the base of the unique quadric cone in the singular pencil with vertex q and $X = Q_0 \cap \{X_1 = X_2 = 0\}$. The latter is isomorphic to $M_{\underline{L}}$, $\underline{L}$ being the generalized parabolic line bundle on X corresponding to L on Y (Remark 3.15). The injective evaluation map $H^0(L) \to L \otimes \mathcal{O}_W$ induces a projection to $\mathbf{P}^3$ mapping Q to a surface containing Y. The space of maximum isotropic spaces of one system for Q_1 is isomorphic to $\mathbf{P}^3$. There is a bijective morphism from this space to $U_{\mathcal{O}}(2, 0)$ which is an isomorphism outside the singular set (*a* $\mathbf{P}^1$) corresponding to nonlocally free sheaves, while the singular set is isomorphic to $U_{\mathcal{O}}^X(2, 0)$. $M_{\underline{L}}(2, 0)$ is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ (see 3.16). The latter is isomorphic to the blow up of $\mathbf{P}^3$ along a line.

Generalizations of these results to hyperelliptic curves of higher genera are possible [B6].

4.9 *Variation of* α. Let $M(\alpha; n, d)$ (respectively $M_{\underline{L}}(\alpha; n, d)$) denote the moduli spaces $M(n, d)$ (resp. $M_{\underline{L}}(n, d)$) for weights $(0, \alpha)$.

(A) $\alpha = 0$. In this case the semistability, stability of a QPB is the same as that of the underlying vector bundle. Hence $M(0; n, d) \approx U_X(n, d) \times \mathrm{Gr}(n, 2n)$.

(B) $n = 2$, $d = 1$, $0 < \alpha \leqslant 1$.

(1) Let $0 < \alpha < 1/2$. Then α-semistability is equivalent to the stability of the underlying bundle and hence there is a surjective morphism from $M(\alpha; 2, 1) \to U_X(2, 1)$. α-stability coincides with α-semistability but it does not imply 1-stability. Consequently there is only a rational map $f: M(\alpha; 2, 1) \to U_Y(2, 1)$. $M(\alpha; 2, 1)$. is nonsingular.
(2) $\alpha = \frac{1}{2}$. The α-stability is equivalent to the stability of the underlying bundle. But α-semistability does not imply stability of the underlying bundle, so h is only a rational map. It is defined on α-stable bundles and is surjective. α-semistability does not imply 1-semistability, but 1-stability implies α-semistability. Hence f is also a rational surjective map. As α-semistability does not imply α-stability, $M(\frac{1}{2}; 2, 1)$ could be singular.
(3) Let $1/2 < \alpha < 1$. The undering bundle of a semistable QPB can be non-stable. Hence there is only a rational map $h: M(\alpha; 2, 1) \to U_X(2, 1)$. However the morphism f is surjective and birational. $M(\alpha; 2, 1)$ is nonsingular (Theorem 4).
(4) Let $\alpha = 1$. Then $M(1; 2, 1)$ is not nonsingular since it includes sheaves which have torsion over D_j, s [RN]. The maps f, h have the same properties as in case (2). Under the assumptions of 3.8, one can see that $M_L(1; 2, 1)$ is a blow up (possibly a double blow up) of $\mathbf{P}^3$ along Y, where Y is embedded in $\mathbf{P}^3$ by the linear system $|L|$, $\deg L = 5$. $M_L(\alpha; 2, 1)$ for case (3) has been described in 4.8.

(C) $n = 2$, $d = 0$, $0 < \alpha \leqslant 1$.

(1) Let $0 < \alpha < 1$. Then α-stability implies α-semistability, the converse is not true. Also α-semistability implies 1-semistability. Hence there is a morphism $f: M(\alpha; 2, 0) \to U_Y(2, 0)$, it is surjective and birational. Since the underlying bundle is semistable, there is a surjective morphism $h: M(\alpha; 2, 0) \to U_X(2, 0)$.
(2) Let $\alpha = 1$. Then the morphism f is as in (1) above, but h is only a rational map, the underlying bundle of a 1-semistable QPB may not be semistable.

5. Generalized parabolic orthogonal bundles

5.1. Let the base field be algebraically closed and of characteristic different from 2. For simplicity of exposition, we assume that Y has a single node y_0 as its only singularity. Let $\pi^{-1}(y_0) = x_1 + x_2$. For a vector bundle E, we denote the rank and degree of E by $r(E)$ and $d(E)$ respectively. We indentify an orthogonal bundle with a pair (E, q) where E is a vector bundle and q a nondegenerate quadratic form on E (with values in the trivial line bundle $\mathcal{O}$). For a closed point x, let q_x denote the induced quadratic form on the fibre E_x. Let $q_1 = \frac{1}{2}(q_{x_1} \oplus q_{x_2})$, $q_2 = q_{x_1} \oplus (-q_{x_2})$.

Lemma 5.2. *Let $\sigma: E_{x_1} \approx E_{x_2}$ be a quadratic isomorphism (i.e. preserving q_{x_1}, q_{x_2}). Then the following holds.*
(a) *The graph Γ_σ of σ is isotropic for q_2.*
(b) *$p_i: \Gamma_\sigma \to E_{x_i}$, $(i = 1, 2)$ is an isotropy for q_1 on Γ_σ and q_{x_i} on E_{x_i}(i.e. $q_{x_i}(p_i(v)) = q_i(v)$ for $v \in \Gamma_\sigma$). In particular, q_1 is nondegenerate on Γ_σ.*

Proof. Easy.

Remark 5.3. Γ_σ is in fact a maximum isotropic space for q. The space S of maximum isotropic spaces for a nondegenerate quadratic form Q on a vector space of dimension $2n$ has two components, each being a smooth variety of dimension $n(n-1)/2 = \dim$

$O(n)$. The above lemma means that S may be regarded as a compactification of $O(n)$. This suggests the following.

DEFINITION 5.4

A generalized quasiparabolic orthogonal bundle (orthogonal QPB in short) on X is an orthogonal bundle (E, q) of rank n together with an n-dimensional vector subspace $F_1(E)$ of $E_{x_1} \oplus E_{x_2}$ which is isotropic for $q_2 = q_{x_1} \oplus (-q_{x_2})$.

For a subbundle N of E define $F_1(N) = F_1(E) \cap (N_{x_1} \oplus N_{x_2})$ and $f_1(N) = \dim F_1(N)$.

DEFINITION 5.5

Let α be a real number, $\alpha \in [0, 1]$. An orthogonal QPB $(E, F_1(E), q)$ is α-stable (resp. α-semistable) if for every isotropic proper subbundle N of E, one has $(d(N) + \alpha f_1(N))/r(N) <$ (resp. $\leqslant$) α.

Remark 5.6. If one considers special orthogonal QPBs, one should take the underlying bundle an $SO(n)$-bundle and $F_1(E)$ belonging to the unique component of S (see 5.3) which contains the graphs of isomorphisms $\sigma: E_{x_1} \to E_{x_2}$ which preserve the $SO(n)$-structure modulo a maximum parabolic subgroup. Recall that an $SO(n)$-bundle can be identified with a vector bundle E of rank n with a nondegenerate quadratic form q and a given trivialization of $\Lambda^n E$ whose square is the trivialization of $(\Lambda^n E)^{\otimes 2}$ given by $\Lambda^n q$.

Example 5.7. Generalized parabolic $SO(2)$-bundles on X are in bijective correspondence with generalized parabolic line bundles on X (recall that the latter give a desingularization of the compactified Jacobian of Y).

Proof. Since $SO(2)_k = k^*$, every $SO(2)$-bundle is of the form $E = L \oplus L^{-1}$, $L \in \mathrm{Pic}^0(X)$ with the natural quadratic map $q = L \oplus L^{-1} \to L \otimes L^{-1} = \mathcal{O}$. L, L^{-1} are isotropic subbundles. Each system S of lines in the quadric q_2 in $\mathbf{P}^3 = \mathbf{P}(E_{x_1} \oplus E_{x_2})$ is isomorphic to $\mathbf{P}^1$. We can choose nonzero elements $e_1 \in L_{x_1}$, $e_2 \in L_{x_2}$, $f_1 \in L_{x_1}^{-1}$, $f_2 \in L_{x_2}^{-1}$ such that $B_{x_i}(e_i, f_i) = 1$, $i = 1, 2$; where B_{x_i} is the bilinear form associated to q_{x_i}. With respect to (ordered) bases (e_1, f_1) of E_{x_1} and (e_2, f_2) of E_{x_2}, any isomorphism $\sigma: F_{x1} \to F_{x2}$ preserving the $SO(2)$-structures is of the form $\sigma(e_1) = ae_2, \sigma(f_1) = a^{-1} f_2$, $a \in k^*$. So $\Gamma_\sigma =$ span of $\{e_1 + ae_2, f_1 + a^{-1} f_2\}$. Thus

$$(E, F_1(E)) = (L, F_1(L)) \oplus (L^{-1}, F_1(L^{-1})),$$

where $F_1(L)$ is spanned by $\lambda e_1 + \mu e_2$ and $F_1(L^{-1})$ by $\mu f_1 + \lambda f_2$, $(\lambda, \mu) \in \mathbf{P}^1$.

DEFINITION 5.8

An orthogonal sheaf on Y is a pair (F, q_F) where F is a torsionfree sheaf on Y and Q_F is a nondegenerate quadratic form on F with values in $\mathcal{O}_Y$. An orthogonal sheaf (F, q_F) is semistable (resp. stable) if for every nonzero proper (totally) isotropic subsheaf N of F, $d(N)/r(N) \leqslant$ (resp. $<$) 0.

DEFINITION 5.9

An isomorphism of orthogonal sheaves is a sheaf isomorphism which preserves the quadratic forms. In case of orthogonal QPBs we also demand (in addition) that the quasiparabolic structures $F_1(E)$ should be preserved.

PROPOSITION 5.10

(a) *There is a map f from the set of isomorphism classes of orthogonal QPBs on X to the set of isomorphism classes of orthogonal sheaves on Y. Let $f(E, F_1(E), q) = (F, q_F)$.*
(b) *If p_i, $i = 1, 2$ are both isomorphisms, then (F, q_F) is an orthogonal bundle. f gives a bijection between orthogonal QPBs on X with p_1, p_2 isomorphisms and orthogonal bundles on Y.*
(c) *$(E, F_1(E), q)$ is 1-semistable (resp. 1-stable) if and only if (F, q_F) is semistable (resp. stable).*

Proof. (a) To a QPB $(E, F_1(E))$ one can associate a torsionfree sheaf F given by the exact sequence $0 \to F \to \pi_* E \to \pi_* E \otimes k(y_0)/\pi_* F_1(E) \to 0$. The quadratic form q on E induces one on $\pi_* E$ and on F. The quadratic form q_F on F is nondegenerate outside y_0 and *a priori* has values in $\pi_* \mathcal{O}_X$. Consider $q_{x_1} \oplus q_{x_2}$ as a form on $E_{x_1} \oplus E_{x_2}$ with values $k(x_1) \oplus k(x_2)$. Since $F_1(E)$ is isotropic for q_2, one sees that $q_{x_1} \oplus q_{x_2}$ maps $F_1(E)$ into $k(y_0)$ contained diagonally in $k(x_1) \oplus k(x_2)$. This means that the form q_F on F has values in $\mathcal{O}_Y \subset \pi \mathcal{O}_X$ and (F, q_F) is an orthogonal sheaf (5.8).
(b) F is locally free if and only if p_i are both isomorphisms. Moreover, $E = \pi^* F$ and hence gets a nondegenerate quadratic form with values in $\mathcal{O}_X$. Since the correspondence $(E, F_1(E)) \mapsto F$, is bijective for F locally free (4.7) the result follows.
(c) This can be checked similarly in 4.2 [B3]. One has only to notice that a subsheaf is totally isotropic if and only if it is generically totally isotropic.

PROPOSITION 5.11

An orthogonal QPB is α-semistable if and only if the underlying QPB is so.

Proof. We only have to check that if $(E, F_1(E), q)$ is α-semistable, then $(E, F_1(E))$ is α-semistable. Let F be a subbundle of E. We may assume F is nonisotropic. By the proof of Proposition 4.2 [RS] we have an exact sequence $0 \to N \to F \oplus F^{\perp} \to N^{\perp} \to 0$ where N is the isotropic subbundle generated by $F \cap F^{\perp}$, $\perp$ denoting orthogonal complement. Also, $d(F) = d(F^{\perp}) = d(N)$.
Case (i) When $N = 0$. Then $E = F \oplus F^{\perp}$, $d(F) = 0$. Since $q|\ F$ is nondegenerate so is $q_2 = q_{x_1} \oplus (-q_{x_2})$. Since $F_1(F) \subset F_1(E)$ is isotropic for q_2, $f_1(F) \leqslant r(F)$. Thus $(d(F) + \alpha f_1(F))/r \leqslant \alpha$.
Case (ii) When $N \neq 0$. We need to show $d(F) + \alpha(f_1(F) - r(F)) \leqslant 0$. Since N is isotropic, orthogonal α-semistability implies $d(N) + \alpha(f_1(N) - r(N)) \leqslant 0$. Since $d(F) = d(N)$, it suffices to check that $(*) f_1(F) - r(F) \leqslant f_1(N) - r(N)$. Now F/N is a vector bundle of degree 0 with induced nondegenerate quadratic form $\bar{q}$. The image of $F_1(F)$ in $(F/N)_{x_1} \oplus (F/N)_{x_2}$ is isomorphic to $F_1(F)/F_1(N)$ and is an isotropic subspace for $\bar{q}_{x_1} \oplus (-\bar{q}_{x_2})$. Hence $\dim F_1(F)/F_1(N) \leqslant r(F/N)$, i.e., $f_1(F) - r(F) \leqslant f_1(N) - r(N)$. This finishes the proof.

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Moduli for principal bundles over algebraic curves: II

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Abstract. We classify principal bundles on a compact Riemann surface. A moduli space for semistable principal bundles with a reductive structure group is constructed using Mumford's geometric invariant theory.

Keywords. Principal bundles; compact Riemann surface; geometric invariant theory; reductive algebraic groups.

4. Reduction to a quotient space problem

In this section we reduce the problem of constructing coarse moduli schemes for the functors $\mathbf{F}^{\tau}_{ss}$ to one of proving the existence of a good quotient of certain universal spaces under the action of a full linear group.

We recall the definition of a good quotient ([22], Definition 1.5, p. 516).

4.1. DEFINITION

Let $\alpha: H \times T \to T$ be an action of the algebraic group H on the scheme T. A morphism $p: T \to Y$ is called a *good quotient* of T modulo H if the conditions (i), (ii) and (iii) below are satisfied.

i) p is surjective, affine and H-invariant.
ii) $p_*(O^H_T) = O_Y$, where O^H_T is the sheaf of H-invariant functions on T.
iii) If Z is a closed H-stable subset of T then $p(Z)$ is closed in Y; further if Z_1, Z_2 are two closed H-stable subsets of T such that $Z_1 \cap Z_2 = \phi$, then $p(Z_1) \cap p(Z_2) = \phi$.

If in addition the condition (iv) below is also satisfied we call $p: T \to Y$ a *geometric quotient.*

iv) $p(x_1) = p(x_2) \Leftrightarrow$ orbit of x_1 = orbit of x_2 (or equivalently, in view of (iii), all orbits are closed).

4.2. *Remark.* A good quotient is a *categorical quotient*, i.e. given any H-invariant morphism $f: T \to Z$ there is a unique morphism $\tilde{f}: Y \to Z$ such that $f = \tilde{f} \circ p$ ([[22] p. 516).

4.3. *Notation.* Let $\alpha: H \times T \to T$ be an action of the algebraic group H on the scheme T. Then for morphisms $h: S \to H$ and $t: S \to T$ we denote by $h[t]$ the composite $S \xrightarrow{h \times t} H \times T \xrightarrow{\alpha} T$. For any morphism $f: S_1 \to S_2$ we denote by $\bar{f}$ the product $f \times id_X: S_1 \times X \to S_2 \times X$.

This is the second and concluding part of the thesis of late Professor A Ramanathan; the first part was published in the previous issue.

–Editor

If E is a bundle (or more generally a scheme) over a scheme M and $m:S\to M$ is a morphism we denote by E_m the pull back m^*E. For $E'\to M\times X$ we write E'_m instead of $E'_{\bar{m}}$.

4.4. DEFINITION

Let S be a set of isomorphism classes of G-bundles on X. Let $\tilde{\mathbf{F}}^S$ be the sheaf associated to the functor $\mathbf{F}^S$: (Sch)$\to$(Sets) which associates to a scheme T the set of isomorphism classes of families of G-bundles in S parametrized by T. On morphisms $\mathbf{F}^S$ is defined to be pulling back. Let M be a scheme and H an algebraic group acting on M by $\alpha:H\times M\to M$. Let $\mathbf{H}\backslash\mathbf{M}$ be the sheaf associated to the presheaf $H\backslash M(T)=$ the quotient set $\mathrm{Hom}(T,H)\backslash\mathrm{Hom}(T,M)$. We call M a *universal space with group H for the set S* if there is an isomorphism of sheaves Φ: $\tilde{\mathbf{F}}^S\to\mathbf{H}\backslash\mathbf{M}$.

4.5. PROPOSITION

Let S^τ be the set of isomorphism classes of semistable G-bundles of topological type τ. Suppose there is a universal space M with group H for the set S^τ and Φ: $\tilde{\mathbf{F}}^{S^\tau}\to\mathbf{H}\backslash\mathbf{M}$ is the isomorphism of sheaves. Then a good quotient of M modulo H, if it exists, gives a coarse moduli scheme for the functor $\tilde{\mathbf{F}}^\tau_{ss}$ (see Definitions 3.2 and 3.9 : in Part I) in a natural way.

Proof. Suppose $\pi:M\to Y$ is a good quotient of M modulo H.

Clearly $\mathbf{F}^{S^\tau}=\mathbf{F}^\tau_{ss}$ (see Definition 3.1). Therefore we have a morphism Φ: $\mathbf{F}^\tau_{ss}\to\mathbf{H}\backslash\mathbf{M}$. Let h_M, h_Y be the functors represented by M, Y respectively. The morphism $h_\pi:h_M\to h_Y$ induced by $\pi:M\to Y$ gives rise to a morphism Ψ: $\mathbf{H}\backslash\mathbf{M}\to h_Y$ because of the H-invariance of π. We claim that the morphism $\eta=\Psi\circ\Phi:\mathbf{F}^\tau_{ss}\to h_Y$ goes down to a morphism $\bar{\eta}:\bar{\mathbf{F}}^\tau_{ss}\to h_Y$ making Y the coarse moduli scheme for $\bar{\mathbf{F}}^\tau_{ss}$.

Suppose the family $\mathscr{F}\to S\times X$ in S^τ has an admissible reduction of structure group to $P=M\cdot U$. Then by Proposition 3.5 we have a family $\mathscr{F}'\to(\mathbb{C}\times S)\times X$ in S^τ such that $p_S^*(\mathscr{F})|_{\mathbb{C}^*\times S\times X}\approx\mathscr{F}'|_{\mathbb{C}^*\times S\times X}$, where p_S: $\mathbb{C}\times S\times X\to S\times X$ is the projection and $\mathscr{F}'_0\to S\times X$ is isomorphic to $\mathscr{F}[P,M](G)\to S\times X$. Therefore $\eta_S(\mathscr{F}')$: $\mathbb{C}\times S\to Y$ coincides with $\eta_S(p_S^*(\mathscr{F}))$ on $\mathbb{C}^*\times S$ and hence on the whole of $\mathbb{C}\times S$. In particular $\eta_S(\mathscr{F})=\eta_S(\mathscr{F}'_0)$: $S\to Y$. It follows that η goes down to a morphism $\bar{\eta}$: $\bar{\mathbf{F}}^\tau_{ss}\to h_Y$.

That $\bar{\eta}:\bar{\mathbf{F}}^\tau_{ss}(\mathrm{Spec}\,\mathbb{C})\to\mathrm{Hom}(\mathrm{Spec}\,\mathbb{C},Y)$ is surjective follows from the fact that $\pi:M\to Y$ is surjective. To check injectivity we only have to show that if E_1 and E_2 are two semistable G-bundles of type τ on X (considered as a family parametrized by $\mathrm{Spec}\,\mathbb{C}$) such that $\eta(E_1)=\eta(E_2)$ then E_1 and E_2 are equivalent. Let the point $m_i\in M$ represent $\Phi_{\mathbb{C}}(E_i)$. Then by the property (iii) in the definition of a good quotient (Definition 4.1), $\bar{C}_1$ and $\bar{C}_2$, the closures of the H-orbits C_1 and C_2 of m_1 and m_2 respectively, intersect in M. Let $m\in\bar{C}_1\cap\bar{C}_2$. We take the canonical reduced scheme structures on C_i and $\bar{C}_i$, $i=1,2$. Let $[id_M]\in\mathbf{H}\backslash\mathbf{M}(M)$ be the class of the identity morphism of M. The element $\Phi_M^{-1}([id_M])\in\tilde{\mathbf{F}}^\tau_{ss}(M)$ then gives for some neighbourhood U of $m\in M$ a faithfully flat morphism $f:U'\to U$ of schemes and a family of G-bundles $\mathscr{F}\to U'\times X$ in S^τ. Let $C'_i=f^{-1}(C_i)$ and $\bar{C}'_i=f^{-1}(\bar{C}_i)$. Since f is faithfully flat, C'_i is dense open in $\bar{C}'_i$. Since Φ is a morphism it follows easily that for the family $\mathscr{F}|\bar{C}'_i\to\bar{C}'_i\times X$, $\mathscr{F}_x\approx E_i\,\forall x\in C'_i$. Therefore by Proposition 3·24(i) $\mathscr{F}_{m'}$ is equivalent to $E_i\,\forall m'\in\bar{C}'_i$, and in particular for $m'\in\bar{C}'_i$ such that $f(m')=m$. This proves that E_1 and E_2 are equivalent.

To verify the condition (ii) for coarse moduli scheme (Definition 3.2) suppose Z is a scheme and $\chi:\bar{\mathscr{F}}^{\tau}_{ss} \to h_Z$ a morphism. Then it is easy to see that corresponding to $\Phi_M^{-1}([id_M])$ the morphism χ gives a morphism $g:M \to Z$ which is H-invariant. Then Y being the categorical quotient of M modulo H(Remark 3.2), g induces $\tilde{g}: Y \to Z$ such that $g = \tilde{g} \circ \pi$. If $h_g: h_Y \to h_Z$ is the corresponding morphism of functors it follows that $h_{\tilde{g}}$ is the unique morphism which satisfies $\chi = h_{\tilde{g}} \circ \bar{\eta}$.

This proposition reduces the problem of constructing coarse moduli schemes for $\bar{\mathbf{F}}^{\tau}_{ss}$ to one of constructing suitable universal spaces and then proving the existence of quotients. To construct universal spaces for G-bundles we will start with the universal spaces for vector bundles provided by the Quot schemes ([19], § 6). These spaces have a stronger universal property (which we have formulated as a definition; see Definition 4·6 below) which is essential for our construction. By taking an embedding of G in some $GL(n, \mathbb{C})$ we will consider a G-bundle as a vector bundle (or $GL(n, \mathbb{C})$-bundle) with a reduction of structure group to G and thus will construct universal spaces for G-bundles as schemes over the universal spaces for $GL(n, \mathbb{C})$-bundles.

4.6. DEFINITION

Let $\mathscr{S}$ be a set of isomorphism classes of G-bundles on X. Let $\mathscr{E} \to T \times X$ be a family of G-bundles in $\mathscr{S}$. Suppose an algebraic group H acts on T by $\alpha: H \times T \to T$ and also on $\mathscr{E}$ as a group of G-bundle isomorphisms compatible with α, we have the commutative diagram

$$\begin{tikzcd} H \times \mathscr{E} \arrow[rr, "\tilde{\alpha}"] \arrow[dr] & & \tilde{\alpha}^*(\mathscr{E}) \arrow[dl] \\ & H \times T \times X & \end{tikzcd}$$

where $\tilde{\alpha} = \alpha \times id_X$ (cf. 4.3). We call $\mathscr{E} \to T \times X$ a *universal family with group H for the set* $\mathscr{S}$ if the following conditions hold.

i) Given any family of G-bundles $\mathscr{F} \to S \times X$ in $\mathscr{S}$ and a point $s_0 \in S$ there exists an open neighbourhood U of s_0 in S and a morphism $t: U \to T$ such that $\mathscr{F}|_{U \times X} \approx \mathscr{E}_t$ (cf. § 4.3 for notation).

ii) Given two morphisms $t_1, t_2: S \to T$ and an isomorphism $\varphi: \mathscr{E}_{t_1} \approx \mathscr{E}_{t_2}$ there exists a *unique* morphism $h: S \to H$ such that $t_2 = h[t_1]$ and $\varphi = (h \times t_1)^* (\alpha)$ (noting that $(h \times t_1)^* (H \times \mathscr{E}) = \mathscr{E}_{t_1}$ and, since $t_2 = h[t_1]$, $(h \times t_1)^* (\tilde{\alpha}^* \mathscr{E}) = \mathscr{E}_{t_2}$).

4.7. *Remark.* The condition (ii) in particular implies that the isotropy group H_α at $x \in T$ is precisely the automorphism group of $\mathscr{E}_x$.

4.7.1. *Remark.* If $\mathscr{E} \to T \times X$ is a universal family with group H for $\mathscr{S}$ it is clear that T is a universal space with group H for $\mathscr{S}$.

4.8. Let A, A' be two algebraic groups and $\rho: A' \to A$ a homomorphism. Let $\mathscr{E} \to T \times X$ be a family of A bundles. Let $\Gamma(\rho, \mathscr{E})$:(Sch/$T$)$\to$(Sets) be the functor defined by

$\Gamma(\rho,\mathscr{E})(t{:}S\to T)=$ the set of isomorphism classes of pairs $(\mathscr{E}',\varphi)$ where $\mathscr{E}'\to S\times X$ is an A' bundle and φ: $\rho_*\mathscr{E}'\to\mathscr{E}_t$ is an isomorphism of A bundles. The pair $(\mathscr{E}'_1,\varphi_1)$ is isomorphic to the pair $(\mathscr{E}'_2,\varphi_2)$ if there is an A' bundle isomorphism $\psi{:}\mathscr{E}'_1\xrightarrow{\approx}\mathscr{E}'_2$ such that the diagram

$$\begin{tikzcd} \rho_*\mathscr{E}'_1 \arrow[rr, "\rho_*\psi"] \arrow[dr, "\varphi_1"'] & & \rho_*\mathscr{E}'_2 \arrow[dl, "\varphi_2"] \\ & \mathscr{E}_t & \end{tikzcd}$$

commutes. Note that if ρ is injective such a ψ, if it exists, is unique for, since A' acts faithfully on A $\rho_*\psi=\varphi_2^{-1}\circ\varphi_1$ uniquely determines ψ. On morphisms $\Gamma(\rho,\mathscr{E})$ is defined as pulling back.

Let τ be a topological A' bundle on X. Let $\Gamma^\tau(\rho,\mathscr{E})$ be the subfunctor of $\Gamma(\rho,\mathscr{E})$ defined by

$$\Gamma^\tau(\rho,\mathscr{E})(S)=\left\{(\mathscr{E}',\varphi)\in\Gamma(\rho,\mathscr{E})(S)\;\middle|\;\begin{array}{l}\mathscr{E}_S \text{ is topologically}\\ \text{isomorphic to } \tau\,\forall\, s\in S.\end{array}\right\}$$

We then have the following lemma (cf. [17], Proposition 9, §3·5, p. 18).

4.8.1. *Lemma. If $\rho{:}A'\to A$ is injective the functor $\Gamma(\rho,\mathscr{E})$ is representable by a T-scheme $T''\to T$ of locally finite type and a universal pair $(\mathscr{U},u)\in\Gamma(\rho,\mathscr{E})(T'')$. The functor $\Gamma^\tau(\rho,\mathscr{E})$ is representable by an algebraic subscheme T' of T'' and the restriction of $(\mathscr{U},u)$ to T'.*

Proof. Since ρ is injective we identify A' with its image in A. Let $\Gamma=\Gamma(\rho,\mathscr{E})$.

Let $\Gamma'{:}(\mathrm{Sch}/T)\to(\mathrm{Sets})$ be the functor such that $\Gamma'(f{:}S\to T)=\mathrm{Hom}_{S\times X}(S\times X,\bar{f}^*(\mathscr{E}/A'))=\mathrm{Hom}_{T\times X}(S\times X,\ \mathscr{E}/A')$. We define a morphism of functors $\Phi{:}\Gamma'\to\Gamma$ as follows.

Let $\sigma\in\Gamma'(S)=\mathrm{Hom}_{S\times X}(S\times X,\mathscr{E}_f/A')$. Define $\Phi_S(\sigma)=(\sigma^*\mathscr{E}_f,\ \varphi_\sigma)$ where $\varphi_\sigma{:}\rho_*\sigma^*\mathscr{E}_f\to\mathscr{E}_f$ is induced by $((S\times X)_{\mathscr{E}_{f/A}}\times\mathscr{E}_f)\times A\to\mathscr{E}_f$, $(s,x,e,a)\mapsto e.a$ where $s\in S$, $x\in X$, $e\in\mathscr{E}_f$ and $a\in A$.

We can also define an inverse morphism $\Psi{:}\Gamma\to\Gamma'$. Let $(\mathscr{E}',\varphi)\in\Gamma(f{:}S\to T)$. Then the fiber bundle associated to $\rho_*\mathscr{E}'$ with fiber A/A' is canonically isomorphic to the fiber bundle associated to $\mathscr{E}$ with fiber A/A'. Since A' leaves the coset (A') of A/A' invariant we have a canonical section σ of $(\rho_*\mathscr{E}')/A'$. Using φ this gives a section, again denoted by σ, of $\mathscr{E}_f/A'$. Define $\Psi_S((\mathscr{E}',\varphi))=\sigma$.

It is easy to check that $\Phi\circ\Psi=id_\Gamma$ and $\Psi\circ\Phi=id_{\Gamma'}$. Thus the functors Γ and Γ' are isomorphic. We shall show that the functor Γ' is representable using the results of ([TDTE, IV]).

By Chevalley's semi-invariants theorem ([2], Theorem 5.1, p. 161) there is a representation of A on a vector space V with a line $l\subset V$ such that A' is the stabilizer of l in A. Let χ^{-1} be the character by which A' acts on l. Then the line bundle L on A/A' associated to the A'-bundle $A\to A/A'$ is the ample line bundle corresponding to the embedding of A/A' in $\mathbb{P}(V)$. Then the line bundle $\mathscr{L}$ on $\mathscr{E}/A'$ associated to the A'-bundle $\mathscr{E}\to\mathscr{E}/A'$ by

the character χ is relatively ample for the morphism $\mathscr{E}/A' \to T \times X$, for it corresponds to the embedding $\mathscr{E}/A' \subsetneq \mathbb{P}(\mathscr{E}(V))$ induced by $A/A' \subsetneq \mathbb{P}(V)$. Therefore, we see that $\mathscr{E}/A'$ is quasi-projective over $T \times X$ and hence $\mathscr{E}/A' \to T$ is also quasi-projective. Therefore it follows from ([TDTE, IV] §4, C. pp. 19–20) that Γ' is representable by a scheme T'' $(= \Pi_{T \times X/T}(\mathscr{E}/A')/T \times X$, in the notation of ([TDTE, II], C. n° 2, pp. 12, 13)) of locally finite type. In fact T'' is an open subscheme of $\mathrm{Hilb}_{(\mathscr{E}/A')/T}$ whose closed points correspond to subschemes of $\mathscr{E}/A'$ which map isomorphically onto $t \times X$, for some $t \times T$, under the projection $\mathscr{E}/A' \to T \times X$ (*loc. cit.*). Therefore if a section $\sigma: t \times X \to \mathscr{E}/A'$ in $\Gamma'(t \subsetneq T)$ is such that the A'-bundle $\sigma^*(\mathscr{E}_t)$ on $t \times X \simeq X$ is of topological type τ then the Hilbert polynomial of the subscheme $\sigma(X) \simeq X$ of $\mathscr{E}_t/A'$ corresponding to the section σ (with respect to the ample line bundle $\mathscr{L}_t$ on $\mathscr{E}_t/A'$) is determined by τ since the restriction of $\mathscr{L}_t$ to $\sigma(X) \simeq X$ is topologically isomorphic to $\chi_*(\tau)$ so that its degree depends only on τ. Since subschemes with a fixed Hilbert polynomial are represented by an algebraic subscheme of Hilb ([TDTE, IV] pp. 17, 20) it follows that Γ^τ is represented by an algebraic subscheme T' of T''.

4.8.2. *Remark.* If A' and A are reductive groups then A/A' is affine and we can take a representation of A in V such that the character χ is trivial, so that A/A' is embedded in V itself. In this case, therefore, it follows that T'' is itself already algebraic.

4.8.3. *Remark.* If P is a parabolic subgroup of G and $\mathscr{E} \to Y$ is a G-bundle on a scheme Y then $\mathscr{E}/P \to Y$ is a projective morphism. Since G/P is projective, this follows as in the proof of the lemma above (taking $A = G$ and $A' = P$).

4.9 If ρ is not an injection the functor $\Gamma(\rho, \mathscr{E})$ may not be a sheaf. Let $\tilde{\Gamma}(\rho, \mathscr{E})$ be the sheaf associated to the functor $\Gamma(\rho, \mathscr{E})$. The following lemma shows that we can construct a universal *space* for A'-bundles starting from a universal *family* $\mathscr{E}$ of A-bundles when $\tilde{\Gamma}(\rho, \mathscr{E})$ is representable and if $\Gamma(\rho, \mathscr{E})$ itself is representable, then we can actually construct a universal *family* for A'-bundles. So taking an embedding $G \subsetneq GL(n, \mathbb{C})$ and starting with a universal family for vector bundles we can construct a universal family for G-bundles. But then to prove the existence of coarse moduli scheme for $\tilde{\mathbf{F}}^\tau_{ss}$ we have to prove the existence of a good quotient of the parameter scheme. For this it is convenient to take the adjoint representation. The existence of a good quotient reduces to proving that a certain morphism is proper and if we take the adjoint representation this follows from the (local) rigidity of the Lie algebra structure of a semisimple Lie algebra (see §5 below). But the adjoint representation is not faithful and hence we construct universal families in two steps, first from vector bundles to G/Z-bundles and then from G/Z-bundles to G-bundles. This involves the representability of the functor $\Gamma(\rho, \mathscr{E})$ where (essentially) ρ is the projection $G \to G/Z$. But this functor is not a sheaf (e.g. $\mathbb{C}^* \to 1 = \mathbb{C}^*/\mathbb{C}^*$, cf. ([TDTE, V, §1])) and we are forced to take the associated sheaf $\tilde{\Gamma}(\rho, \mathscr{E})$ which we can prove to be representable by identifying it with a suitable Picard functor (Lemma 4.15.1) If $\tilde{\Gamma}(\rho, \mathscr{E})$ alone is representable we can construct only a universal *space* for G-bundles even starting from a universal *family* for G/Z-bundles. But by Proposition 4.5 this is enough to prove the existence of a coarse moduli scheme for $\bar{\mathbf{F}}^\tau_{ss}$.

4.10. *Lemma. Suppose the family $\mathscr{E} \to T \times X$ is a universal family with group H for a set $\mathscr{S}$ of A-bundles. Also suppose that the sheaf $\tilde{\Gamma}(\rho, \mathscr{E})$ is representable by a scheme M.*

(i) *The group H can be made to act on M in a natural way and M with this action of H then becomes a universal space with group H for the set $\mathcal{S}'$ of A'-bundles which give A-bundles in $\mathcal{S}$ on extending the structure group by $\rho: A' \to A$.*

(ii) *Moreover if $\rho: A' \to A$ is an injection so that $\Gamma(\rho, \mathcal{E})$ itself is representable and we have a universal pair $(\mathcal{U}, u) \in \Gamma(\rho, \mathcal{E})(M)$* (Lemma 4.8.1.) *the group H can be made to act in a natural way, on $\mathcal{U}$ as a group of A'-bundle isomorphisms compatible with its action on M ((i) above). With this action of H, $\mathcal{U}$ then becomes a universal family with group H for $\mathcal{S}'$.*

Proof. (i) To give the action of H on M we describe the action of $\mathrm{Hom}(S, H)$ on $\mathrm{Hom}(S, M)$ for any scheme S. Let $h \in \mathrm{Hom}(S, H)$ and $m \in \mathrm{Hom}(S, M)$. Let $\pi: M \to T$ be the structural morphism and $t = \pi \circ m$. Since M represents the sheaf $\tilde{\Gamma}(\rho, \mathcal{E})$ corresponding to the morphism m we have an open covering $\{U_i\}$ of S, faithfully flat morphisms $f_i: U_i' \to U_i$, A'-bundles $\mathcal{E}_i' \to U_i' \times X$ and A-bundles isomorphisms $\varphi_i: \rho_* \mathcal{E}_i' \to \overline{(t \circ f_i)}^* \mathcal{E}$. We define $h[m]$ to be the morphism from S to M corresponding to the element $(\mathcal{E}_i', \tilde{\alpha}_{h \times t} \circ \varphi_i)$ in $\tilde{\Gamma}(\rho, \mathcal{E})(S)$ where $\tilde{\alpha}_{h \times t} = \overline{(h \times t)}^*(\tilde{\alpha})$ and $\tilde{\alpha}: H \times \mathcal{E} \to \bar{\alpha}^* \mathcal{E}$ gives the action of H on $\mathcal{E}$ (Definition 4.6). Then it is easy to see that we have indeed an action of H on M and that $h[t] = \pi \circ h[m]$. To prove that M is a universal space let $\mathcal{E}' \to S \times X \in \mathbf{F}^{\mathcal{S}'}(S)$ i.e. a family of A'-bundles in $\mathcal{S}'$. By extending the structure group by ρ we get a family of A-bundles $\rho_* \mathcal{E}'$ in $\mathcal{S}$. Since T is universal this gives an open covering $\{U_i\}$ of S and morphisms $f_i: U_i \to T$ such that $\mathcal{E}_{f_i} \xrightarrow{\approx} \rho_* \mathcal{E}'|_{U_i \times X}$. This then gives morphisms $f_i': U_i \to M$. Using condition (ii) of Definition 4.6 satisfied by $\mathcal{E} \to T \times X$ these f_i' are seen to define an element of $\mathbf{H} \backslash \mathbf{M}(S)$. It is easy to check that by associating this element of $\mathbf{H} \backslash \mathbf{M}(S)$ to $\mathcal{E}' \in \mathbf{F}^{\mathcal{S}'}(S)$ we have an isomorphism of sheaves $\tilde{\mathbf{F}}^{\mathcal{S}'} \to \mathbf{H} \backslash \mathbf{M}$ (see proof of (ii) below, locally, in the faithfully flat topology, the arguments run on the same lines).

(ii) In this case we can define $h[m]$ as the morphisms from S to M corresponding to the pair $(\mathcal{U}_m, \tilde{\alpha}_{h \times t} \circ u_m)$. Therefore by definition the pair $(\mathcal{U}_m, \tilde{\alpha}_{h \times t} \circ u_m)$ is isomorphic to $(\mathcal{U}_m, u_{h[m]})$, and hence there is an isomorphism (which is unique since ρ is injective, cf. § 4.8) $\tilde{\beta}_{h \times t}: \mathcal{U}_m \to \mathcal{U}_{h[m]}$ making the diagram

$$\begin{tikzcd}
\rho_* \mathcal{U}_m \arrow[r, "\rho_* \tilde{\beta}_{h \times m}"] \arrow[d, "u_m"'] & \rho_* \mathcal{U}_{h[m]} \arrow[d, "u_{h[m]}"] \\
\mathcal{E}_t \arrow[r, "\tilde{\alpha}_{h \times t}"'] & \mathcal{E}_{h[t]}
\end{tikzcd}$$

commutative. Taking $S = H \times M$ and h, m to be the projections $p_H: H \times M \to H$, $p_M: H \times M \to M$ respectively, in the above we get the action $p_H[p_M] = \beta: H \times M \to M$ of H on M and $\tilde{\beta}_{p_H \times p_M} = \tilde{\beta}: H \times \mathcal{U} \to \bar{\beta}^* \mathcal{U}$ which gives the action of H on $\mathcal{U}$.

The condition (i) of Definition 4.6 for $\mathcal{U}$ follows immediately from the universal properties of M and $\mathcal{E}$. To check condition (ii) of Definition 4.6, let $m_1, m_2: S \to M$ and $\varphi: \mathcal{U}_{m_1} \to \mathcal{U}_{m_2}$ an isomorphism be given. Let $t_1 = \pi \circ m_1$ and $t^2 = \pi \circ m^2$. Define φ' by the commutative diagram:

$$\begin{tikzcd} p_* \mathscr{U}_{m_1} \arrow[r, "p_*\varphi"] \arrow[d, "u_{m_1}"'] & p_* \mathscr{U}_{m_2} \arrow[d, "u_{m_2}"] \\ \mathscr{E}_{t_1} \arrow[r, "\varphi'"'] & \mathscr{E}_{t_2} \end{tikzcd} \tag{$*$}$$

By the universal property of $\mathscr{E}$ there is a unique morphism $h:S \to H$ such that $t_2 = h[t_1]$ and $\varphi' = \tilde{\alpha}_{h \times t_1}$. Now m_2 is defined by the pair $(\mathscr{U}_{m_2}, u_{m_2})$ and $h[m_1]$ by the pair $(\mathscr{U}_{m_1}, \tilde{\alpha}_{h \times t_1} \circ u_{m_1})$. But from the diagram $(*)$ and the observation above, it follows that $\varphi:\mathscr{U}_{m_1} \to \mathscr{U}_{m_2}$ gives an isomorphism of the pairs $(\mathscr{U}_{m_1}, \tilde{\alpha}_{h \times t_1} \circ u_{m_1})$ and $(\mathscr{U}_{m_2}, u_{m_2})$. Therefore $m_2 = h[m_1]$, as required to be shown.

4.11. We shall now construct universal families for semistable vector bundles. This is essentially contained in ([19], § 6; [20], § 3). We have made it a little more explicit to suit our purposes.

Let $\mathscr{S}^{r,d}$ be the set of isomorphism classes of semistable vector bundles of rank r and degree d. Let L be an ample line bundle on X of degree d_0. Since $\mathscr{S}^{r,d}$ is bounded i.e. there is a family of vector bundles on X parametrized by an algebraic scheme in which every element of $\mathscr{S}^{r,d}$ occurs ([19], Proposition 3.2, p. 307), we can find an integer m_0 such that for any $m \geqslant m_0$ and every $V \in \mathscr{S}^{r',d}$, $r' \leqslant r$, $H^i(X, V \otimes L^m) = 0$ for $i > 0$ and $H^0(X, V \otimes L^m)$ generates $V \otimes L^m$. Then $H^0(X, V \otimes L^m)$ has same rank, say n, for all $V \in \mathscr{S}^{r,d}$.

4:11.1. Let $P \subset GL(n, \mathbb{C})$ be the parabolic subgroup defined as the stabilizer of the subspace $\mathbb{C}^{n-r} \subset \mathbb{C}^n$. The decomposition $\mathbb{C}^n = \mathbb{C}^{n-r} \oplus \mathbb{C}^r$ gives a homomorphism $P \to GL(r, \mathbb{C})$. Given a P-bundle on a scheme Y the representations $P \subsetneq GL(n, \mathbb{C})$ and $P \to GL(r, \mathbb{C})$ give rise to vector bundles V_1 and V_2 (of ranks n and r respectively) on Y and there is a surjective homomorphism $V_1 \to V_2$ induced by the P-equivariant projection $\mathbb{C}^n \to \mathbb{C}^r$. Conversely given two vector bundles V_1 and V_2 of rank n and r respectively and a surjective homomorphism $V_1 \to V_2 \to 0$ we can construct a P-bundle as follows. For any open set $U \subset Y$ and a faithfully flat morphism $f: U' \to U$ associate the set of all vector bundle isomorphisms φ_1, φ_2 making the diagram

$$\begin{tikzcd} U' \times \mathbb{C}^n \arrow[r] \arrow[d, "\varphi_1"] & U' \times \mathbb{C}^r \arrow[r] \arrow[d, "\varphi_2"] & 0 \\ f^* V_1 \arrow[r] & f^* V_2 \arrow[r] & 0 \end{tikzcd}$$

commutative. Thus we get a sheaf $\mathscr{F}$ for the faithfully flat topology and the trivial group scheme $Y \times P$ over Y given by P acts on $\mathscr{F}$ by $(\varphi_1, \varphi_2)p = (\varphi_1 \circ p, \varphi_2 \circ p)$, $p \in P$. Then $\mathscr{F}$ is a principal homogeneous space under $Y \times P$ and by descent ([SGA, 3] exposé XXIV) it is representable by a P-bundle over Y. Thus we can consider a P-bundle as a surjective homomorphism $V_1 \to V_2 \to 0$ of vector bundles. Then an isomorphism of the P-bundles $V_1 \to V_2 \to 0$ and $V'_1 \to V'_2 \to 0$ is given by a commutative diagram

$$\begin{array}{ccccc} V_1 & \longrightarrow & V_2 & \longrightarrow & 0 \\ \downarrow \approx & & \downarrow \approx & & \\ V'_1 & \longrightarrow & V'_2 & \longrightarrow & 0 \end{array}$$

Since $GL(n,\mathbb{C}) \to GL(n,\mathbb{C})/P$ is locally trivial ([3], Theorem 4.13, p. 90) any P-bundle is in fact locally trivial ([17], Theorem 2, §4.3, p. 1–24).

4.11.2. The trivial bundle $I_n = X \times \mathbb{C}^n \to X$ thought of as a family of vector bundles on X parametrized by a point, with the group $GL(n,\mathbb{C})$ acting in the natural way on I_n gives a universal family with group $GL(n,\mathbb{C})$ for the singleton set $\{I_n\}$. (Condition (ii) of Definition 4.6 is obviously satisfied and condition (i) follows from ([11], Lecture 7, p. 51, (ii) and (iii)).) So applying Lemma 4.8.1 with $A' = P \subset GL(n,\mathbb{C}) = A$ and fixing the topological type of the P-bundle such that we get by the extension of structure group $P \to GL(r,\mathbb{C})$ vector bundles of degree d (since the topological type of a vector bundle on X is determined by its degree and rank and any extension splits topologically this condition fixes the topological type of the P-bundle; cf. 4.11.1), we get a universal family of P-bundles $\mathscr{U} \to M \times X$ with group $GL(n,\mathbb{C})$ for the set of P-bundles of the fixed topological type which give I_n on extending the structure group by $P \to GL(n,\mathbb{C})$, parametrized by an algebraic scheme M. Let $\mathcal{O}$ be the $GL(r,\mathbb{C})$-bundle obtained from $\mathscr{U}$ by the extension of structure group $P \to GL(r,\mathbb{C})$.

4.11.3. By our constructions in §4.8 it follows that M is an open subscheme of the locally finite type scheme $\mathrm{Hom}(X, G_{n,r})$ which represents the functor Γ', $\Gamma'(S) = \mathrm{Hom}(S \times X, G_{n,r})$ where $G_{n,r} = GL(n,\mathbb{C})/P$ is the Grassmannian of r-dimensional quotients of $\mathbb{C}^n$. Then $\mathscr{U}$ is the pull back of the P-bundle $GL(n,\mathbb{C}) \to GL(n,\mathbb{C})/P$ by the universal section in $\mathrm{Hom}(M \times X, G_{n,r})$ and $\mathcal{O}$ is the pull back of the $GL(r,\mathbb{C})$-bundle* $Q \to G_{n,r}$ obtained from $GL(n,\mathbb{C}) \to G_{n,r}$ by the extension of structure group $P \to GL(r,\mathbb{C})$. The P-bundle $\mathscr{U}$ corresponds to the surjection $I_n \to \mathcal{O} \to 0$ which is the pull back of the surjection $G_{n,r} \times \mathbb{C}^n \to Q \to 0$ induced by $\mathbb{C}^n \to \mathbb{C}^r$. (Note that $GL(n,\mathbb{C}) \to G_{n,r}$ becomes trivial when we extend the structure group by $P \subsetneq GL(n,\mathbb{C})$.)

4.11.4. Let

$$R = \left\{ q \in M \,\middle|\, \begin{array}{l} \mathcal{O}_q \text{ is semistable and the canonical map} \\ I_n \to H^0(X, \mathcal{O}_q) \text{ is an isomorphism} \end{array} \right\}.$$

It follows from the semi-continuity theorem and the fact that the points corresponding to semistable bundles form an open subset of the parameter scheme in any family of vector bundles ([19], Corollary 7.2, p. 332) that R is an open subset of M. We take R with the open subscheme structure induced from M. Clearly R is stable under the action of $GL(n,\mathbb{C})$. We denote the restriction of $\mathcal{O}$ to $R \times X$ also by $\mathcal{O}$. Let $\mathcal{O}(-m) = \mathcal{O} \otimes p_X^*(L^{-m})$ where $p_X : R \times X \to X$ is the projection. Let $GL(n,\mathbb{C})$ act on $\mathcal{O}(-m)$ by its action on $\mathcal{O}$ and the trivial action on $p_X^*(L^{-m})$.

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*Editor's Note: For the definition of Q see Introduction, in part I. It is the principal $GL(r)$ bundle associated to the universal quotient bundle on $G_{n,r}$.

4.11.5. PROPOSITION

The family $\mathcal{O}(-m)\to R\times X$ with this action of $GL(n,\mathbb{C})$ gives a universal family with group $GL(n,\mathbb{C})$ for the set $\mathcal{S}^{r,d}$ of semistable vector bundles of rank r and degree d.

Proof. Let $\mathcal{F}\to S\times X$ be a family in $\mathcal{S}^{r,d}$ and $s_0\in S$. The direct image $p_{S_*}(\mathcal{F}(m))$ is locally free ([11], Lecture 7, p. 51). Choose a trivialization $P_{S_*}(\mathcal{F}(m))|_U\approx U\times\mathbb{C}^n$ in a neighbourhood U of s_0. Then we have a surjection $p_S^*(U\times\mathbb{C}^n)=I_n\to\mathcal{F}(m)|_{U\times X}\to 0$. This gives a reduction of structure group of the trivial $GL(n,\mathbb{C})$ bundle I_n to P (cf. 4.11.1). Then by the universal property of the P-bundle $\mathcal{U}$ on $R\times X$ we get a morphism $f:U'\to R$, $s_0\in U'\subset U$, inducing the P-bundle $I^n\to\mathcal{F}(m)|_{U'}\to 0$. Extending the structure group by $P\to GL(r,\mathbb{C})$ and tensoring by L^{-m} we get $\bar{f}(\mathcal{O}(-m))\approx\mathcal{F}|_{U'\times X}$.

To check condition (ii) of Definition 4.6, let $r_1, r_2:S\to R$ and $\varphi:\mathcal{O}_{r_1}(-m)\to\mathcal{O}_{r_2}(-m)$ be given. Let $\varphi':\mathcal{O}_{r_1}\to\mathcal{O}_{r_2}$ be the isomorphism induced by φ.

Then φ' induces an isomorphism $\varphi'':p_{S_*}(\mathcal{O}_{r_1})\xrightarrow{\approx}p_{S_*}(\mathcal{O}_{r_2})$. But by commutativity under base change (since $H^1(X,\mathcal{O}_q)=0\,\forall q\in R$([11], Lecture 7, p. 51)), $p_{S_*}(\mathcal{O}_{r_i})$ is canonically isomorphic to $r_i^*(p_{R_*}\mathcal{O})$, $i=1,2$. Since $p_{R_*}\mathcal{O}$ is canonically isomorphic to the trivial bundle we have the commutative diagram

$$\begin{tikzcd}
r^*{}_1(I_n) \arrow[r] \arrow[d, "\varphi''"] & \mathcal{O}_{r_1} \arrow[r] \arrow[d, "\varphi'"] & 0 \\
r^*{}_2(I_n) \arrow[r] & \mathcal{O}_{r_2} \arrow[r] & 0
\end{tikzcd}$$

Now use the universal property for the P-bundle $\mathcal{U}$(cf, 4.11.1).

4.11.6. *Remark.* If $\mathcal{U}$ is any universal family of vector bundles with group H for a set $\mathcal{S}$ of vector bundles then $\mathcal{U}\otimes L_1$ is a universal family with group H(H acting trivially on L_1) for the set $\mathcal{S}\otimes L_1=\{V\otimes L_1|V\in\mathcal{S}\}$, L_1 being any line bundle. Therefore it follows from Proposition 4.11.5 that $\mathcal{O}(q)$ is a universal family for $\mathcal{S}^{r,d+rd_0(m+q)}$ the set of isomorphism classes of semistable vector bundles of rank r and degree $d+rd_0(m+q)$, $q\in\mathbb{Z}$.

4.11.7. *Remark.* It is easy to see that the scheme R above is the same as scheme R^{ss} which Seshadri constructs in ([20], § 3; [19], § 6).

4.12. PROPOSITION

The set of isomorphism classes of semistable G-bundles of a fixed topological type τ is bounded, i.e. there exists a family $\mathcal{E}\to M\times X$ of G-bundles such that given any semistable G-bundle E of type τ there is an $m\in M$ such that $E\approx\mathcal{E}_m$.

Proof. Let $\rho:G\to GL(V)$ be a faithful representation. Let $V=V_1\oplus\cdots\oplus V_q$ be a decomposition into irreducible subspaces. Let r_i be the rank of V_i and d_i be the degree of the vector bundle $\rho_{i_*}(\tau)$ where $\rho_i:G\to GL(V_i)$ gives the action of G on V_i. Let $u_i\to M_i\times X$ be

the universal family for the set $\mathscr{S}^{r_i,d_i}$ and $U = U_1 \underset{X}{\times} \cdots \underset{X}{\times} U_r \to (M_1 \times \cdots \times M_r) \times X$. Let $\mathscr{E} \to M \times X$ represent the functor $\Gamma(\rho', U)$ (§§4.8, 4.8.1), where $\rho' = \rho_1 \times \cdots \times \rho_r$: $G \to GL(V_1) \times \cdots \times GL(V_r)$; use Proposition 3.17 to see that this ρ satisfies what we require.

4.13. Let Aut $\mathscr{G}'$ be the group of Lie algebra automorphisms of $\mathscr{G}'$. Let dim $\mathscr{G}' = r$. The group Aut $\mathscr{G}'$ may not be connected and Ad $G = G/Z$ is the connected component of identity of Aut $\mathscr{G}'$. Let $\rho: \mathbb{C}^* \times \text{Aut}\,\mathscr{G}' \hookrightarrow GL(\mathscr{G}')$ be the natural inclusion. Note that, in $GL(\mathscr{G}')$, $\mathbb{C}^* \cap \text{Aut}\,\mathscr{G}'$ is trivial. Applying Lemma 4.8.1. for this ρ and the universal family $\mathscr{O} \to R \times X$ for $\mathscr{S}^{r,rmd_0}$ to get a universal family $\mathscr{E}' \to R_1' \times X$ with group $GL(n, \mathbb{C})$ for $\mathbb{C}^* \times \text{Aut}\mathscr{G}'$-bundles which under the extension of structure group ρ give semistable vector bundles of degree rmd_0. Since any Aut$\mathscr{G}'$-bundle gives a vector bundle of degree zero on extension of structure group by Aut$\mathscr{G}' \hookrightarrow GL(\mathscr{G}')$ (Aut$\mathscr{G}'$ is contained in the orthogonal group corresponding to the Killing form) it follows that for the extension of structure group $\mathbb{C}^* \times \text{Aut}\mathscr{G}' \to \mathbb{C}^*$, $\mathscr{E}'$ gives a family of line bundles $\mathscr{E}'(\mathbb{C}^*)$ of degree md_0 on X. Let J^{md_0} be the Jacobian of X of line bundles of degree md_0. Then by the universal property of J^{md_0} we have a morphism $R_1' \to J^{md_0}$ corresponding to $\mathscr{E}'(\mathbb{C}^*) \to R_1' \times X$. Let R_1 be the fiber over $L^m \in J^{md_0}$ under this morphism. Restricting $\mathscr{E}'$ to $R_1 \times X$ and extending the structure group by $\mathbb{C}^* \times \text{Aut}\mathscr{G}' \to \text{Aut}\mathscr{G}'$ we get an Aut$\mathscr{G}'$-bundle $\mathscr{E}_1 \to R_1 \times X$. The action of $GL(n, \mathbb{C})$ on $\mathscr{E}' \to R_1' \times X$ gives an action of $GL(n, \mathbb{C})$ on $\mathscr{E}_1 \to R_1 \times X$.

4.13.1. PROPOSITION

The Aut$\mathscr{G}'$*-bundle* $\mathscr{E}_1 \to R_1 \times X$, *with the natural action of* $GL(n, \mathbb{C})$, *constructed above is a universal family with group* $GL(n, \mathbb{C})$ *for the set of isomorphism classes of* Aut$\mathscr{G}'$*-bundles which give semistable vector bundles of degree zero on extension of structure group by the inclusion* Aut$\mathscr{G}' \hookrightarrow GL(\mathscr{G}')$. *Moreover the parametrizing scheme* R_1 *is non-singular.*

Proof. The universal property of $\mathscr{E}_1$ is clear from the above discussion. The non singularity of R_1 is proved in the following two lemmas (4.13.3 and 4.13.4).

4.13.2. *Remark.* The elements of the tensor space $\mathscr{H} = \mathscr{G}'^* \otimes \mathscr{G}'^* \otimes \mathscr{G}' = \text{Hom}$ $(\mathscr{G}' \otimes \mathscr{G}', \mathscr{G}')$ give algebra structures on $\mathscr{G}'$. Those elements of $\mathscr{H}$ which give algebra structures which satisfy the Jacobi identity and skew symmetry form a closed subvariety of $\mathscr{H}$ and give Lie algebra structures on $\mathscr{G}'$. The points of the variety $Y = GL(\mathscr{G}')/\mathbb{C}^* \times \text{Aut}\mathscr{G}' \subset \mathbb{P}(\mathscr{H})$ then give Lie algebra structures on $\mathscr{G}'$, determined up to a scalar and isomorphic to the original Lie algebra structure of $\mathscr{G}'$. If $0 \neq h \in \mathscr{H}$ is such that $\{h\} \in \bar{Y}$ and the Lie algebra structure of $\mathscr{G}'$ given by h is semisimple then $\{h\} \in Y$ ([16], Corollary 4.3, p. 514). This fact will be crucial for us in proving the existence of quotient space in the next section (cf. proof of Lemma 5.6) and is the reason why we have chosen the adjoint representation for constructing universal families.

Let $r_1 \in R_1 \subset R_1'$ and $r \in R$ its image. Then the $\mathbb{C}^* \times \text{Aut}\mathscr{G}'$-bundle $\mathscr{E}'_{r_1}$ gives the vector bundle $\mathscr{O}_r$ under the extension of structure group $\mathbb{C}^* \times \text{Aut}\mathscr{G}' \to GL(\mathscr{G}')$ and the Aut$\mathscr{G}'$-bundle $\mathscr{E}_{r_1}$ gives the vector bundle $\mathscr{O}_r(-m)$ under Aut$\mathscr{G}' \hookrightarrow GL(\mathscr{G}')$. If we take the embedding $Y \subset \mathbb{P}(\mathscr{H})$, r_1 gives a section $r_1: X \to Q_r(Y) \subset \mathbb{P}(\mathscr{O}_r^* \otimes \mathscr{O}_r^* \otimes \mathscr{O}_r)$ such that $r_1^*(\Lambda) = L^{-m}$ where Λ is the tautological line bundle on $\mathbb{P}(\mathscr{O}_r^* \otimes \mathscr{O}_r^* \otimes \mathscr{O}_r)$

corresponding to the vector bundle $\mathcal{O}_r^* \otimes \mathcal{O}_r^* \otimes \mathcal{O}_r$. Since $\mathbb{P}(\mathcal{O}_r^* \otimes \mathcal{O}_r^* \otimes \mathcal{O}_r) = \mathbb{P}(\mathcal{O}_r(-m)^* \otimes \mathcal{O}_r(-m)^* \otimes \mathcal{O}_r(-m))$, r_1 also gives a section $r_1: X \to \mathbb{P}(\mathcal{O}_r^* \otimes \mathcal{O}_r^* \otimes \mathcal{O}_r \otimes L^m)$, and $r_1(\Lambda') = L^{-m} \otimes L^m = 1$, where $\Lambda' = \Lambda \otimes p_X^* L^m$. Therefore r_1 gives a section s, determined up to a scalar, in $H^0(X, \mathcal{O}_r(-m)^* \otimes \mathcal{O}_r(-m)^* \otimes \mathcal{O}_r(-m))$ such that $s(x) \neq 0\ \forall x \in X$ and $s(x)$ is a Lie algebra structure on the fiber of $\mathcal{O}_r(-m)$ over x, isomorphic to the natural Lie algebra structure of $\mathcal{G}'$.

Since Y is embedded in the *projective* space $\mathbb{P}(\mathcal{H})$ it is easy to see that the scalars in $GL(n, \mathbb{C})$ act trivially on R_1. This is a reason why we have constructed a universal family for Aut $\mathcal{G}'$-bundles first working with $\mathbb{C}^* \times \operatorname{Aut}\mathcal{G}'$ instead of directly using the inclusion $\operatorname{Aut}\mathcal{G}' \hookrightarrow GL(\mathcal{G}')$.

4.13.3. *Lemma. The schemes R and R_1' are nonsingular and* $\dim R = n^2 + r^2(g-1)$ *and* $\dim R_1' = n^2 + (r+1)(g-1)$.

Proof. We use the notation of §4.11.3. Let $Y = GL(\mathcal{G}')/\mathbb{C}^* \times \operatorname{Aut}\mathcal{G}'$ and $Q(Y)$ be the fiber bundle with fiber Y associated to Q considered as a $GL(\mathcal{G}')(= GL(r, \mathbb{C}))$-bundle. The scheme R is an open subscheme of $\operatorname{Hom}(X, G_{n,r})$ and R_1 is an open subscheme of $\operatorname{Hom}(X, Q(Y))$. The morphism $R_1 \to R$ is induced by $\varphi: Q(Y) \to G_{n,r}$. By associating a morphism $f: X \to Q(Y)$ to its graph Γ_f in $X \times Q(Y)$, $\operatorname{Hom}(X, Q(Y))$ becomes an open subscheme of $\operatorname{Hilb}(X \times Q(Y))$ ([TDTE, IV], §4, pp. 19, 20). The graph $\Gamma_f \approx X$ of $f \in R_1'$ is a nonsingular complete subvariety of the non-singular variety $X \times Q(Y)$. Therefore the obstruction to the smoothness of the Hilbert scheme at Γ_f is an element of $H^1(\Gamma_f, N_{\Gamma_f})$ where N_{Γ_f} is the normal bundle of Γ_f in $X \times Q(Y)$ ([TDTE, IV], §5; cf. also [9]). Identifying Γ_f with X by the projection $X \times Q(Y) \to X$, $N_{\Gamma_f} \approx f^*(T_q)$ where T_q is the tangent bundle of $Q(Y)$. We shall show that $H^1(X, f^*(T_q)) = 0$ from which it will follow that R_1' is nonsingular.

We have the following diagram of vector bundles on X, which is commutative and exact (in the obvious sense)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to K \to & & (\varphi f)^*(\operatorname{Ad} Q) & \longrightarrow & f^*(T'_q) & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
0 \to K \to & & (\varphi f)^*(M) & \longrightarrow & f^*(T_q) & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & (\varphi f)^*(T_g) & \underset{=}{\longrightarrow} & (\varphi f)^*(T_g) & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{1}$$

where T_q (respectively T_g) is the tangent bundle of $Q(Y)$ (resp. $G_{n,r}$) and T_q' is the tangent bundle along fibers of $\varphi: Q(Y) \to G_{n,r}$, the canonical map.

The first column is the pull back by φf of the Atiyah exact sequence of $Q \to G_{n,r}$. The horizontal arrows from the first to the second column are induced by the differential of the projection $Q \to Q(Y) = Q/\mathbb{C}^* \times \operatorname{Aut}\mathcal{G}'$.

Let $\varphi f \in R$ correspond to the quotient

$$0 \to H_f \to I_n \to F_f \to 0, \tag{2}$$

i.e. $F^*(Q) = F_f$. Then $(\varphi f)(T_g) = H_f^* \otimes F_f$ ([TDTE] IV, § 5).

It is easy to check the following identifications:

$$\begin{array}{ccccccccc} 0 \to & (\varphi f)^*(\mathrm{Ad}\, Q) & \longrightarrow & (\varphi f)^*(M) & \longrightarrow & (\varphi f)^*(T_g) & \to 0 \\ & \Big\downarrow \approx & & \Big\downarrow \approx & & \Big\downarrow \approx & \\ 0 \to & F^*{}_f \otimes F_f & \longrightarrow & I_n \otimes F_f & \longrightarrow & H^*{}_f \otimes F_f & \to 0 \end{array} \tag{3}$$

where the second row is obtained from (2) by dualizing and tensoring by F_f.

Also, K is the adjoint bundle of the $\mathbb{C}^* \times \mathrm{Aut}\,\mathscr{G}'$ bundle obtained from the reduction of structure group of F_f to $\mathbb{C}^* \times \mathrm{Aut}\,\mathscr{G}'$ corresponding to f.

From the cohomology exact sequence of the bottom row of (3) we get $H^1(X, (\varphi f)^* M) = H^1(X, I_n \otimes F_f) = n$-copies of $H^1(X, F_f)$. But $F_f \in \mathscr{S}^{r, rmd_0}$ and hence $H^1(X, F_f) = 0$. Therefore $H^1(X, (\varphi f)^* M) = 0$. Also $H^1(X, H_f^* \otimes F_f) = 0$ and therefore $H^1(X, (\varphi f)^*(T_g)) = 0$. This proves that R is nonsingular (for the same reason that $H^1(X, f^*(T_g)) = 0$ proves R_1' is nonsingular).

From diagram (1), taking cohomology, we have

$$H^1((\varphi f)^* M) \to H^1(f^* T_q) \to 0.$$

(All cohomologies over X). This proves that $H^1(X, f^* T_q) = 0$ as claimed.

By ([TDTE, IV], § 5) the Zariski tangent space to R (resp. R_1') at φf (resp. f) can be canonically identified with $H^0(X, H_f^* \otimes F_f)$ (resp. $H^0(X, f^* T_q)$). From the bottom row of (3) we get that $\deg H_f^* \otimes F_f = n \deg F_f = n \cdot rmd_0$. Applying Riemann–Roch: $\dim H^0(X, H_f^* \otimes F_f) = n \cdot rmd_0 + r(n-r)(1-g)$. Since $n = \dim H^0(X, F_f) = rmd_0 + r(1-g)$ we have $\dim R = \dim H^0(X, H_f^* \otimes F_f) = n^2 + r^2(g-1)$.

From the exact sequence

$$0 \to H^0(f^* T_q') \to H^0(f^* T_q) \to H^0(f^* T_g) \to H^1(f^* T_q') \to 0$$

(all cohomologies over X) we get

$$\dim H^0(f^* T_q) = \dim H^0(f^* T_g) + \dim H^0(f^* T_q') - \dim H^1(f^* T_q').$$

We apply Riemann–Roch to $f^* T_q'$, noting that $\deg f^* T_q' = 0$ (since in the first row of (1) both K and $(\varphi f)^* \mathrm{Ad}\, Q$ being adjoint bundles have degree zero) and $\mathrm{rk}\, f^* T_q' = r^2 - r - 1 (= \dim Y)$, to get $\dim R_1' = \dim H^0(X, f^* T_q) = n^2 + r^2(g-1) + (r^2 - r - 1)(1-g) = n^2 + (r+1)(g-1)$.

4.13.4. *Lemma. The scheme* R_1 *is nonsingular and* $\dim R_1 = n^2 + (r+1)(g-1) - g$.

Proof. Since R_1' and J^{md_0} are non-singular and R_1 is the fiber over $L^m \in J^{md_0}$, it is enough to check that the differential of $\psi: R_1' \to J^{md_0}$ is surjective at any point $r \in R_1$. Let $t: \mathrm{Spec}\, \mathbb{C}[\varepsilon] \to J^{md_0}$ be a tangent to J^{md_0} at L^m. We have to show that we can lift this morphism to R_1' such that the unique closed point of $\mathrm{Spec}\, \mathbb{C}[\varepsilon]$ goes to r. For this we use the universal properties of R and R_1'. Let $r \in R_1'$ correspond to the $\mathbb{C}^* \times \mathrm{Aut}\,\mathscr{G}'$-bundle $M \times E$ where M is a line bundle on X and $E \to X$ is an $\mathrm{Aut}\,\mathscr{G}'$-bundle. Let $\mathscr{P} \to J^{md_0} \times X$ be the Poincaré bundle. Consider the $\mathbb{C}^* \times \mathrm{Aut}\,\mathscr{G}'$-bundle $\bar{t}^* \mathscr{P} \times_X E_\varepsilon \to \mathrm{Spec}\, \mathbb{C}[\varepsilon] \times X$ where $E_\varepsilon = \mathrm{Spec}\, \mathbb{C}[\varepsilon] \times E$. Then $H^1(X, (\bar{t}^*(\mathscr{P}) \otimes E_\varepsilon(\mathscr{G}'))_o) = H^1(X, L^m \otimes E(\mathscr{G}')) = 0$, where o denotes the closed point of $\mathrm{Spec}\, \mathbb{C}[\varepsilon]$. Therefore the direct image

$p_{\varepsilon *}(\bar{t}^* \mathscr{P} \underset{X}{\times} E_\varepsilon(\mathscr{G}')) = \mathscr{I}$ on $\operatorname{Spec} \mathbb{C}[\varepsilon]$ is locally free and hence free and the natural map $p_\varepsilon^*(\mathscr{I}) \to \bar{t}^*(\mathscr{P}) \otimes E_\varepsilon(\mathscr{G}')$ is surjective ([11], pp. 51–52). If we choose a trivialization of $\mathscr{I}$ we get a morphism $\operatorname{Spec} \mathbb{C}[\varepsilon] \to R$ by the universal property of R, and a lift of this to R_1' by the universal property of R_1'. It is easy to see that for a suitably chosen trivialization of $\mathscr{I}$ the latter gives the required lift of $\operatorname{Spec} \mathbb{C}[\varepsilon] \to J^{md_0}$ to R_1'. Since $\dim R_1 = \dim R_1' - \dim J^{md_0}$ we get, by lemma 4.13.3, $\dim R_1 = n^2 + (r+1)(g-1) - g$.

4.14. PROPOSITION

Let $\mathscr{E}_2 \to R_2 \times X$ be the Ad *G-bundle representing the functor $\Gamma(\rho, \mathscr{E}_1)$ where $\rho: \operatorname{Ad} G \to \operatorname{Aut} \mathscr{G}'$ is the inclusion and $\mathscr{E}_1 \to R_1 \times X$ is the universal* $\operatorname{Aut}\mathscr{G}'$*-bundle constructed in the preceding article. Then $\mathscr{E}_2$ is a universal family with group $GL(n, \mathbb{C})$ (for the action of $GL(n, \mathbb{C})$ provided by Lemma* 4.10) *for the set of all isomorphism classes of semistable* Ad G-bundles.

The natural morphism $\pi_2: R_2 \to R_1$ is étale and finite and hence R_2 is nonsingular.

Proof. Since Ad G is semisimple there are only finitely many isomorphism classes of topological Ad G-bundles ($\pi_1(\operatorname{Ad} G)$ is finite, cf. [14], § 5). Therefore, by Lemma 3.8.1, R_2 is a scheme of finite type. The universal property for $\mathscr{E}_2$ follows from Lemma 4.10. We note that $\mathscr{E}_2$ is universal for all semistable Ad G-bundles because by Corollary 3.18 for any Ad G-bundle $E, E(\mathscr{G}')$ is semistable if and only if E is semistable. (Note that $E(\mathscr{G}')$ is semistable if and only if $E(\mathscr{G})$ is semistable.)

Since Ad G is the connected component of identity of $\operatorname{Aut}\mathscr{G}'$ the morphism $GL(\mathscr{G}')/\operatorname{Ad} G \to GL(\mathscr{G}')/\operatorname{Aut}\mathscr{G}'$ is an étale covering. Therefore it follows that $\mathscr{E}_1/\operatorname{Ad} G \to R_1 \times X$ is an étale covering. Taking $Y = R_1 \times X$, $\tilde{Y} = \mathscr{E}_1/\operatorname{Ad} G$ and $T = R_1$ in the following lemma we see that $\pi_2: R_2 \to R_1$ is étale and finite.

4.14.1. *Lemma. Let $Y \to T$ and $\tilde{Y} \to T$ be schemes over a scheme T and $\tilde{Y} \to Y$ a T-morphism which is an étale (surjective) covering. Suppose Y is flat and projective over T and $\tilde{Y}$ is quasi-projective over T so that the functor π:(locally noetherian schemes/T)$\to$(Sets) defined by*

$$\pi(f: S \to T) = \operatorname{Hom}_{S \underset{T}{\times} Y}(S \underset{T}{\times} Y, S \underset{T}{\times} \tilde{Y}) = \operatorname{Hom}_T(S \underset{T}{\times} Y, \tilde{Y})$$

is representable by a locally finite type T-scheme $\pi: \Pi \to T$ ([TDTE, IV], § 4). *Assume also that $Y \to T$ is faithfully flat. Then π is étale and finite.*

Proof. To prove $\pi: \Pi \to T$ is étale, let A be a scheme, A_0 a subscheme of A defined by a nilpotent ideal and

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, hook] & \Pi \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] & T \end{tikzcd}$$

a commutative diagram. We only have to show that the morphism corresponding to the broken arrow exists uniquely ([EGA IV], Definitions 17.1.1, 17.3.1). Consider the commutative diagram

$$\begin{tikzcd}
A_0 \underset{T}{\times} Y \arrow[r] \arrow[d, hook] & \tilde{Y} \arrow[d] \\
A \underset{T}{\times} Y \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

where the morphism $A_0 \times Y \to \tilde{Y}$ is induced by $A_0 \to \Pi$. Since $A_0 \underset{T}{\times} Y$ in $A \underset{T}{\times} Y$ is also defined by a nilpotent ideal and $\tilde{Y} \to Y$ is étale the broken arrow in the diagram can be realized uniquely and hence by the universal property of Π the broken arrow in the preceding diagram can also be realized.

It follows from ([SGA I], Exposé I, § 5, Corollary 5.3) that $\pi : \Pi \to T$ is quasi finite (i.e. has finite fibers). Since a proper and quasi finite morphism is finite ([EGA III], Corollary 4.4.11) it now suffices to show that π is proper. We use the valuation criterion for this.

Let A be a discrete valuation ring over $\mathbb{C}$ with residue field $\mathbb{C}$ and quotient field K. Suppose we are given morphisms

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, hook] & \Pi \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dotted] & T
\end{tikzcd}$$

Again by the universal property of Π this gives

$$\begin{tikzcd}
\operatorname{Spec} K \underset{T}{\times} Y \arrow[r] \arrow[d, hook] & \tilde{Y} \arrow[d] \\
\operatorname{Spec} A \underset{T}{\times} Y \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

By taking the base change of the étale covering $\tilde{Y} \to Y$ by $\operatorname{Spec} A \underset{T}{\times} Y \to Y$ we get an étale covering over $\operatorname{Spec} A \underset{T}{\times} Y$ which, by the above diagram, has a section over the open subset $\operatorname{Spec} K \underset{T}{\times} Y$ which is dense ($Y \to T$ being faithfully flat) and hence is trivial. This proves that the broken arrow in this diagram, and hence in the preceding diagram, can be realized.

4.15. Let G' be the commutator subgroup $[G, G]$ of G. Let τ be a topological G-bundle on X. The group G/G' is a torus isomorphic to $\mathbb{C}^{*q}$. Therefore fixing an isomorphism $G/G' \approx \mathbb{C}^q$, a G/G'-bundle can be considered as a q-tuple of line bundles. Let $d_1, \ldots, d_q$ be the degrees of the topological line bundles $L_1, \ldots, L_q$ corresponding to the G/G'-bundle obtained from τ by the extension of structure group $G \to G/G'$. Let $J_\tau = J^{d_1} \times \cdots \times J^{d_q}$ where J^{d_i} is the Jacobian of X of line bundles of degree d_i. Let $U_i \to J^{d_i} \times X$ be the Poincaré bundle and $U_\tau \to J_\tau \times X$ be $(id_{J_\tau} \times \Delta_X)^* \ (U_1 \times \cdots \times U_r)$ where $\Delta_X : X \to X \times \cdots \times X$ is the diagonal embedding and $U_1 \times \cdots \times U_q \to J_\tau \times X \times \cdots \times X$ is

the external product of $U_i \to J_i \times X$. Then U_τ is a G/G'-bundle and making G/G' act trivially on J_τ and in the natural way on U_τ (since G/G' is abelian this action gives G/G'-bundle automorphisms) it is easy to see, using the universal property of the Jacobian, that U_τ is a universal family with group G/G' for the set of isomorphism classes of G/G'-bundles of the topological type determined by the degrees $d_1, \ldots, d_q$. The $G/G' \times G/Z$-bundle $\mathscr{E}'_2 = U_\tau \times_X \mathscr{E}_2 \to J_\tau \times R_2 \times X$ is then a universal family with group $G/G' \times GL(n, \mathbb{C})$ for semistable $G/G' \times G/Z$- bundles of suitable topological type, in the obvious way.

4.15.1. *Lemma. The sheaf* $\tilde{\Gamma}^\tau(\rho, \mathscr{E}'_2)$ *corresponding to the natural projection* $\rho: G \to G/G' \times G/Z$ *and the family* $\mathscr{E}'_2 \to R'_2 \times X$, $R'_2 = J_\tau \times R_2$, (§§ 4.8, 4.9) *is representable by a scheme* R_3, *étale and finite over* R'_2.

Proof. We will prove the lemma by identifying $\tilde{\Gamma}(\rho, \mathscr{E}'_2)$ with a product of Picard functors and using the representability theorems for Picard functors. The idea is simply to generalize the fact that given a projective bundle $P \to X$ to give a vector bundle $V \to X$ such that $\mathbb{P}(V) = P$ is equivalent to giving a tautological line bundle on P. For this purpose we construct an algebraic group H which is related to Ad G similar to the way in which $GL(n, \mathbb{C})$ is related to $PGL(n, \mathbb{C}) = \mathrm{Ad}\, SL(n, \mathbb{C})$.

Let T be a maximal torus of $G' = [G, G]$. Then $Z \cdot T$ is a maximal torus for G and $Z \cap T$ is the finite group $Z' = Z[G']$. Let $\chi_1, \ldots, \chi_l$ be a set of characters of $Z \cdot T$ such that the homomorphism $t \mapsto (\chi_1(t), \ldots, \chi_l(t))$ from $Z \cdot T$ to $\mathbb{C}^{*l}$ is injective on Z'. Let F be the finite subgroup $\{(\chi_1(t), \ldots, \chi_l(t), t^{-1}) | t \in Z'\}$ of $\mathbb{C}^{*l} \times G$. Define H to be the quotient group $(\mathbb{C}^{*l} \times \mathrm{G})/F$. Let A be the quotient group $(\mathbb{C}^{*l} \times Z')/F$. Then we have the diagram which is commutative and exact.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & Z' \arrow[r] \arrow[d] & G \arrow[r, "\rho"] \arrow[d] & G/G' \times G/Z \arrow[r] \arrow[d, "''"] & 1 \\
1 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & H \arrow[r, "\rho'"] \arrow[d] & G/G' \times G/Z \arrow[r] & 1 \\
 & A/Z' \arrow[r, "\approx"] \arrow[d] & H/G \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd} \tag{1}$$

Note that both A and A/Z' are isomorphic to $\mathbb{C}^{*l}$ since Z' is a finite subgroup of A and we can choose isomorphisms such that we have the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & Z' \arrow[r] \arrow[d, "\approx"] & A \arrow[r] \arrow[d, "\approx"] & A/Z' \arrow[r] \arrow[d, "\approx"] & 1 \\
1 \arrow[r] & C_{\underline{n}} \arrow[r] & \mathbb{C}^{*l} \arrow[r, "\times \underline{n}"'] & \mathbb{C}^{*l} \arrow[r] & 1
\end{tikzcd} \tag{2}$$

where $\underline{n} = (n_1, \ldots, n_l)$, $n_i \in \mathbb{N}$,

$$C_{\underline{n}} = \{(z_1, \ldots, z_l) \in \mathbb{C}^{*l} | z_i^{n_i} = 1, \forall i\}$$

and $\times \underline{n}$ is the homomorphism $(z_1, \ldots, z_l) \mapsto (z_1^{n_1}, \ldots, z_l^{n_l})$.

Let B be a Borel subgroup of G containing $Z\cdot T$. The characters $\chi_1,\ldots,\chi_l$ extend to B and give a homomorphism $B\to A\approx(\mathbb{C}^{*l}\times Z')/F$ by sending $b\in B$ to $(\chi_1(b),\ldots,\chi_l(b),1)\cdot F$. Clearly H is a connected reductive algebraic group and $B_H=(\mathbb{C}^{*l}\times B)/F$ is a Borel sub-group of H. Again the characters $\chi_1,\ldots,\chi_l$ extend to B_H by defining $\chi_1((z_1,\ldots,z_l,b)\cdot F)\approx z_1\cdot\chi_i(b)$ and we have a homomorphism $B_H\to A$ defined by $(z_1,\ldots,z_l,b)\cdot F\to(z_1\chi_1(b),\ldots,z_l\chi_l(b),1)\cdot F$. We then have the commutative diagram

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ B_H \arrow[rr] \arrow[d] & & A \arrow[d] \\ H/G \arrow[rr, "\approx"'] & & A/Z' \end{tikzcd} \tag{3}$$

The image $\bar{B}=B/Z$ of $\bar{G}=G/Z$ is a Borel subgroup of $\bar{G}$ and $G/B\to\bar{G}/\bar{B}$ is an isomorphism. The projection $\mathbb{C}^{*l}\times G\to G$ induces an isomorphism $H/B_H\to G/B$ which is the inverse of the isomorphism $G/B\to H/B_H$ induced by the inclusion $G\hookrightarrow H$. We then have the commutative diagram

$$\begin{tikzcd} H \arrow[r, hookleftarrow] \arrow[d] & G \arrow[r] \arrow[d] & \bar{G} \arrow[d] \\ H/B_H \arrow[r, phantom, "\approx"] & G/B \arrow[r, phantom, "\approx"] & \bar{G}/\bar{B} \end{tikzcd} \tag{4}$$

Let $M\to G/B$ be the A-bundle obtained from the B-bundle $G\to G/B$ by the extension of structure group $B\to A$. Note that M is also the A-bundle obtained from the B_H-bundle $H\to H/B_H$ by the extension of structure group $B_H\to A$ as follows from the diagrams (3) and (4). The group H operates on G/G' (by left multiplication through ρ') and on $G/B\approx H/B_H$. Since $M\to G/B\approx H/B_H$ is a bundle associated to $H\to H/B_H, H$ operates on M also, compatibly with its action on H/B_H. Further H is precisely the automorphism group of the 'structure' consisting of $G/G'\times G/B$ and M, i.e., to put it more precisely, given an isomorphism $\varphi: G/G'\times G/B\to G/G'\times G/B$ induced by an element of G and an A-bundle isomorphism $\varphi_M: M\to M$ over φ_2, where $\varphi_2: G/B\to G/B$ is given by φ, then there exists an unique $h\in H$ whose action gives φ and φ_M. The existence of such an h is clear and for uniqueness note that since $\cap_{g\in G} gBg^{-1}=Z$, only Z' acts trivially on $G/G'\times G/B$ and the action of Z' on M is faithful since it is given by the characters $\chi_1,\ldots,\chi_l$.

Let $\tilde{\mathbf{P}}_M$ be the sheaf associated to the functor $\mathbf{P}_M:(\mathrm{Sch}/R_2')\to(\mathrm{Sets})$ which associates to $f: S\to R_2'$ the set of isomorphism classes of A-bundles on $\bar{f}^*(\mathscr{E}_2/B)$, where $\bar{f}=f\times id_X: S\times X\to R_2'\times X$, such that for every point $(s,x)\in S\times X$, and for any trivialization $\varphi_{(s,x)}:\mathscr{E}_{2,(s,x)}/B\xrightarrow{\approx} G/B$ there exists an isomorphism $\tilde{\varphi}_{(s,x)}$ of A-bundles over $\varphi_{(s,x)}$:

$$\begin{tikzcd} \mathscr{E}_{(s,x)} \arrow[r, "\tilde{\varphi}_{(s,x)}"] \arrow[d] & M \arrow[d] \\ \mathscr{E}_2(s,x)/B \arrow[r, "\varphi_{(s,x)}"'] & G/B \end{tikzcd}$$

We define $\mathbf{P}_M$ on morphisms to be pull back. We shall show that $\tilde{\mathbf{\Gamma}}(\rho', \mathscr{E}'_2)$ is isomorphic to $\tilde{\mathbf{P}}_M$.

Let $\Lambda \in \mathbf{P}_M(f:S \to R'_2)$. Consider the functor F:(Sch/$S \times X$)$\to$(Sets) which associates to $f':S' \to S \times X$ the set of pairs (φ, φ_M) where φ is an isomorphism of the trivial $(G/G' \times G/Z)$-bundle on S' with $f'^*(\bar{f}^*(\mathscr{E}'_2))$ and φ_M is an isomorphism of A-bundles such that

$$\begin{tikzcd} S' \times M \arrow[r, "\varphi_M"] \arrow[d] & f'^*(\Lambda) \arrow[d] \\ S' \times G/B \arrow[r, "\varphi"'] & f'^*(\bar{f}^*(\mathscr{E}'_2/B)) \end{tikzcd}$$

commutes. On morphisms $\mathbf{F}$ is defined to be pull back. Let $\mathbf{H} = S \times X \times H$ be the constant group scheme over $S \times X$ (considered as a functor) defined by H. Since H is the automorphism group of the structure consisting of $G/G' \times G/B$ and $M \to G/B$, as explained above, it is easy to see that $\mathbf{F}$ is a principal homogeneous space under $\mathbf{H}$. Since $\mathbf{H}$ is an affine algebraic group scheme over $S \times X$ it follows by descent ([SGA 3], exposé XXIV) that $\mathbf{F}$ is representable. Thus we get an H-bundle $\mathscr{E}_\Lambda$ on $S' \times X$ and, from the construction, there is a natural isomorphism $\varphi:\rho'_*\mathscr{E}_\Lambda \to \bar{f}^*(\mathscr{E}'_2)$. We define the morphism $\Phi:\tilde{\mathbf{P}}_M \to \tilde{\mathbf{\Gamma}}(\rho', \mathscr{E}'_2)$ by setting $\Phi_S(\Lambda) = (\mathscr{E}_\Lambda, \varphi)$.

We now construct an inverse $\Psi:\tilde{\mathbf{\Gamma}}(\rho', \mathscr{E}'_2) \to \tilde{\mathbf{P}}_M$ for Φ. Let $(\mathscr{E}, \varphi) \in \tilde{\mathbf{\Gamma}}(\rho', \mathscr{E}'_2)$ $(f:\mathring{S} \to R'_2)$. Thus $\mathscr{E} \to S \times X$ is an H-bundle and $\varphi:\rho'_*\mathscr{E} \to \bar{f}^*(\mathscr{E}_2)$ is an isomorphism. Now φ induces an isomorphism $\mathscr{E}/B_H \approx \bar{f}^*(\mathscr{E}_2(G/B))$ (see diagrams (3) and (4)). Note that $\mathscr{E}'_2(G/B) = \mathscr{E}'_1/B$. Define $\Lambda \to \bar{f}(\mathscr{E}_2/B)$ to be the A-bundle obtained from the B_H-bundle $\mathscr{E} \to \mathscr{E}/B_H$ by the extension of structure group $B_H \to A$ (see diagram (3)). Define $\Psi_S((\mathscr{E}, \varphi)) = \Lambda$. Then it is straightforward to check that Ψ is a morphism inverse to Φ.

Let $\tilde{\mathbf{P}}$ be the sheaf associated to the functor $\mathbf{P}$:(Sch/R'_2)$\to$(Sets) which associates to $f:S \to R'_2$ the set of isomorphism classes of A-bundles on $\bar{f}^*(\mathscr{E}'_2(G/B))$. Since $A \approx \mathbb{C}^{*l}$ (diagram (2)) an A-bundle is just an l-tuple of line bundles and hence $\mathbf{P} \approx \mathbf{Pic} \times \cdots \times \mathbf{Pic}$ the l-fold product of the relative Picard functor of $\mathscr{E}'_2(G/B)/R'_2$ ([TDTE, V], §1). Since the morphism $\mathscr{E}'_2(G/B) = \mathscr{E}_2/B \to R'_2$ is projective (Remark 4.8.3), flat and smooth with irreducible fibers, **Pic** is representable by a locally finite type scheme P over R'_2 ([TDTE V], Theorem 3.1; [TDTE VI], Theorem 4.1, Corollary 4.2) and hence so is $\tilde{\mathbf{P}}$. Clearly $\mathbf{P}_M$ is a subfunctor of $\mathbf{P}$.

Let $\tilde{\mathbf{F}}_1$ (resp. $\tilde{\mathbf{F}}_2$) be the sheaf associated to the functor $\mathbf{F}_1$ (resp. $\mathbf{F}_2$):(Sch R'_2)$\to$(Sets) which associates to $f:S \to R'_2$ the set of isomorphism classes of H G-bundles on

$S \times X$ (resp. on $\bar{f}^*(\mathscr{E}'_2(G/B))$). The morphism $\mathscr{E}'_2(G/B) \to R'_2 \times X$ induces a morphism $\mathbf{F}_1 \to \mathbf{F}_2$. Since $\mathscr{E}'_2(G/B) \to R'_2 \times X$ is projective and flat with irreducible fibers by ([TDTE, V], Proposition 2.1 and [EGA, III], Corollary 7.8.8) it follows that $\mathbf{F}_1 \to \mathbf{F}_2$ is a monomorphism.

The exact sequence $1 \to G \to H \to H/G \to 1$ gives morphisms $\Gamma(\rho, \mathscr{E}'_2) \to \Gamma(\rho', \mathscr{E}'_2) \to \mathbf{F}_1$ and an exact sequence (part of which is)

$$H^0(S \times X, H/G) \to H^1(S \times X, G) \to H^1(S \times X, H) \to H^1(S \times X, H/G)$$

([17], §3.6, Propositions 11 and 12; see also [SGA I], exposé XI, §4). Since $H = Z[H] \cdot G$ it follows that $H^0(S \times X, H/G)$ operates trivially on $H^1(S \times X, G)$. Therefore $H^1(S \times X, G) \to H^1(S \times X, H)$ is an injection. This means that $\Gamma(\rho, \mathscr{E}_2) \to \Gamma(\rho', \mathscr{E}_2)$ is a monomorphism.

We use the isomorphism $H/G \approx A/Z'$ of diagram (1) and the isomorphism $A/Z' \approx \mathbb{C}^{*l}$ of diagram (2) to get an isomorphism of $\mathbf{F}_2$ with $\mathbf{P}$. We have the commutative diagram

$$\begin{tikzcd} & & \mathbf{F}_1 \arrow[d, hook] \\ \Gamma(\rho, \mathscr{E}'_2) \arrow[r, hook] & \Gamma(\rho', \mathscr{E}'_2) \arrow[r] \arrow[d] & \mathbf{F}_2 \arrow[d] \\ \mathbf{P}_M \arrow[r, hook] & \mathbf{P} \arrow[r] & \mathbf{P} \end{tikzcd}$$

where $\mathbf{P} \to \mathbf{P}$ is induced by $\times \underline{n}$: of diagram (2).

It follows from ([TDTE VI], Theorem 2.5) that the morphism $\mathbf{P} \to \mathbf{P}$ corresponding to $\mathbf{P} \to \mathbf{P}$ is étale. Corresponding to the trivial H/G-bundle on $R'_2 \times X$ we get a morphism $R'_2 \to \mathbf{P}$. Let R'_3 be the fiber product

$$\begin{tikzcd} R'_3 \arrow[r] \arrow[d] & \mathbf{P} \arrow[d] \\ R'_2 \arrow[r] & \mathbf{P} \end{tikzcd}$$

Then it follows from ([TDTE VI], Corollary 4.2 and [EGA II], Corollary 5.4.3) that R'_3 is proper over R'_2. Also R'_3 is étale over R'_2.

Using the fact that an H-bundle $\mathscr{E}$ comes from a G-bundle if and only if $\mathscr{E}$ (H/G) is trivial, it is straight forward to check that R'_3 represents the sheaf $\tilde{\Gamma}(\rho, \mathscr{E}'_2)$. Let R_3 be the subscheme of R'_3 corresponding to G-bundles of type τ. Then clearly $R_3 \to R'_2$ is again étale and finite (since topological type is a discrete invariant).

4.15.2. PROPOSITION

The scheme R_3 with the action of $GL(n, \mathbb{C}) \subset G/G' \times GL(n, \mathbb{C})$ (given by Lemma 4.10) is a universal space with group $GL(n, \mathbb{C})$ for semistable G-bundles of topological type τ.

The $GL(n, \mathbb{C})$-equivariant morphism $R_3 \to R'_2$ is étale and finite and hence R_3 is nonsingular.

Proof. This follows immediately from the preceding lemma and Lemma 4.10 noting that G/G' acts trivially on R_2' and hence trivially on R_3.

5. Existence of a quotient space

5.1. *Lemma. Suppose a reductive algebraic group H acts on the schemes Y and Z. If $f:Y\to Z$ is an affine H-equivariant morphism and Z has a good quotient $p:Z\to\bar{Z}$ modulo H then Y has a good quotient $q:Y\to\bar{Y}$ modulo H and the induced morphism $\bar{f}:\bar{Y}\to\bar{Z}$ is affine.*

If moreover f is finite then $\bar{f}$ is also finite. When f is finite and $p:Z\to\bar{Z}$ is a geometric quotient then $q:Y\to\bar{Y}$ is also a geometric quotient.

Proof. Let $\{U_i\}$ be a covering of $\bar{Z}$ by affine open sets. Let $U_i'=(p\circ f)^{-1}(U_i)$. Then $\{U_i'\}$ is a covering of Y by affine H-invariant open sets. Since U_i' is affine there exists a good quotient $q_i:U_i'\to\bar{U}_i'$ of U_i' modulo H, with $\bar{U}_i'$ affine ([22], Theorem 1.1(A)). We shall now give a patching up data for $\{\bar{U}_i'\}$. Let $\bar{f}_i:\bar{U}_i'\to U_i$ be the morphism induced by f. Let $\bar{U}_{ij}'=\bar{f}_i^{-1}(U_i\cap U_j)$. Then $q_i^{-1}(\bar{U}_{ij}')=(p\circ f)^{-1}(U_i\cap U_j)=U_i'\cap U_j'$. Since, being a good quotient is local with respect to the base ([20], § 3, Property 1, p. 356) $q_i:U_i'\cap U_j'\to\bar{U}_{ij}'$ is a good quotient of $U_i'\cap U_j'$ modulo H. Interchanging i and j, $q_j:U_i'\cap U_j'\to\bar{U}_{ji}'$ is also a good quotient of $U_i'\cap U_j'$ modulo H. A good quotient, being a categorical quotient, is unique. Therefore we have natural isomorphisms $h_{ij}:\bar{U}_{ij}'\to\bar{U}_{ji}'$. The h_{ij}'s satisfy the cocycle condition and we can patch up the $\bar{U}_i'$ by h_{ij} along $\bar{U}_{ij}'$ to get a prescheme $\bar{Y}$. The q_i patch up to give a morphism $q:Y\to\bar{Y}$ and the $\bar{f}_i$ a morphism $\bar{f}:\bar{Y}\to\bar{Z}$. Clearly $\bar{f}$ is affine and $\bar{Z}$ being a (separated) scheme $\bar{Y}$ is also a (separated) scheme ([EGA II], Proposition 1.2.4). Again since being a good quotient is local with respect to the base $q:Y\to\bar{Y}$ is a good quotient.

To show that $\bar{f}$ is finite if f is finite we can assume that Y and Z are affine. So let $Y=\operatorname{Spec}A$ and $Z=\operatorname{Spec}B$ and f be given by the homomorphism $f:B\to A$ making A into a finite B-module. Then $\bar{Z}=\operatorname{Spec}B^H$ and $\bar{Y}=\operatorname{Spec}A^H$ ([22], Theorem 1.1(A)) and $\bar{f}$ is given by $B^H\to A^H$ the restriction of f (where A^H, B^H are the rings of invariants under H). We have to show that A^H is a finite B^H-module. Since A is a finite B-module there exists $a_1,\dots,a_r\in A^H$ such that any $a\in A^H$ can be written as $a=\Sigma_{i=1}^r f(b_i)a_i$ with $b_i\in B$. Applying Reynold's operator P on both sides we get $a=\Sigma f(P(b_i))\cdot a_i$ by Reynold's identity and functoriality (cf. [10], Chapter 1, Theorem 1.19). Since $P(b_i)\in B^H$ this proves that A^H is a finite B^H-module. The last assertion of the lemma is easily verified.

5.2. It follows immediately from the preceding lemma and the results of section 4 that to prove the existence of a good quotient for R_3 modulo $GL(n,\mathbb{C})$ it is enough to prove the existence of a good quotient of R_1 modulo $GL(n,\mathbb{C})$ (or equivalently, modulo $SL(n,\mathbb{C})$ since the scalars act trivially on R_1. See Remark 4.13.2). We shall prove this by using Mumford's theory of stable and semistable points for actions of reductive groups ([10]).

5.3. Let $ad\in\mathscr{G}'^*\otimes\mathscr{G}'^*\otimes\mathscr{G}'=\operatorname{Hom}(\mathscr{G}'\otimes\mathscr{G}',\mathscr{G}')=\mathscr{H}$ be the tensor which gives the Lie algebra structure of $\mathscr{G}'$, i.e. for $x,y\in\mathscr{G}'$, $ad(x,y)=[x,y]$. For the natural action of $GL(\mathscr{G}')$ on the tensor space $\mathscr{H}$ the stabilizer of the line (ad) generated by ad is $\mathbb{C}^*\times\operatorname{Aut}\mathscr{G}'$, where $\operatorname{Aut}\mathscr{G}'$ is the group of Lie algebra automorphisms of $\mathscr{G}'$. Therefore

$GL(\mathscr{G}')/\mathbb{C}^* \times \operatorname{Aut}\mathscr{G}' = Y$ gets embedded as a locally closed subscheme of $\mathbb{P}(\mathscr{H})$. Let $\bar{Y}$ be the closure of Y in $\mathbb{P}(\mathscr{H})$. We take on $\bar{Y}$ the canonical reduced subscheme structure.

Now by our constructions we have the 'universal morphisms' $f: R \times X \to G_{n,r}$ and $f_1: R_1 \times X \to Q(Y)$ such that

$$\begin{tikzcd} R_1 \times X \arrow[r, "f_1"] \arrow[d] & Q(Y) \arrow[d] \\ R \times X \arrow[r, "f"'] & G_{n,r} \end{tikzcd}$$

commutes.

Now the morphisms $Y \subset \bar{Y} \subset \mathbb{P}(\mathscr{H})$ give rise to the morphisms $Q(Y) \subset Q(\bar{Y}) \subset Q(\mathbb{P}(\mathscr{H}))$. Since $Y \subset \bar{Y}$ is an open immersion $Q(Y) \subset Q(\bar{Y})$ is an open immersion and since $\bar{Y} \subset \mathbb{P}(\mathscr{H})$ is a closed immersion $Q(\bar{Y}) \subset \mathbb{P}(Q(\mathscr{H}))$ is also a closed immersion (§ 2.4). Let $\mathbb{R} = Q(\mathscr{H}) = Q^* \otimes Q^* \otimes Q$. Let Λ be the relatively ample line bundle $O_{\mathbb{P}(\mathbb{R})}(1)$ on $\mathbb{P}(\mathbb{R})$. Let Λ denote also its restriction to $Q(\bar{Y})$. Consider the functor Γ:(Sch/R)$\to$(Sets) defined by

$$\Gamma(S \to R) = \{\sigma \in \operatorname{Hom}_{G_{n,r}}(S \times X, Q(\bar{Y})) \mid \sigma^*(\Lambda)|_{s \times X} \approx L^{-m}, s \in S\}.$$

This is representable by an algebraic scheme S_1 over R. Let $g_1: S_1 \times X \to Q(\bar{Y})$ be the universal section. It is easy to see that the functor represented by R_1 is a subfunctor of Γ (Remark 4.13.2). Since $Q(Y)$ is open in $Q(\bar{Y})$ it follows that $R_1 \subset S_1$ is an open immersion. We have the commutative diagram

$$\begin{tikzcd} R_1 \times X \arrow[r, "f_1"] \arrow[d, hook] & Q(Y) \arrow[d, hook] \\ S_1 \times X \arrow[r, "g_1"] \arrow[d] & Q(\bar{Y}) \arrow[d] \\ R \times X \arrow[r, "f"] & G_{n,r} \end{tikzcd}$$

5.3.1. *Lemma. The morphism $S_1 \to R$ is proper.*

Proof. We make use of the valuative criterion for properness. Let A be a discrete valuation ring over $\mathbb{C}$ with residue field $\mathbb{C}$ and quotient field K. Suppose we have morphisms

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\varphi_K"] \arrow[d, hook] & S_1 \arrow[d] \\ \operatorname{Spec} A \arrow[r, "\varphi"'] \arrow[ur, dotted] & R \end{tikzcd} \tag{1}$$

By the universal properties of R and R_1 (recall that R and R_1 represent the corresponding functors of sections over the category of locally noetherian schemes ([TDTE IV])). We have

$$\begin{tikzcd}
\operatorname{Spec} K \times X \arrow[r, "\tilde{\varphi}_K"] \arrow[d, hook] & Q(\bar{Y}) \arrow[r, phantom, "\subset"] \arrow[d] & \mathbb{P}(\mathbb{R}) \arrow[dl] \\
\operatorname{Spec} A \times X \arrow[r, "\tilde{\varphi}"'] \arrow[ur, dotted] & G_{n,r} &
\end{tikzcd} \tag{2}$$

On Spec $K \times X$, we have the exact sequence

$$0 \to \tilde{\varphi}_K^*(\Lambda) \to \tilde{\varphi}_K^*(\mathbb{R}) \to \tilde{\varphi}_K^*(\mathbb{R}/\Lambda) = V_K \to 0.$$

We can extend the quotient vector bundle V_K to Spec $A \times X$ as a quotient coherent sheaf V of $\tilde{\varphi}^*(\mathbb{R})$ flat over Spec A(cf. [EGA IV], Proposition 2.8.1). Let V_0 be the restriction of V to Spec $\mathbb{C} \times X \simeq X$ corresponding to the closed point of Spec A. We then have the surjection $\tilde{\varphi}_0^*(\mathbb{R}) \to V_0 \to 0$ of coherent sheaves on X. Note that $\tilde{\varphi}_0^*(\mathbb{R})$ is a semistable vector bundle (by Proposition 3.17 since $\tilde{\varphi}_0^*(Q)$ is semistable) of degree $-r^3 md_0$ and therefore $\mu(\tilde{\varphi}_0^*(\mathbb{R})) = -md_0$. From the definition of S_1 and the flatness of V over Spec A it follows that deg $V_0 = -r^3 md_0 - md_0$. Since X is a curve we can write $V_0 = V'_0 \oplus T$ where V'_0 is locally free and T is torsion. Since $\tilde{\varphi}_0^*(\mathbb{R})$ is semistable, V'_0 being a quotient $\mu(V'_0) \geqslant -\mathrm{md}_0$. Therefore deg $V'_0 \geqslant -(r^3+1)md_0$. On the other hand T being a torsion sheaf deg $T \geqslant 0$ and since deg $V_0 = -(r^3+1)md_0$ we have deg $V'_0 \leqslant -(r^3+1)md_0$. This shows that deg $V'_0 = -(r^3+1)\cdot md_0$. Therefore $T = 0$ and $V'_0 = V_0$. Hence V_0 is locally free. Then V is a vector bundle on Spec $A \times X$ (cf. [19], Lemma 6.17). By the universal property of $\mathbb{P}(\mathbb{R})$, V determines a morphism Spec $A \times X \to \mathbb{P}(\mathbb{R})$ (cf. [EGA II], Theorem 4.2.4; note that in our notation $\mathbb{P}$ stands for the dual of what $\mathbb{P}$ stands for in this reference). But since $Q(\bar{Y})$ is closed in $\mathbb{P}(\mathbb{R})$ this morphism goes into $Q(\bar{Y})$ proving that the broken arrow in diagram (2) can be realized. This immediately implies that the broken arrow in diagram (1) can be realized.

5.4. We now recall briefly some definitions and results from Mumford's 'geometric invariant theory' ([10]; see also [22]). Let a reductive group H act on a projective algebraic scheme Y. Let $\Lambda \to Y$ be an ample line bundle on Y and H act on Λ also as a group of line bundle isomorphisms compatible with its action on Y i.e. Λ has a H-linearization ([10], Chapter I, § 3). Then a point $y \in T$ is called *semistable* if for some $m > 0$ there is a H-invariant section $s \in H^0(Y, \Lambda^m)$ such that $s(y) \neq 0$. If moreover every orbit of H in $Y_s = \{x \in Y \mid s(x) \neq 0\}$ is closed and of the same dimension as H, y is called a *(properly) stable* point ([10], Chapter I, § 4).

5.4.1. The set Y_{ss}(resp. Y_s) of semistable (resp. stable) points is open in Y and there exists a good quotient $p: Y_{ss} \to \bar{Y}_{ss}$ modulo H such that $\bar{Y}_{ss}$ is projective.

There is an open subscheme $\bar{Y}_s \subset \bar{Y}_{ss}$ such that $p: p^{-1}(\bar{Y}_s) \to \bar{Y}_s$ is a 'geometric quotient' ([10], Chapter I, Theorem 1.10 and the remark on p. 40; [22], Theorem 1.1(B)).

5.4.2. Let $\lambda: \mathbb{C}^* \to G$ be a 1-PS (i.e. 1-parameter subgroup) and $y \in Y$. Then $\lim_{t \to 0} \lambda(t) y = y_0$ exists (since Y is projective) and y_0 is fixed by λ. Let $t \mapsto t^r$ be the character by which λ operates on Λ_{y_0}. Then we define $\mu^\Lambda(y, \lambda) = -r$. (In this definition Λ can be an arbitrary line bundle not necessarily ample.) Then a point $y \in Y$ is semistable

(resp. stable) if and only if $\mu^{\Lambda}(y,\lambda)\geqslant 0$ (resp. >0) for every 1-PS λ of G ([10], Chapter 2, § 1, Theorem 2.1).

5.4.3. Let H act on the projective algebraic scheme Y' and $f: Y'\to Y$ an H-equivariant morphism. Then $\mu^{f^*\Lambda}(y',\lambda)=\mu^{\Lambda}(f(y'),\lambda), y'\in Y'$ ([10], p.49).

5.4.4. Given a 1-PS λ of G there is a parabolic subgroup $P(\lambda)\supset\lambda$ such that $\mu^{\Lambda}(y,\lambda)=\mu^{\Lambda}(y,g\lambda g^{-1})$ for every $g\in P(\lambda)$. This $P(\lambda)$ depends only on λ and not on the scheme Y or Λ ([10], Chapter 2, § 2, [22], Lemma 3.1, Proposition 3.1).

5.5. Now by the usual 'diagonal argument' we can choose an N-tuple $(x_1,\ldots,x_N)$ of points in X for N sufficiently large such that the morphism $R_1\to Q(\bar{Y})^N=Q(\bar{Y})\times\cdots\times Q(\bar{Y})$, N factors, given by evaluating f_1 at the points x_i, i.e. $R_1\ni r\to(f_1(r,x_1),\ldots, f_1(r,x_N))$, is an injection ([19], p. 326). It is proved in ([20], § 3, Lemma 2) that we can choose the N-tuple $(x_1,\ldots,x_N)$ such that under the morphism $R\to G^N_{n,r}=Z$ (defined by $R\ni r\to(f(r,x_1),\ldots,f(r,x_N)))$ R goes into the open subscheme of semistable points Z_{ss} of Z for the action of $SL(n,\mathbb{C})$ (and the natural polarization of the Grassmannian) and moreover such that the morphism $R\to Z_{ss}$ is proper.

We have the commutative diagram

$$\begin{array}{ccc}
R_1 & \longrightarrow & Q(Y)^N \\
\downarrow & & \downarrow \\
S_1 & & Q(\bar{Y})^N\subset\mathbb{P}(\mathbb{R})^N \\
\downarrow & & \downarrow \\
R & \longrightarrow & G^N_{n,r}
\end{array}$$

We shall give a suitable ample line bundle on $Q(\bar{Y})^N$ and prove that R_1 goes into the semistable points of $Q(\bar{Y})^N$ for the natural action of $SL(n,\mathbb{C})$ on $Q(\bar{Y})^N$.

Let M be the very ample line bundle on the Grassmannian $G_{n,r}$ corresponding to the natural embedding of $G_{n,r}$ in $\mathbb{P}(\Lambda^{n-r}\mathbb{C}^n)$. There is a natural $SL(n,\mathbb{C})$ linearization of M given by the action of $SL(n,\mathbb{C})$ on $\Lambda^{n-r}\mathbb{C}^n$. As defined in § 5.3, let Λ be the tautological line bundle on $\mathbb{P}(\mathbb{R})$ corresponding to the vector bundle $\mathbb{R}$. Then Λ is relatively ample for the morphism $p:Q(\bar{Y})\to G_{n,r}$. Therefore the line bundle $p^*(M)^a\otimes\Lambda^b$ is ample on $Q(\bar{Y})$ for $a\gg b$ ([EGA II] § 4.6, [22], § 5).

The action of $SL(n,\mathbb{C})$ on $G_{n,r}$ has a natural lift to an action on the bundle Q and hence on the associated bundle $\mathbb{R}=Q^*\otimes Q^*\otimes Q$. This gives an $SL(n,\mathbb{C})$-linearization of Λ. By ([22], Proposition 5.1) we can choose the positive integers a,b with b/a sufficiently small such that $p(Q(\bar{Y})^N_{ss})\subset Z_{ss}$ where $Q(\bar{Y})^N_{ss}$ is the set of semistable points of $Q(\bar{Y})^N$ for the action of $SL(n,\mathbb{C})$ and the ample line bundle on $Q(\bar{Y})^N$ which is the product of the line bundles $p^*(M)^a\otimes\Lambda^b$ on each factor $Q(\bar{Y})$.

5.5.1. *Lemma. The point* $(ad)\in\mathbb{P}(\mathscr{G}'^*\otimes\mathscr{G}'^*\otimes\mathscr{G}')$ (cf. § 5.3) *is semistable for the natural action of* $SL(\mathscr{G}')$ *and the line bundle* $O(1)$ *on* $\mathbb{P}(\mathscr{G}'^*\otimes\mathscr{G}'^*\otimes\mathscr{G}')$.

Proof. Let $\varphi:(\mathscr{G}'^*\otimes\mathscr{G}'^*\otimes\mathscr{G}')=\mathrm{Hom}(\mathscr{G}',\mathrm{End}\,\mathscr{G}')\to\mathscr{G}'^*\otimes\mathscr{G}'^*=(\mathscr{G}'\otimes\mathscr{G}')^*$ be defined by $\varphi(f)(x\otimes y)=\mathrm{trace}(f(x)\circ f(y))$, $f\in\mathrm{Hom}(\mathscr{G}',\mathrm{End}\,\mathscr{G}')$ and $x,y\in\mathscr{G}'$. Then φ is an $SL(\mathscr{G}')$-equivariant morphism. Choose an arbitrary linear space isomorphism of $\mathscr{G}'^*$

with $\mathscr{G}'$. Then we get an isomorphism $\mathscr{G}'^* \otimes \mathscr{G}'^* \approx \mathrm{End}(\mathscr{G}')$. Let det:$\mathscr{G}'^* \otimes \mathscr{G}'^* \approx \mathrm{End}(\mathscr{G}') \to \mathbb{C}$ be the determinant map. Then $\det \circ \varphi$ is an $SL(\mathscr{G}')$-invariant polynomial on $\mathscr{G}'^* \otimes \mathscr{G}'^* \otimes \mathscr{G}'$ and $(\det \circ \varphi)(ad)$ is the determinant of the Killing form of $\mathscr{G}'$ and hence is non-zero, $\mathscr{G}'$ being semisimple.

5.5.2. *Lemma. The point* $(x_1, \ldots, x_N) \in Q(Y)^N \subset Q(\bar{Y})^N$ *is semistable if and only if* $(p(x_1), \ldots, p(x_N))$ *is semistable in* $G_{n,r}^N$.

Proof. Let $(p(x_1), \ldots, p(x_N)) = (y_1, \ldots, y_N)$. If $(x_1, \ldots, x_N)$ is semistable then by our choice of $p^*(M)^a \otimes \Lambda^b$, $(y_1, \ldots, y_N)$ is semistable ([22] Proposition 5.1).

Suppose $(y_1, \ldots, y_N)$ is semistable. Let λ be a 1-PS of $SL(n, \mathbb{C})$. Let $P(\lambda)$ be the canonical parabolic subgroup associated to λ (§ 5.4.4). Let $x \in Q(\bar{Y})$ and $y = p(x) \in G_{n,r}$. Then y is an r-dimensional quotient of $\mathbb{C}^n$. Let P_y be the maximal parabolic subgroup of $SL(n, \mathbb{C})$ which is the stabilizer of $y \in G_{n,r}$. By Bruhat's lemma $P_y \cap P(\lambda)$ contains a maximal torus T of $SL(n, \mathbb{C})$. Since $\lambda \subset P(\lambda)$ there is a $g \in P(\lambda)$ such that $g\lambda g^{-1} \subset T$. Then

$$\mu(x, \lambda) = \mu(x, g\lambda g^{-1}), \tag{1}$$

where μ stands for $\mu p^{*(M)^a \otimes \Lambda^b}$ (§ 5.4.4). Since $T \subset P_y$ we can choose a basis $e_1, \ldots, e_r$, $e_{r+1}, \ldots, e_n$ for $\mathbb{C}^n$ such that the images $\bar{e}_i$, of e_i in y, $i = 1, \ldots, r$ form a basis for y and such that T becomes the group of diagonal matices with respect to $e_1, \ldots, e_n$. Now

$$\mu(x, \lambda) = a\mu^M(y, \lambda) + b\mu^\Lambda(x, \lambda) = a\mu^M(y, g\lambda g^{-1}) + b\mu^\Lambda(x, g\lambda g^{-1}). \tag{2}$$

We shall first calculate $\mu^\Lambda(x, \lambda) = \mu^\Lambda(x, \lambda_g)$ where $\lambda_g = g\lambda g^{-1}$. Note that

$$\mu^\Lambda(x, \lambda_g^r) = r\mu^\Lambda(x, \lambda_g) \tag{3}$$

([22], Proposition 2.1). Let $\lambda_g(z) = z^{a_i} e_i$, $z \in \mathbb{C}^*$, $s_i \in \mathbb{Z}$, $i = 1, \ldots, r$. Define the 1-PS λ' of the center of $GL(y)$ by $\lambda'(z)\bar{e}_i = z^s \bar{e}_i$, $z \in \mathbb{C}^*$, $s = s_1 + \cdots + s_r$.

Since λ_g leaves y invariant it follows easily from definitions (cf. § 5.4.3) that for calculating $\mu^\Lambda(x, \lambda_g)$ we can restrict our attention to y or equivalently the subspace generated by $e_1, \ldots, e_r$. For the action on the quotient space y we have $\lambda_g^r = \lambda' \cdot \lambda''$ where $\lambda''(z)e_i = z^{rs_i - s} e_i$, $z \in \mathbb{C}^*$, $i = 1, \ldots, r$. Note that $\lambda'' \subset SL(y)$. Since λ' is in the center of $GL(y)$ it is easy to see that

$$\mu^\Lambda(x, \lambda_g^r) = \mu^\Lambda(x, \lambda') + \mu^\Lambda(x, \lambda'') \tag{4}$$

and that

$$\mu^\Lambda(x, \lambda') = s. \tag{5}$$

Moreover we get from ([10], Chap. 4, § 4, eq.(*), p. 67) that

$$\mu^M(y, \lambda_g) = s. \tag{6}$$

Therefore

$$\begin{aligned}\mu^\Lambda(x, \lambda) &= \frac{1}{r}\{\mu^M(y, \lambda_g) + \mu^\Lambda(x, \lambda'')\} \\ &= \frac{1}{r}\{\mu^M(y, \lambda) + \mu^\Lambda(x, \lambda'')\}\end{aligned}$$

with λ'' a 1-PS of $SL(y)$, by (1), (4), (5) and (6). This gives

$$\begin{aligned}\mu(x,\lambda) &= a\mu^M(y,\lambda) + b\mu^\wedge(x,\lambda)\\ &= \left(a+\frac{b}{r}\right)\mu^M(y,\lambda) + \frac{b}{r}\mu^\wedge(x,\lambda''). \qquad (7)\end{aligned}$$

Writing x_i for x in (7) and summing through i we get

$$\begin{aligned}\mu(x_1,\ldots,x_N),\lambda &= \left(a+\frac{b}{r}\right)\mu^M((y_1,\ldots,y_N),\lambda)\\ &\quad + \frac{b}{r}\cdot\sum_{i=1}^{N}\mu^\wedge(x_i,\lambda_i''). \qquad (8)\end{aligned}$$

Since $(y_1,\ldots,y_N)$ was assumed to be semistable $\mu^M((y_1,\ldots,y_N),\lambda)\geqslant 0$. Since $x_i \in Q(Y)$ it follows from Lemma 5.5.1 that $\mu^\wedge(x_i,\lambda_i'')\geqslant 0\ \forall i$ (cf. 5.4.3). Therefore $\mu((x_1,\ldots,x_N),\lambda)\geqslant 0$ which proves $(x_1,\ldots,x_N)$ is semistable (§ 5.4.2).

5.5.3. *Lemma. Under the morphism* $R_1 \to Q(Y)^N \subset Q(\bar{Y})^N$ R_1 *maps into the open subscheme* $Q(\bar{Y})^N_{ss}$ *and hence* $R_1 \to Q(\bar{Y})^N$ *factors as* $R_1 \to Q(\bar{Y})^N_{ss} \hookrightarrow Q(\bar{Y})^N$.

Proof. Since under the morphism $R \to G_{n,r} = Z$, R maps into Z_{ss} this follows immediately from the preceding lemma.

5.6. *Lemma. The injective morphism* $R_1 \to Q(\bar{Y})^N_{ss}$ *is proper.*

Proof. We use the valuation criterion. Let A be a discrete valuation ring over $\mathbb{C}$ with residue field $\mathbb{C}$ and quotient field K. Suppose we are given

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & R_1 \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dotted] & Q(\bar{Y})^N_{ss} \end{tikzcd} \qquad (1)$$

We have to complete the broken arrow.

From (1) and the diagram in § 5.5, using the facts that $p(Q(\bar{Y})^N_{ss}) \subset Z_{ss}$ and the morphism $R \to Z_{ss}$ and $S_1 \to R$ are proper ([20], § 3, Lemma 2 and Lemma 5.3.1) it follows that we can get a lift $\operatorname{Spec} A \to S_1$ giving the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[rr] \arrow[dd, hook] \arrow[dr] & & R_1 \arrow[dl] \arrow[dd] \\ & S_1 \arrow[dr] & \\ \operatorname{Spec} A \arrow[rr] \arrow[ur] & & Q(\bar{Y})^N_{ss} \end{tikzcd}$$

We will be through if we show that the closed point of Spec A maps into R_1 under this morphism $\operatorname{Spec} A \to S_1 \supset R_1$.

Let $V = \mathcal{E}_0(-m)$ be the vector bundle (semistable of degree zero) corresponding to the image of the closed point of Spec A in R under the composite $\operatorname{Spec} A \to S_1 \to R$. The image

of the closed point of Spec A in S_1 then gives a section of $Q_0(\bar{Y}) \subset \mathbb{P}(V^* \otimes V^* \otimes V) \to X$ which actually comes from a section $s \in H^0(X, V^* \otimes V^* \otimes V)$ with $s(x) \neq 0\ \forall x \in X$. This section s, since its image is in $Q_0(\bar{Y})$, gives a Lie algebra structure on the fibers of V (cf. Remark 4.13.2). Let $\bar{s} \in H^0(X, V^* \otimes V^*)$ correspond to the Killing form of the Lie algebra structure given by s. We shall show that $\bar{s}$ is nondegenerate on all fibers.

Suppose $\bar{s}$ is not nondegenerate on all fibers. Then the homomorphism $V \to V^*$ induced by $\bar{s}$ has a non-trivial kernel sheaf. Since both V and V^* are semistable vector bundles of degree zero the kernel is actually a subbundle V_1, semistable and of degree zero ([19], Proposition 3.1). Then V_1 is a solvable ideal in V, i.e. the fibers of V_1 are solvable ideals in the fibers of V ([18], Chapter VI, proof of Theorem 2.1). Again since $V_1 \otimes V_1$ and V_1 are semistable vector bundles of degree zero the image $[V_1, V_1]$ of the morphism $V_1 \otimes V_1 \to V_1, x \otimes y \mapsto [x, y]$, given by the Lie bracket operation, is a subbundle V_2 of V_1 of degree zero and semistable. Similarly $V_3 = [V_2, V_2]$ etc. are all semistable vector bundles of degree zero. Since V_1 is solvable we arrive after a certain stage at a non-zero subbundle V', of degree zero and semistable, which is an abelian ideal in V.

The inclusion $V' \otimes L^m \hookrightarrow V \otimes L^m$ induces $W = H^0(X, V' \otimes L^m) \hookrightarrow H^0(X, V \otimes L^m) = I_n$. Let W' be a supplement for W in I_n so that $I_n = W \oplus W'$. Let λ be the 1-PS of $SL(n, \mathbb{C})$ which acts on W by the character $\lambda(z) = z^{\mathrm{rk}W'}$ and on W' by the character $\lambda(z) = z^{-\mathrm{rk}W}$, $z \in \mathbb{C}^*$. Let $(x_1, \ldots, x_N)$ be the image of the closed point of Spec A in $Q(\bar{Y})^N_{\mathrm{ss}}$ and $(y_1, \ldots, y_N) = (p(x_1), \ldots, p(x_N))$. We shall now compute $\mu((x_1, \ldots, x_N), \lambda)$ and show that it is < 0 contradicting semistability of $(x_1, \ldots, x_N)$.

It follows from ([10], Chapter 4, §4, equation $(***)_N$, p. 88; cf. also [19], p. 309) that $\mu^M((y_1, \ldots, y_N), \lambda) = n \cdot \Sigma_{i=1}^N \mathrm{rk}(W_i) - r \cdot N(\mathrm{rk}W)$ where W_i is the image of W in y_i. (In [10] the calculation is made for the Grassmannian of subspaces. It is easy to translate it to the Grassmannian of *quotient* spaces which we need here.) Since $W = H^0(X, V' \otimes L^m)$ generates V' (by our choice of m, cf. §3.11), rk $W_i = \mathrm{rk}(V')\forall i$. Therefore

$$\mu^M((y_1, \ldots, y_N), \lambda) = n \cdot N(\mathrm{rk}V') - r \cdot N(\mathrm{rk}W). \tag{1}$$

Applying Riemann–Roch we get

$$\mathrm{rk}\, W = \mathrm{rk}\, H^0(X, V' \otimes L^m) = (\mathrm{rk}\, V')(md_0 + 1 - g)$$

and

$$n = \mathrm{rk}\, H^0(X, V \otimes L^m) = r \cdot (md_0 + 1 - g).$$

Therefore (rk W)$/n$ = (rk V')$/r$. Hence from (1) we have

$$\mu^M((y_1, \ldots, y_N), \lambda) = 0. \tag{2}$$

To calculate $\mu^\wedge((x_1, \ldots, x_N), \lambda)$ let $x = x_i$ and $y = y_i$ and let $g \in P(\lambda)$ such that $\lambda_g = g \lambda g^{-1} \subset P_y$ (cf. Proof of Lemma 5.5.2). Then $\mu^\wedge(x, \lambda) = \mu^\wedge(x, \lambda_g)$. It follows from ([10] Chapter 2, §2, pp. 55–56) that $P(\lambda) = P_W$, the stabilizer of W in $SL(n, \mathbb{C})$. Therefore $I_n = gW \oplus gW' = W \oplus gW'$ and λ_g acts on W by the character $\lambda_g(z) = z^{\mathrm{rk}W'}$ and on gW' by $\lambda_g(z) = z^{-\mathrm{rk}W}$, $z \in \mathbb{C}^*$. Since $\lambda_g \subset P_y, \lambda_g$ leaves invariant $\ker(I_n \to y)$ and hence we can find a set of linearly independent elements $e_1, \ldots, e_q, e_{q+1}, \ldots, e_r$ such that $e_1, \ldots, e_q \in W$ and $e_{q+1}, \ldots, e_r \in gW'$ and $\bar{e}_1, \ldots, \bar{e}_q$ the images of $e_1, \ldots, e_q$ under $I_n \to y$ form a basis for the fiber $V'_{x'} \subset y$ of V' over $x' \in X$ (where y is the fiber of V over x') and $\bar{e}_1, \ldots, \bar{e}_q, \bar{e}_{q+1}, \ldots, \bar{e}_r$ form a basis for y. Then

$$\lambda_g^r(z) \cdot \bar{e}_i = \begin{cases} z^{r(\mathrm{rk}W)} \bar{e}_i & \text{for } 1 \leqslant i \leqslant q \\ z^{-r(\mathrm{rk}W')} \bar{e}_i & \text{for } q+1 \leqslant i \leqslant r. \end{cases}$$

For the action of λ_g on y we then have $\lambda_g^r = \lambda' \cdot \lambda''$ where $\lambda'(z)\bar{e}_i = z^t \bar{e}_i$; $t = q(\mathrm{rk} W) - (r-q)\mathrm{rk} W'$ and

$$\lambda''(z)\bar{e}_i = \begin{cases} z^{r(\mathrm{rk}W)-t}\bar{e}_i = z^{(r-q)\cdot n}\bar{e}_i, & 1 \leqslant i \leqslant q \\ z^{-r(\mathrm{rk}W')-t}\bar{e}_i = z^{-qn}\bar{e}_i, & q+1 \leqslant i \leqslant r. \end{cases}$$

As in Lemma 5.5.2, eq. (6)

$$\mu^{\wedge}(x, \lambda') = \mu^M(y, \lambda_g) = \mu^M(y, \lambda).$$

To calculate $\mu^{\wedge}(x, \lambda'')$ we shall use ([10] Proposition 2.3; cf. also [22], §2). Let $\tilde{x}$ be a point in $y^* \otimes y^* \otimes y$ which lies above $x \in Q_y(\bar{Y}) \subset \mathbb{P}(y^* \otimes y^* \otimes y)$. Let

$$\tilde{x} = \sum_{i,j,k=1}^{r} x_{ijk}\, \bar{e}_i^* \otimes \bar{e}_j^* \otimes \bar{e}_k,$$

where $\bar{e}_i^*$ form the dual basis to $\bar{e}_i$. If we think of $\tilde{x}$ as a Lie algebra structure on y then the x_{ijk} are the 'structure constants' and we have

$$[\bar{e}_i, \bar{e}_j] = \sum_{k=1}^{r} x_{ijk}\bar{e}_k.$$

From the fact that $\bar{e}_1, \ldots, \bar{e}_q$ span an abelian ideal in y we get that $x_{ijk} = 0$ whenever i, j, k satisfy any one of the following three conditions: (1) both i and j are $\leqslant q$ and k arbitrary (abelian) (2) $i \leqslant q, j$ arbitrary and $k \geqslant q+1$ (ideal) (3) i arbitrary, $j \leqslant q$ and $k \geqslant q+1$ (symmetric to (2)). Therefore x_{ijk} may not be zero only in the following cases:

Case i. $i \leqslant q, j \geqslant q+1$ and $k \leqslant q$.
Case ii. $i \geqslant q+1, j \leqslant q$ and $k \leqslant q$.
Case iiia. $i \geqslant q+1, j \geqslant q+1$ and $k \leqslant q$.
Case iiib. $i \geqslant q+1, j \geqslant q+1$ and $k \geqslant q+1$.

Let

$$\lambda''(z)(\bar{e}_i^* \otimes \bar{e}_j^* \otimes \bar{e}_k) = z^{a_{ijk}} \cdot \bar{e}_i^* \otimes \bar{e}_j^* \otimes \bar{e}_k, a_{ijk} \in \mathbb{Z}.$$

Then it is easy to see that $a_{ijk} = q$ in cases (i), (ii), and (iiib) and $a_{ijk} = q + r$ in case (iiia). Therefore in every case when $x_{ijk} \neq 0$, $\lambda''(z)$ acts by a strictly positive power, viz. q or $q + r$, of z. It follows by ([10], Proposition 2.3) that $\mu^{\wedge}(x, \lambda'') < 0$. Now using eq. (8) of Lemma 5.5.2 and (2) above, we get $\mu((x_1, \ldots, x_N), \lambda) < 0$ contradicting the semistability of $(x_1, \ldots, x_N)$. Therefore we conclude that the Killing form $\bar{e}$ must be nondegenerate on all fibers. Then the Lie algebra structure of all fibers is semisimple. But the Lie algebra structure of a semisimple Lie algebra is (locally) rigid, i.e., interpreting $\bar{Y}$ as Lie algebra structures on y, if $x \in \bar{Y}$ gives a semisimple Lie algebra structure on y then $x \in Y$ ([16], §§3,4, Corollary 4.3, pp. 413–415). This shows that the image of s lies in $Q_0(Y)$. Therefore the image of the closed point of Spec A under Spec $A \to S_1$ lies in R_1 as was to be shown.

5.7. *Lemma. Let* $\pi: R_3 \to Q(\bar{Y})^N_{\mathrm{ss}}$ *be the composite* $R_3 \xrightarrow{\pi_3} J_\tau \times R_2 \to R_2 \xrightarrow{\pi_2} R_1 \to Q(\bar{Y})^N_{\mathrm{ss}}$. *Then for* $r_3 \in R_3$, $\pi(r_3)$ *is a stable point of* $Q(\bar{Y})^N_{\mathrm{ss}}$ *if and only if the G-bundle* $E \to X$ *corresponding to* r_3 *is stable.*

Proof. Let $\pi_3(r_3) = (j, r_2)$. Then the Ad G-bundle E' corresponding to $r_2 \in R_2$ is obtained from the G-bundle E by the extension of structure group $G \to \operatorname{Ad} G = G/Z$. Therefore E' is stable if and only if E is stable ([14], Proposition 7.1, p. 146).

Then point $\pi(r_3)$ in $Q(\bar{Y})^N_{ss}$ is stable if and only if the orbit of $\pi(r_3)$ under $SL(n, \mathbb{C})$ is closed and the isotropy of $\pi(r_3)$ in $SL(n, \mathbb{C})$ is finite ([10], Amplification 1.11; [12], § 2, Theorem 2(a), p. 193).

The morphism $R_2 \to Q(\bar{Y})^N_{ss}$ is an $SL(n, \mathbb{C})$-equivariant finite morphism. Moreover it is easy to see that if two Ad G-bundles E'_1 and E'_2 give rise to isomorphic Aut $\mathscr{G}'$-bundles under the extension of structure group Ad $G \hookrightarrow$ Aut $\mathscr{G}'$ then E'_1 is stable if and only if E'_2 is stable. It then follows that it is enough to show that the Ad G-bundle E' is stable if and only if the $SL(n, \mathbb{C})$ orbit of r_2 in R_2 is closed and the isotropy of r_2 in $SL(n, \mathbb{C})$ is finite.

Suppose E' is a stable Ad G-bundle. Then gr $E' = E'$ (Proposition 3.12) and it follows from Proposition 3.24(i) that the $SL(n, \mathbb{C})$ orbit of r_2 is closed (cf. proof of Proposition 4.5). Since the group Aut E' of Ad G-bundle automorphisms of E' is finite ([14], Proposition 3.2, p. 136) it follows from Remark 4.7 that the isotropy of r_2 in $SL(n, \mathbb{C})$ is finite.

Conversely suppose that the $SL(n, \mathbb{C})$ orbit of r_2 is closed and the isotropy of r_2 finite. From Proposition 3.24(ii) it follows that the closure of the orbit of r_2 always contains a point r'_2 which is isomorphic to gr E'(cf. proof of Proposition 3.5). Therefore if the orbit of r_2 is closed then $E' \approx$ gr E'. By Proposition 3.15 this implies that E' is a unitary bundle E'_ρ corresponding to a unitary representation $\rho: \pi_1(X) \to \bar{K}$, where $\bar{K}$ is a maximal compact subgroup of Ad G. If ρ is not irreducible then there is a subgroup $S \subset \bar{K}$, with dim $S > 0$, which commutes with ρ ([14], Definition 1.2, cf. also § 2, p. 131). Then the group S gives rise to a group of automorphisms of $E'_\rho = E'$ of dimension > 0. This contradicts the finiteness of the isotropy at r_2 (Remark 4.7). Therefore ρ is irreducible. Then $E' = E'_\rho$ is stable by ([14], Proposition 2.2, p. 133).

5.8. PROPOSITION

Let $\mathscr{E} \to S \times X$ be an arbitrary family of G-bundles on X parametrized by a scheme S. Then the set S_{ss} (resp. S_s) of points $s \in S$ such that $\mathscr{E}_s$ is semistable (resp. stable) is an open subset of S.

Proof. Since a G-bundle E is semistable (resp. stable) if and only if the Ad G-bundle E' obtained from E by the extension of structure group $G \to \operatorname{Ad} G$ is so ([14], Proposition 7.1, p. 147), we can assume that $G = \operatorname{Ad} G$. Let $\mathscr{E}(\mathscr{G})$ be the vector bundle associated to $\mathscr{E}$ by the adjoint representation $G \hookrightarrow GL(\mathscr{G})$. Since the question is local on S we can assume that for $m \gg 0$, $\mathscr{E}(\mathscr{G}) \otimes p_X^*(L^m)$ is a quotient of I_n for a suitable $n, I_n \to \mathscr{E}(\mathscr{G}) \otimes p_X^*(L^m) \to 0$. This then gives a morphism $f: S \times X \to G_{n,r}$ the Grassmannian of r dimensional quotients of I_n and the G-bundle $\mathscr{E} \to S \times X$ gives a morphism $\tilde{f}: S \times X \to Q(Y) \subset Q(\bar{Y})$ in the obvious way. By choosing an N-tuple $(x_1, \ldots, x_N)$ of points of X we get

$$\begin{tikzcd} & Q(Y)^N \subset Q(\bar{Y})^N \arrow[d] \\ S \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & G_{n,r} \end{tikzcd}$$

We can choose the N-tuple $(x_1,\ldots,x_N)$, $N\gg 0$, such that $f(s)$ is a semistable (resp. stable) point of $G^N_{n,r}$ if and only if $\mathscr{E}_s(\mathscr{G})$ is a semistable (resp. stable) vector bundle ([19], Theorem 7.1(3), also Corollary 7.2). It follows then from Corollary 3.18 and Lemma 5.5.2 that the set S_{ss} is $\bar{f}^{-1}(Q(\bar{Y})^N_{ss})$. Since $Q(\bar{Y})^N_{ss}$ is open in $Q(\bar{Y})^N$ this proves that S_{ss} is open is S.

Again making S smaller if necessary we can assume that the family $\mathscr{E}|S_{ss}$ is induced by a morphism $S_{ss}\to R_2$. It follows from Lemma 5.7 that the points in R_2 corresponding to stable bundles is open since it is the inverse image of the open subset $Q(\bar{Y})^N_s$ of $Q(\bar{Y})^N_{ss}$ under the morphism $R_2\to Q(\bar{Y})^N_{ss}$. Therefore S_s is open in S.

5.9. *Theorem. The functor* $\mathbf{F}^\tau_{ss}$ *(Definition* 3.9) *has a coarse moduli scheme* M^τ. *The scheme* M^τ *is irreducible, projective, normal and Cohen-Macaulay. The dimension of* M^τ *is* $\dim Z+(g-1)\cdot\dim G$. *The set* M^τ_s *of points in* M^τ *corresponding to stable G-bundles is an open (and hence dense) subset of* M^τ.

Proof. By Propositions 4.5 and 4.15.2 it follows that to prove the existence of a coarse moduli scheme for $\mathbf{F}^\tau_{ss}$ it is enough to prove that a good quotient of R_3 modulo $GL(n,\mathbb{C})$ exists. Since $R_3\to J_\tau\times R_2$, $R_2\to R_1$ and $R_1\to Q(\bar{Y})^N_{ss}$ are all finite $GL(n,\mathbb{C})$-equivariant morphisms it follows from Lemma 5.1 that a good quotient of R_3 modulo $GL(n,\mathbb{C})$ exists if a good quotient of $Q(\bar{Y})^N_{ss}$ modulo $GL(n,\mathbb{C})$ (or $SL(n,\mathbb{C})$, since the scalars act trivially on $Q(\bar{Y})^N_{ss}$) exists. We know that a good quotient of $Q(\bar{Y})^N_{ss}$ exists and is projective ([10], Theorem 1.10; [22], Theorem 1.1(B)). Therefore $\mathbf{F}^\tau_{ss}$ has a coarse moduli scheme M^τ. Moreover since $R_3\to J_\tau\times R_2$ etc. are finite morphisms it follows from Lemma 5.1 that M^τ is projective (noting that a scheme finite over a projective scheme is projective).

Since M^τ is a categorical quotient of the non-singular (and hence normal) scheme R_3 it follows that M^τ is normal ([10], Chapter 0, §2, p. 5). Since $q:R_3\to M^\tau$ is a good quotient R_3 can be covered by G-invariant affine open subsets $\operatorname{Spec} A_i$ such that the corresponding quotients $\operatorname{Spec} A_i^G$ form an affine open covering for M^τ.

Since R_3 is nonsingular the $\mathbb{C}$-algebras A_i are regular and hence by ([8], Main theorem) it follows that A_i^G are Cohen-Macaulay. Therefore M^τ is Cohen-Macaulay.

It follows easily from § 5.4.1 and Lemmas 5.7 and 5.1 that M^τ_s is open in M^τ and $q:R^s_3=q^{-1}(M^\tau_s)\to M^\tau_s$ is a geometric quotient.

Since $R^s_3\to M^\tau_s$ is a geometric quotient modulo $SL(n,\mathbb{C})$ we have $\dim M^\tau=\dim R_3-\dim SL(n,\mathbb{C})$. Since $R_3\to J_\tau\times R_2$ and $R_2\to R_1$ are étale and finite $\dim R_3=\dim J_\tau+\dim R_1$. Therefore using Lemma 4.13.4, $\dim M^\tau=(g\cdot\dim Z)+(n^2+(r+1)(g-1)-g)-(n^2-1)$. Noting that $r=\dim G-\dim Z$, we get $\dim M^\tau=\dim Z+(g-1)\dim G$.

We need the following lemma to complete the proof of the theorem.

5.9.1. *Lemma. Let S be a complex analytic space and* $\mathscr{E}\to S\times X$ *be a complex analytic family of semistable G-bundles of topological type* τ *parametrized by S. Then there is an analytic morphism* $f_{\mathscr{E}}:S\to M^\tau$ *such that for any* $s\in S$, $f_{\mathscr{E}}(s)$ *is the equivalence class of* $\mathscr{E}_s$.

Proof. This follows easily from the fact that the functors **Pic** and '$\Gamma(\rho,\mathscr{E})$' etc. used in our construction of universal families are representable in the analytic category also and are represented by the same universal spaces as in the algebraic category (cf. [5]).

We will now continue with the proof of Theorem 5.9. Since we have shown that M^τ is normal, to show that it is irreducible it is enough to show that it is connected. Let

$m_1, m_2 \in M^\tau$ and let E_1, E_2 be semistable G-bundles belonging to the equivalence classes m_1, m_2 respectively. Then by ([14], Proposition 4.2 p. 142) there is a complex analytic family of semistable G-bundles $\mathscr{E} \to S \times X$ parametrized by an open *connected* subspace S of the complex plane $\mathbb{C}$ such that for some $s_1, s_2 \in S$ we have $\mathscr{E}_{s_i} = E_i, i = 1, 2$. (In [14], Proposition 4.2 this is stated only for stable bundles but the same proof goes through for semistable bundles also, noting that the proof of ([14], Proposition 4.1, p. 138) with a little modification gives that for a complex analytic family $\mathscr{F} \to S \times X$ the set of $s \in S$ such that $\mathscr{F}_s$ is *semi*stable is an analytic open subset of S.) Now applying Lemma 5.9.1 we get that M^τ is connected and hence irreducible.

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